A modular axiomatization of fuzzy type theory

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Abstract

We present a modular, and thus flexibly modifiable, axiomatization of Novák's fuzzy type theory over IMTL-algebras with the Delta operator. The axioms and rules are separated into a basic many-valued type-theoretic layer, modules for the constituent parts of the underlying fuzzy logic, and a module governing fuzzy equality. We prove syntactically that the resulting modular system and Novák's original axiomatics, supplemented with a missing symmetry axiom for fuzzy equality, interpret one another, and that the resulting round-trip translations recover the original formulas up to the expected language-induced identifications. We also show that the added symmetry axiom is necessary for completeness with respect to Novák's intended semantics, and analyze the restrictions on fuzzy equality that make the axiom redundant in his later EQ-algebra-based fuzzy type theory.

Keywords: fuzzy type theory, axiomatization, fuzzy equality, syntactic translation, many-valued type theory, fuzzy logic, IMTL.

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1 Introduction

Novák's fuzzy type theory [8] is a many-valued version of simple type theory in the Church–Henkin tradition. Its language is a typed λ-calculus with primitive types for urelements and truth values, and with higher types interpreted as types of functions. In this respect it belongs to the same family of type theories as the classical systems of Church and Henkin (see, e.g., [1] for a standard presentation). The fuzzy aspect consists in interpreting the type of truth values not by the two-valued Boolean algebra but by an algebra of truth degrees: Novák's original system uses $\mathrm{IMTL}_{\triangle}$-algebras, though several variants based on other fuzzy logics have also been defined [7]. Fuzzy type theory, or FTT, has been used as a formalism for linguistic modeling, including the models of trichotomous evaluative expressions, fuzzy quantifiers, and fuzzy syllogisms [9, 10, 6].

Novák's original axiomatization of fuzzy type theory as given in [8] provides a compact presentation of FTT as a higher-order fuzzy logic over $\mathrm{IMTL}_{\triangle}$. However, its axioms are organized as a single system with various interdependencies, which makes later modifications of the background logic or of the requirements on fuzzy equality less straightforward. In this paper, we therefore provide an alternative axiomatization designed to have a modular structure while preserving the proof-theoretic strength of Novák's intended system. The modularity

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consists in dividing the axioms and rules into several largely independent groups, or 'modules', which separately axiomatize (1) the type-theoretic machinery, (2a) the propositional logic IMTL, (2b) the Delta connective, (2c) the quantifiers, and (3) fuzzy equality.

Novák's later variant of FTT based on EQ-algebras [11] already points in this modular direction, since its axioms are likewise divided into several blocks with only limited interaction. Another alternative is the Russell-style simple fuzzy type theory of [3], which uses crisp logical equality and allows theories of additional crisp or fuzzy structures to be added axiomatically. Both frameworks, however, differ substantially from the $\mathrm{IMTL}_{\triangle}$-based FTT, in particular in their treatment of equality. We therefore focus here primarily on Novák's original fuzzy type theory, with possible extensions to variants over other residuated fuzzy logics.

The modular or layered perspective on $\mathrm{IMTL}_{\triangle}$-based FTT was already outlined in [2, Rem. 3.18]; the present paper supplies an explicit syntactic reconstruction. We define a modular fuzzy type theory MFTT, establish mutual interpretability with Novák's axiomatics by translations in both directions, and verify that the compositions of these translations recover the original formulas modulo provable equality and language unification.

In carrying out this reconstruction, we also identify a gap in the published axiomatics of FTT over $\mathrm{IMTL}_{\triangle}$: its intended semantics requires equality to be a symmetric fuzzy relation, but the printed axiom list in [8] does not force the symmetry condition at the required strength. Therefore, we take the reference system FTT used throughout the paper to include the element-type symmetry axiom. We further show, by constructing a countermodel, that the system without the added axiom is incomplete with respect to Novák's intended $\mathrm{IMTL}_{\triangle}$-based semantics.

The article is organized as follows. Section 2 recalls the background logic $\mathrm{IMTL}_{\triangle}$ as well as Novák's FTT, and fixes notation and terminology. Section 3 introduces the modular fuzzy type theory MFTT and discusses some alternative forms of its axioms and rules. Section 4 proves the mutual interpretability of MFTT and FTT by explicit syntactic translations in both directions, and records the corresponding round-trip comparison. Section 5 establishes the necessity of the symmetry axiom for fuzzy equality in the $\mathrm{IMTL}_{\triangle}$-based FTT and compares this situation with EQ-based FTT. The concluding section then summarizes the results.

2 Preliminaries

We first recall two background systems used in the following sections: the predicate fuzzy logic $\mathrm{IMTL}_{\triangle}\forall$ and Novák's fuzzy type theory $\mathrm{FTT}(\mathrm{IMTL}_{\triangle})$. The formal presentations are given mainly for reference, so that the later comparison with the modular axiomatics can be stated syntactically without ambiguity.

2.1 The logic $\mathrm{IMTL}_{\triangle}\forall$

The logic $\mathrm{IMTL}_{\triangle}\forall$ is the first-order extension of the propositional logic $\mathrm{IMTL}_{\triangle}$ that underlies Novák's original version of fuzzy type theory [8]. We recall here the axiomatization used below. For its propositional IMTL core we use Pei's concise axiomatization from [14].

Definition 2.1 (The logic $\mathrm{IMTL}_{\triangle}\forall$)**.** The primitive propositional connectives of $\mathrm{IMTL}_{\triangle}\forall$ are $\rightarrow, \wedge, \bot, \triangle$. The following connectives are defined:

$$\neg\varphi \coloneqq \varphi \rightarrow \bot \qquad\qquad \top \coloneqq \neg\bot$$

$$\varphi \leftrightarrow \psi \coloneqq (\varphi \to \psi) \wedge (\psi \to \varphi) \qquad \varphi \vee \psi \coloneqq \neg(\neg\varphi \wedge \neg\psi)$$
$$\varphi \,\&\, \psi \coloneqq \neg(\varphi \to \neg\psi) \qquad \varphi \oplus \psi \coloneqq \neg(\neg\varphi \,\&\, \neg\psi)$$

The propositional IMTL core is axiomatized by Pei's six axiom schemata and the rule of modus ponens:

(IMTL-P1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(IMTL-P2) $\varphi \wedge \psi \to \varphi$
(IMTL-P3) $\varphi \wedge \psi \to \psi \wedge \varphi$
(IMTL-P4) $\varphi \to ((\varphi \to \psi) \to \varphi \wedge \psi)$
(IMTL-P5) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(IMTL-P6) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
(MP) $\varphi,\ \varphi \to \psi \ /\ \psi$

The connective $\triangle$ is axiomatized by the following schemata and the $\triangle$-necessitation rule:

($\triangle$1) $\triangle\varphi \vee \neg\triangle\varphi$
($\triangle$2) $\triangle(\varphi \vee \psi) \to (\triangle\varphi \vee \triangle\psi)$
($\triangle$3) $\triangle\varphi \to \varphi$
($\triangle$4) $\triangle\varphi \to \triangle\triangle\varphi$
($\triangle$5) $\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$
($\triangle$-nec) $\varphi \ /\ \triangle\varphi$

The first-order logic IMTL$_\triangle\forall$ is obtained from this propositional axiomatics by adding the universal quantifier $\forall$ to the logical vocabulary and allowing the formation of predicate formulas over a given extralogical language. The existential quantifier is defined as $(\exists x)\varphi \coloneqq \neg(\forall x)\neg\varphi$. Writing $\varphi[t/x]$ for the substitution of a term t for a variable x in φ, the quantifier axiom schemata and the generalization rule of IMTL$_\triangle\forall$ are as follows:

($\forall$1) $((\forall x)\varphi) \to \varphi[t/x]$, where the term t is substitutable for x in φ
($\forall$2) $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, where x is not free in χ
($\forall$3) $(\forall x)(\chi \vee \varphi) \to (\chi \vee (\forall x)\varphi)$, where x is not free in χ
(Gen) $\varphi \ /\ (\forall x)\varphi$

2.2 Novák's fuzzy type theory over IMTL$_\triangle$

We next recall Novák's axiomatics of fuzzy type theory over the logic IMTL$_\triangle$. Apart from minor notational changes, the system is Novák's FTT in the form presented in [8]; however, we supplement it with the element-type symmetry axiom, as discussed below in Remark 2.8 and Section 5. When it is necessary to distinguish this IMTL$_\triangle$-based reference system from other variants of fuzzy type theory, we shall denote it by FTT(IMTL$_\triangle$).

Definition 2.2 (The system of types in FTT)**.** Let ε and o be two distinct primitive type symbols. The set of types is the smallest set Types such that:

- $\{\varepsilon, o\} \subseteq \mathsf{Types}$;

- If $\alpha, \beta \in \mathsf{Types}$, then $(\beta\alpha) \in \mathsf{Types}$.

The primitive type ε is the type of urelements, and the primitive type o is the type of truth values. A complex type $(\beta\alpha)$ is read as the type of unary functions from arguments of type α to values of type β. We associate type expressions to the left and omit parentheses accordingly; thus, $\alpha\beta\gamma$ means $((\alpha\beta)\gamma)$.

Definition 2.3 (The language of FTT)**.** Let Types be the set of types from Definition 2.2. The basic language $\mathcal{L}$ of Novák's fuzzy type theory consists of the following primitive symbols:

- Variables $x_\alpha, y_\alpha, z_\alpha, \ldots$, for each type $\alpha \in \mathsf{Types}$.
- Individual constants $c_\alpha, d_\alpha, \ldots$, for each type $\alpha \in \mathsf{Types}$.
- Distinguished constants $\boldsymbol{C}_{ooo}$, $\boldsymbol{D}_{oo}$, and $\boldsymbol{E}_{o\alpha\alpha}$ for every $\alpha \in \mathsf{Types}$.
- Auxiliary symbols: the abstraction symbol λ and parentheses.

The intended meanings of the distinguished constants are as follows: $\boldsymbol{E}_{o\alpha\alpha}$ is the equality at type α, $\boldsymbol{C}_{ooo}$ is lattice conjunction, and $\boldsymbol{D}_{oo}$ is the Monteiro–Baaz Delta connective.

Definition 2.4 (The formulas of FTT)**.** The sets $\mathsf{Form}_\alpha(\mathcal{L})$ of formulas of type α, for each $\alpha \in \mathsf{Types}$, are defined as the smallest family of sets closed under the following formation rules:

- Variables as formulas: If x_α is a variable of type α, then $x_\alpha \in \mathsf{Form}_\alpha(\mathcal{L})$.
- Constants as formulas: If c_α is a constant of type α, then $c_\alpha \in \mathsf{Form}_\alpha(\mathcal{L})$.
- Application: If $B_{\beta\alpha} \in \mathsf{Form}_{\beta\alpha}(\mathcal{L})$ and $A_\alpha \in \mathsf{Form}_\alpha(\mathcal{L})$, then $(B_{\beta\alpha}A_\alpha) \in \mathsf{Form}_\beta(\mathcal{L})$.
- λ-Abstraction: If $A_\beta \in \mathsf{Form}_\beta(\mathcal{L})$ and x_α is a variable of type α, then $(\lambda x_\alpha A_\beta) \in \mathsf{Form}_{\beta\alpha}(\mathcal{L})$.

The set of all $\mathcal{L}$-formulas is $\mathsf{Form}(\mathcal{L}) = \bigcup_{\alpha\in\mathsf{Types}} \mathsf{Form}_\alpha(\mathcal{L})$. When $A \in \mathsf{Form}_\alpha(\mathcal{L})$, we may write A_α to indicate that A has type α.

Parentheses may be omitted in formulas wherever this does not cause ambiguity. We adopt left association for application, writing ABC for $((AB)C)$. We also take application to bind more strongly than λ-abstraction, so that λxAB means $(\lambda x(AB))$.

Definition 2.5 (Defined symbols of FTT)**.** Let $\mathcal{L}$ be the language from Definition 2.3. The following symbols are introduced by definition:

- For each $\alpha \in \mathsf{Types}$, the *equality* at type α is $\equiv_\alpha \coloneqq \lambda x_\alpha\big(\lambda y_\alpha \boldsymbol{E}_{o\alpha\alpha} y_\alpha\big)x_\alpha$.
  We write $A_\alpha \equiv B_\alpha$ for $\equiv_\alpha B_\alpha A_\alpha$.
- *Lattice conjunction* $\wedge \coloneqq \lambda x_o\big(\lambda y_o \boldsymbol{C}_{ooo} y_o\big)x_o$.
- The *Delta connective* $\triangle \coloneqq \lambda x_o \boldsymbol{D}_{oo} x_o$.
- The *full-truth* constant $\top \coloneqq (\lambda x_o x_o) \equiv (\lambda x_o x_o)$.
- The *full-falsity* constant $\bot \coloneqq (\lambda x_o x_o) \equiv (\lambda x_o \top)$.

- *Negation* $\neg \coloneqq \lambda x_o(\bot \equiv x_o)$.
- *Implication* $\to \coloneqq \lambda x_o \lambda y_o\big((x_o \wedge y_o) \equiv x_o\big)$.
- *Lattice disjunction* $\vee \coloneqq \lambda x_o \lambda y_o\big(((x_o \to y_o) \to y_o) \wedge ((y_o \to x_o) \to x_o)\big)$.
- *Strong conjunction* $\& \coloneqq \lambda x_o \lambda y_o \neg(x_o \to \neg y_o)$.
- *Strong disjunction* $\oplus \coloneqq \lambda x_o \lambda y_o \neg(\neg x_o \,\&\, \neg y_o)$.
- If $A_o \in \mathsf{Form}_o(\mathcal{L})$ and x_α is a variable of type α, the *universal quantifier* is defined by

$$(\forall x_\alpha)A_o \;\coloneqq\; (\lambda x_\alpha A_o) \equiv (\lambda x_\alpha \top).$$

- If $A_o \in \mathsf{Form}_o(\mathcal{L})$ and x_α is a variable of type α, the *existential quantifier* is defined by

$$(\exists x_\alpha)A_o \;\coloneqq\; \neg(\forall x_\alpha)\neg A_o.$$

Thus $\equiv_\alpha \in \mathsf{Form}_{o\alpha\alpha}(\mathcal{L})$ for each $\alpha \in \mathsf{Types}$. Furthermore, $\top, \bot, (\forall x_\alpha)A_o, (\exists x_\alpha)A_o \in \mathsf{Form}_o(\mathcal{L})$; $\neg, \triangle \in \mathsf{Form}_{oo}(\mathcal{L})$; and $\to, \wedge, \vee, \&, \oplus \in \mathsf{Form}_{ooo}(\mathcal{L})$.

We write $A_o \circledast B_o$ for $\circledast A_o B_o$, where $\circledast \in \{\to, \wedge, \vee, \&, \oplus\}$. We use the standard convention on parentheses, with the following precedence rules: (i) application of formulas, including unary connectives, binds more strongly than the infix notation introduced for binary logical symbols; (ii) the conjunctive and disjunctive connectives $\wedge, \&, \vee, \oplus$ bind more strongly than $\to$ and $\equiv$.

Note that in [8], implication $\to$ and strong disjunction $\oplus$ were denoted by $\Rightarrow$ and ∇, respectively. We will additionally use $A_o \leftrightarrow B_o$ as an abbreviation for $(A_o \to B_o) \wedge (B_o \to A_o)$ in FTT-formulas.

Definition 2.6 (Substitution and replacement)**.** We use the standard notion of *subformula*. (Thus, in particular, the variable immediately following λ is not considered an occurrence of a subformula.)

An occurrence of a variable x_α in a formula B_β is *free* if it is not contained in any subformula of B_β of the form $\lambda x_\alpha C_\gamma$; otherwise it is *bound*.

As usual, the formula A_α is *substitutable for a variable* x_α in B_β if no free occurrence of x_α in B_β occurs in a subformula of the form $\lambda y_\delta C_\eta$ such that y_δ is free in A_α.

Similarly, a formula A'_α is *substitutable for an occurrence of a subformula* A_α in B_β if no free variable of A'_α becomes bound in B_β as a result of the replacement.

If $x^1_{\alpha_1}, \ldots, x^n_{\alpha_n}$ are pairwise distinct and $A^i_{\alpha_i}$ is substitutable for $x^i_{\alpha_i}$ in B_β for each $i \leq n$, then $B_\beta[A^1_{\alpha_1}/x^1_{\alpha_1}, \ldots, A^n_{\alpha_n}/x^n_{\alpha_n}]$ denotes *simultaneous (capture-free) substitution* of each $x^i_{\alpha_i}$ by $A^i_{\alpha_i}$, i.e., the formula obtained from B_β by replacing every free occurrence of $x^i_{\alpha_i}$ by $A^i_{\alpha_i}$, simultaneously for all $i = 1, \ldots, n$.

The expression $B_\beta[A'_\alpha /\!\!/ A_\alpha]$ denotes a formula obtained from B_β by replacing one specified occurrence of a subformula A_α in B_β by A'_α. Unlike simultaneous substitution, this replacement need not satisfy the substitutability condition unless said otherwise. If there is more than one occurrence of A_α in B_β, the occurrence to be replaced is regarded as part of the notation (i.e., we should formally write $/\!\!/_n$ for replacing the n-th occurrence, but the subscript n is usually omitted when it is arbitrary).

When the notation $B_\beta[A'_\alpha /\!\!/ A_\alpha]$ occurs inside a larger formula, it binds to the immediately preceding formula B_β (i.e., binds more strongly than application). The same convention applies to the notation for simultaneous substitution.

Definition 2.7 (Logical axioms and rules of FTT)**.** The logical axioms of FTT are given by the following schemata:

(FT1) $\triangle(x_\alpha \equiv y_\alpha) \to (f_{\beta\alpha}x_\alpha \equiv f_{\beta\alpha}y_\alpha)$
(FT2a) $(\forall x_\alpha)(f_{\beta\alpha}x_\alpha \equiv g_{\beta\alpha}x_\alpha) \to (f_{\beta\alpha} \equiv g_{\beta\alpha})$
(FT2b) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \to (f_{\beta\alpha}x_\alpha \equiv g_{\beta\alpha}x_\alpha)$
(FT3) $(\lambda x_\alpha B_\beta)A_\alpha \equiv B_\beta[A_\alpha/x_\alpha]$, where A_α is substitutable for x_α in B_β
(FT4) $(x_\varepsilon \equiv y_\varepsilon) \to ((y_\varepsilon \equiv z_\varepsilon) \to (x_\varepsilon \equiv z_\varepsilon))$
(FT5) $(x_o \equiv y_o) \equiv ((x_o \to y_o) \wedge (y_o \to x_o))$
(FT6) $(A_o \equiv \top) \equiv A_o$
(FT7) $(A_o \to B_o) \to ((B_o \to C_o) \to (A_o \to C_o))$
(FT8) $(A_o \to (B_o \to C_o)) \equiv (B_o \to (A_o \to C_o))$
(FT9) $((A_o \to B_o) \to C_o) \to (((B_o \to A_o) \to C_o) \to C_o)$
(FT10) $(\neg B_o \to \neg A_o) \equiv (A_o \to B_o)$
(FT11) $A_o \wedge B_o \equiv B_o \wedge A_o$
(FT12) $A_o \wedge B_o \to A_o$
(FT13) $(A_o \wedge B_o) \wedge C_o \equiv A_o \wedge (B_o \wedge C_o)$
(FT14) $(g_{oo}(\triangle x_o) \wedge g_{oo}(\neg\triangle x_o)) \equiv (\forall y_o)g_{oo}(\triangle y_o)$
(FT15) $\triangle(A_o \wedge B_o) \equiv \triangle A_o \wedge \triangle B_o$
(FT16) $(\forall x_\alpha)(A_o \to B_o) \to (A_o \to (\forall x_\alpha)B_o)$, where x_α is not free in A_o
(FT17) $(A_\varepsilon \equiv B_\varepsilon) \to (B_\varepsilon \equiv A_\varepsilon)$, see Remark 2.8

FTT has the following inference rules:

(FT-N) $A_o \;/\; \triangle A_o$
(FT-R) $A_\alpha \equiv A'_\alpha,\ B_o \;/\; B_o[A'_\alpha /\!\!/ A_\alpha]$

A *theory* T over FTT is given by a language $\mathcal{L}(T)$ extending the basic language $\mathcal{L}$ and by a set $\mathsf{Ax}(T) \subseteq \mathsf{Form}_o\big(\mathcal{L}(T)\big)$ of *special axioms* of T. The notions of proof and provability of A_o in T, denoted by $T \vdash A_o$, are defined as usual.

Remark 2.8. Note that axiom (FT17) is not included in the original axiomatization of FTT over $\mathrm{IMTL}_\triangle$ in [8]. It is, however, valid in the intended semantics and was implicitly used in the canonical-model construction in [8, Eq. (41)]. We therefore include it in the reference system FTT. Section 5 shows that it is not derivable from the printed axiomatics of [8].

3 The modular type theory over $\mathrm{IMTL}_\triangle$ with fuzzy equality

In this section, we present a modular axiomatization of fuzzy type theory over $\mathrm{IMTL}_\triangle$ with fuzzy equality, following the layered perspective of [2, Rem. 3.18]. We begin with an equivalent presentation of the minimal many-valued type theory TT_0 from [2, Sect. 2]. We then expand its language by the logical symbols of propositional IMTL, the connective $\triangle$, and the universal quantifier, and extend its deductive apparatus by the corresponding axioms and rules. Finally, we add a module axiomatizing the fuzzy equality predicate $\approx$; we call the resulting system

MFTT (or modular FTT) for short. We then discuss some redundancies and alternative forms in the MFTT axiomatics.

3.1 The axiomatic system MFTT

We now present the modular axiomatization of FTT, structured into several packages (or Modules).

The language of MFTT. Although each module below axiomatizes only certain primitive symbols, their axioms and rules apply to the entire language of MFTT, which we therefore present in advance and as a whole. If some modules are changed (e.g., by taking a different background fuzzy logic), the corresponding part of the language must be replaced accordingly.

Definition 3.1 (The language of MFTT)**.** The language of MFTT is that of the typed λ-calculus over the same system of types as in Definition 2.2. The variables, extralogical constants, auxiliary symbols, and formation rules for formulas are as in Definitions 2.3 and 2.4.

For each type $\alpha \in \mathsf{Types}$, the logical language contains the *type-theoretic equality* constant $=_{o\alpha\alpha}$. The remaining primitive logical constants in the language of MFTT over IMTL$_\triangle\forall$ are the following:

- *Implication* $\to_{ooo}$.
- *Lattice conjunction* $\wedge_{ooo}$.
- *Falsity* $\bot_o$.
- The *Delta connective* $\triangle_{oo}$.
- The *universal quantifier* $\forall_{o(o\alpha)}$, for each $\alpha \in \mathsf{Types}$.
- A distinguished *fuzzy equality* $\approx_{o\alpha\alpha}$, for each $\alpha \in \mathsf{Types}$.

We write $A_\alpha = B_\alpha$ and $A_\alpha \approx B_\alpha$, respectively, for $=_{o\alpha\alpha}A_\alpha B_\alpha$ and $\approx_{o\alpha\alpha}A_\alpha B_\alpha$. We usually omit type annotations of propositional connectives. For binary connectives (including those defined below), we usually write $A_o \circledast B_o$ for $\circledast_{ooo}A_oB_o$, where $\circledast \in \{\to, \wedge, \vee, \&, \oplus, \leftrightarrow\}$. The following symbols are introduced by definition:

- *Negation* $\neg := \lambda x_o(x_o \to \bot)$.
- *Truth* $\top := \neg\bot$.
- *Lattice disjunction* $\vee := \lambda x_o\lambda y_o\neg(\neg x_o \wedge \neg y_o)$.
- *Strong conjunction* $\& := \lambda x_o\lambda y_o\neg(x_o \to \neg y_o)$.
- *Strong disjunction* $\oplus := \lambda x_o\lambda y_o\neg(\neg x_o \,\&\, \neg y_o)$.
- *Propositional equivalence* $\leftrightarrow := \lambda x_o\lambda y_o\big((x_o \to y_o) \wedge (y_o \to x_o)\big)$.
- *The existential quantifier* $\exists_{o(o\alpha)} := \lambda p_{o\alpha}\neg\big(\forall_{o(o\alpha)}\lambda x_\alpha\neg(p_{o\alpha}x_\alpha)\big)$, for each type α.

As notational abbreviations, we write $(\forall x_\alpha)A_o$ and $(\exists x_\alpha)A_o$, respectively, for $\forall_{o(o\alpha)}\lambda x_\alpha A_o$ and $\exists_{o(o\alpha)}\lambda x_\alpha A_o$. We use the same precedence conventions as in Definition 2.5, where $=$, $\approx$, and $\leftrightarrow$ bind with the same strength as $\to$. The syntactic notions introduced in Definition 2.6 apply without change to formulas in the language of MFTT and its definitional extensions. We use the same notation $B[A^1/x^1, \ldots, A^n/x^n]$ and $B[A'/\!\!/A]$.

Module 1: The ground many-valued type theory. This fundamental module realizes the minimal many-valued type theory TT_0 from [2]; the deductive equivalence between TT_0 and Module 1 is established below in Section 3.2. In the lists of axioms and rules, we always present axioms as rules with no premises (i.e., in the form / A_o). The symbols for substitution and single-occurrence replacement are as in Definition 2.6. The theory uses the language from Definition 3.1.

Definition 3.2 (Module 1: the ground many-valued type theory)**.** The rules and axioms of Module 1 are given by the following schemata:

(TT1-1)	$A_o \,/\, A_o[B_\alpha/x_\alpha]$	substitution
(TT1-2)	$A_o,\ B_\alpha = B'_\alpha \,/\, A_o[B'_\alpha /\!\!/ B_\alpha]$	replacement
(TT1-3)	$/\ (\lambda x_\alpha A_\beta)B_\alpha = A_\beta[B_\alpha/x_\alpha]$	λ-conversion
(TT1-4)	$/\ (\lambda x_\alpha\, F_{\beta\alpha} x_\alpha) = F_{\beta\alpha}$	η-conversion

Here, schema (TT1-1) assumes that B_α is substitutable for x_α in A_o, schema (TT1-3) assumes that B_α is substitutable for x_α in A_β, and (TT1-4) assumes that x_α is not free in $F_{\beta\alpha}$.

Within MFTT, Module 1 constitutes the minimal many-valued type-theoretic apparatus, akin to an "operating system". The subsequent modules represent changeable extensions that allow for substantial variability of the resulting system.

Module 2: The logic $\mathrm{IMTL}_\triangle\forall$**.** Module 2 extends the ground type theory TT_0 of Module 1 by the axioms and rules for the logical symbols from Definition 3.1, so that these symbols behave according to the logic $\mathrm{IMTL}_\triangle\forall$.

Module 2 can be divided into submodules 2a for the propositional core IMTL, 2b for the connective $\triangle$, and 2c for quantifiers. The type theories axiomatized by Modules 1+2a, Modules 1+2a+2b, Modules 1+2a+2c, and Modules 1+2a+2b+2c can be systematically denoted by TT(IMTL), $\mathrm{TT}(\mathrm{IMTL}_\triangle)$, TT(IMTL$\forall$), and $\mathrm{TT}(\mathrm{IMTL}_\triangle\forall)$, respectively.

Definition 3.3 (Module 2: the logic $\mathrm{IMTL}_\triangle\forall$)**.** Module 2a consists of Pei's axioms for the IMTL core and the modus ponens rule:

(TT2a-1)	$/\ (A_o \to B_o) \to ((B_o \to C_o) \to (A_o \to C_o))$	(IMTL-P1)
(TT2a-2)	$/\ A_o \wedge B_o \to A_o$	(IMTL-P2)
(TT2a-3)	$/\ A_o \wedge B_o \to B_o \wedge A_o$	(IMTL-P3)
(TT2a-4)	$/\ A_o \to ((A_o \to B_o) \to A_o \wedge B_o)$	(IMTL-P4)
(TT2a-5)	$/\ ((A_o \to B_o) \to C_o) \to (((B_o \to A_o) \to C_o) \to C_o)$	(IMTL-P5)
(TT2a-6)	$/\ (\neg A_o \to \neg B_o) \to (B_o \to A_o)$	(IMTL-P6)
(TT2a-7)	$A_o,\ A_o \to B_o \,/\, B_o$	modus ponens

Module 2b consists of the axioms and rule for the Delta connective:

(TT2b-1)	$/\ \triangle A_o \vee \neg\triangle A_o$	($\triangle$1)
(TT2b-2)	$/\ \triangle(A_o \vee B_o) \to (\triangle A_o \vee \triangle B_o)$	($\triangle$2)
(TT2b-3)	$/\ \triangle A_o \to A_o$	($\triangle$3)

(TT2b-4)	$/ \triangle A_o \to \triangle\triangle A_o$	($\triangle$4)
(TT2b-5)	$/ \triangle(A_o \to B_o) \to (\triangle A_o \to \triangle B_o)$	($\triangle$5)
(TT2b-6)	$A_o \,/\, \triangle A_o$	$\triangle$-necessitation

Module 2c consists of the quantifier axioms and the rule of generalization:

(TT2c-1)	$/ ((\forall x_\alpha)A_o) \to (A_o[B_\alpha/x_\alpha])$	($\forall$1)
(TT2c-2)	$/ (\forall x_\alpha)(C_o \to A_o) \to (C_o \to (\forall x_\alpha)A_o)$	($\forall$2)
(TT2c-3)	$/ (\forall x_\alpha)(C_o \vee A_o) \to (C_o \vee (\forall x_\alpha)A_o)$	($\forall$3)
(TT2c-4)	$A_o \,/\, (\forall x_\alpha)A_o$	generalization

In the axiom schema (TT2c-1), the formula B_α must be substitutable for x_α in A_o. In (TT2c-2) and (TT2c-3), the variable x_α must not be free in C_o.

Module 3: Fuzzy equality. Building on Modules 1 and 2, Module 3 provides an axiomatization of the fuzzy equality predicate $\approx$ so as to match Novák's treatment of equality in FTT: at the truth-value type o, fuzzy equality is identified with the equivalence connective; at the element type ε, it is a distinguished fuzzy equality relation; and at higher types $\beta\alpha$ it is induced pointwise from the fuzzy equality on the codomain type β, by quantification over arguments of the domain type α.

Unlike Novák's presentation of fuzzy type theory, we keep the equality $=$ of Module 1 distinct from the fuzzy equality predicate $\approx$. The former is the type-theoretic equality inherited from the ground system TT_0: it governs replacement, λ-conversion, and η-conversion, and its role is confined to derivable equality statements or, semantically, to equality holding to the full degree. By contrast, $\approx$ is the graded equality predicate whose behavior is axiomatized in the present module. The symbol $\equiv$ is reserved for Novák's single equality predicate in FTT.

Remark 3.4. The use of more than one equality predicate already has a precedent within the FTT tradition. In the theory of trichotomous evaluative linguistic expressions, Novák supplements the built-in equality $\equiv$ of FTT with an additional fuzzy equality $\sim$ on truth values, generally distinct from $\equiv_o$ [9, Sects. 3.5.1 and 3.5.3]. In that setting, the two equalities are distinguished for a different reason than $=$ and $\approx$ in MFTT: $\sim$ is introduced specifically for modeling the sorites-based horizons underlying evaluative expressions and is kept separate from $\equiv_o$ to allow a more general formulation of the theory [9, Sect. 3.5.3]. The same additional truth-value equality is subsequently used in the theories of intermediate quantifiers and generalized intermediate syllogisms [10, 6].

Definition 3.5 (Module 3: fuzzy equality)**.** Module 3 extends $\mathrm{TT}(\mathrm{IMTL}_\triangle\forall)$ by the fuzzy equality predicate $\approx$ from Definition 3.1, governed by the following axiom schemata, for all $\alpha, \beta \in \mathsf{Types}$:

(TT3-1)	$/ (A_o \approx B_o) \leftrightarrow (A_o \leftrightarrow B_o)$	truth-value equality
(TT3-2)	$/ A_\varepsilon \approx A_\varepsilon$	element-type reflexivity
(TT3-3)	$/ (A_\varepsilon \approx B_\varepsilon) \to (B_\varepsilon \approx A_\varepsilon)$	element-type symmetry
(TT3-4)	$/ (A_\varepsilon \approx B_\varepsilon) \to ((B_\varepsilon \approx C_\varepsilon) \to (A_\varepsilon \approx C_\varepsilon))$	element-type transitivity
(TT3-5)	$/ \triangle(A_\alpha \approx B_\alpha) \to (A_\alpha = B_\alpha)$	separation
(TT3-6)	$/ \triangle(A_\alpha \approx B_\alpha) \to (F_{\beta\alpha}A_\alpha \approx F_{\beta\alpha}B_\alpha)$	weak extensionality

(TT3-7) $\quad / (F_{\beta\alpha} \approx G_{\beta\alpha}) \leftrightarrow (\forall x_\alpha)(F_{\beta\alpha} x_\alpha \approx G_{\beta\alpha} x_\alpha)$ pointwise function equality

In (TT3-7), the variable x_α is assumed not to occur freely in $F_{\beta\alpha}$ or $G_{\beta\alpha}$.

Axiom (TT3-1) identifies fuzzy equality of truth values with the equivalence connective. Axioms (TT3-2)–(TT3-4) say that fuzzy equality at the element type ε is a fuzzy equivalence relation. Axiom (TT3-5) is 1-faithfulness, or full-degree reflection: if objects of any type are fuzzily equal to the full degree, then they are type-theoretically equal. Axiom (TT3-6) ensures that full-degree fuzzy equality of arguments is preserved by arbitrary functions. Finally, axiom (TT3-7) defines fuzzy equality at function types pointwise.

The type theory that consists of all three modules will be systematically denoted by $\mathrm{TT}_{\approx}(\mathrm{IMTL}_{\triangle}\forall)$ and informally referred to as MFTT (or modular FTT).

For MFTT, we use the usual Henkin-style semantics of many-valued type theory (cf. [2]), with an $\mathrm{IMTL}_{\triangle}$-algebra of truth values, the usual interpretation of the quantifiers, and the fuzzy equality $\approx$ subject to the semantic counterparts of the axioms of Module 3. We refer to such structures as MFTT-*models*. It is routine to verify that all axioms of MFTT are valid in these models and that its inference rules preserve validity. Thus MFTT is sound with respect to the class of MFTT-models.

Several natural axiomatic extensions of MFTT can further constrain the relation between the type-theoretic equality $=$ and the fuzzy equality $\approx$. Their systematic investigation is left for future work.

Remark 3.6. Since Module 3 consists exclusively of axioms and contains no inference rules, it is an axiomatic *theory* (in the sense of Definition 2.7) over Modules 1+2, i.e., over $\mathrm{TT}(\mathrm{IMTL}_{\triangle}\forall)$. Moreover, the fact that $\approx$ does not occur in Modules 1 and 2 shows that fuzzy equality is in fact an additional structure rather than an indispensable part of the many-valued type theory.

More generally, the modular axiomatization organizes fuzzy type theory into a layered structure: Module 1 provides the type-theoretic foundation, independent of the choice of fuzzy logic; Module 2 then supplies the chosen logic; and Module 3 is an axiomatic theory of fuzzy equality in this type-theoretic setting. Module 3 may be omitted, replaced, or supplemented by modules axiomatizing other first- or higher-order fuzzy structures, such as fuzzy orderings, fuzzy preference relations, or fuzzy measures with the induced generalized quantifiers.

The layered structure of MFTT also suggests how partial variants of fuzzy type theory could be constructed. A version of Novák's later EQ-algebra-based fuzzy type theory allowing partial functions has already been developed and used to model the semantics of definite descriptions and fuzzy quantifiers [12, 13]. In the present $\mathrm{IMTL}_{\triangle}\forall$-based setting, one could replace Module 2 with a suitable partial fuzzy logic, e.g., along the lines of [4]. If partiality is represented by total objects together with a definedness indicator, Module 1 can remain essentially unchanged, whereas for genuinely partial λ-abstraction or application, corresponding local changes to Module 1 would be required. Module 3 would then be adapted to the chosen treatment of partial fuzzy equality, and theories of further partial structures could again be added as separate modules.

3.2 Redundancy and alternative forms of axioms and rules in MFTT

Some of the axioms and rules of MFTT are redundant or can be replaced by equivalent alternatives in the presence of the remaining axioms and rules from Modules 1–3. In addition,

modifying or omitting one module may require adjustments to others. Here we gather some of these cases.

Redundancy of substitution in logics with quantifiers. In the presence of basic logical principles, the substitution rule may be omitted:

Proposition 3.7. *In theories that contain modus ponens* (TT2a-7)*, generalization* (TT2c-4)*, and specification* (TT2c-1)*, the substitution rule* (TT1-1) *is derivable.*

Proof. Let B_α be substitutable for x_α in A_o. From the premise A_o, generalization (TT2c-4) yields $(\forall x_\alpha)A_o$; an instance of specification (TT2c-1) gives $((\forall x_\alpha)A_o) \to A_o[B_\alpha/x_\alpha]$; applying modus ponens (TT2a-7) to these two formulas, we obtain $A_o[B_\alpha/x_\alpha]$. □

A related derivation of substitution, using replacement and λ-conversion together with generalization and specification, is given in [2, Cor. 3.8]. Proposition 3.7 also shows that this redundancy is independent of the remaining principles of Module 1, provided that Module 2 includes the requisite logical rules.

Quantifier axioms in logics without involutive negation. While the quantifier axioms (TT2c-1)–(TT2c-3) together with the generalization rule (TT2c-4) are sufficient for Module 2c over the logic $\mathrm{IMTL}_\triangle$, they may not suffice if Module 2a is replaced by a fuzzy logic whose first-order extension does not define $\exists$ from $\forall$. In that case, $\exists$ should be regarded as a primitive quantifier, and Module 2c should be supplemented with the following two axiom schemata:

(TT2c-5)	$/\ A_o[B_\alpha/x_\alpha] \to (\exists x_\alpha)A_o$	$(\exists 1)$
(TT2c-6)	$/\ (\forall x_\alpha)(A_o \to C_o) \to (((\exists x_\alpha)A_o) \to C_o)$	$(\exists 2)$

In (TT2c-5), the formula B_α is assumed to be substitutable for x_α in A_o, and in (TT2c-6), the variable x_α must not be free in C_o.

Equivalence of Module 1 and the minimal many-valued type theory TT_0. Module 1 differs slightly from the axiomatization of the minimal many-valued type theory TT_0 given in [2, Def. 2.8]. While the substitution rule and the λ-conversion axiom have the same form in both systems, TT_0 replaces capture-permitting replacement (TT1-2) and η-conversion (TT1-4) with the following two rules:

(TT1-2′)	$A_o,\ B_\alpha = B'_\alpha\ /\ A_o[B'_\alpha /\!\!/ B_\alpha]$	capture-free replacement
(TT1-4′)	$F_{\beta\alpha}x_\alpha = G_{\beta\alpha}x_\alpha\ /\ F_{\beta\alpha} = G_{\beta\alpha}$	functional extensionality

In (TT1-2′), no free variable of B'_α may become bound as a result of the replacement, and in (TT1-4′), the variable x_α is not free in $F_{\beta\alpha}$ or $G_{\beta\alpha}$.

Proposition 3.8 (Equivalent TT_0-presentation of Module 1)**.** *In Module 1, the pair consisting of replacement* (TT1-2) *and η-conversion* (TT1-4) *can be replaced, without changing derivability, by capture-free replacement* (TT1-2′) *and functional extensionality* (TT1-4′)*.*

Proof. First assume (TT1-2) and (TT1-4). Capture-free replacement (TT1-2′) is an immediate special case of (TT1-2). Reflexivity, symmetry, and transitivity of provable equality are derivable from λ-conversion and replacement exactly as in [2, Thm. 2.11]. Now assume

$F_{\beta\alpha}x_\alpha = G_{\beta\alpha}x_\alpha$, where x_α is not free in $F_{\beta\alpha}$ or $G_{\beta\alpha}$. From reflexivity, we have $\lambda x_\alpha(F_{\beta\alpha}x_\alpha) = \lambda x_\alpha(F_{\beta\alpha}x_\alpha)$, from which replacement (TT1-2) yields $\lambda x_\alpha(F_{\beta\alpha}x_\alpha) = \lambda x_\alpha(G_{\beta\alpha}x_\alpha)$. Two further applications of replacement using η-conversion (TT1-4) on both sides yield $F_{\beta\alpha} = G_{\beta\alpha}$, proving (TT1-4′).

Conversely, assume (TT1-2′) and (TT1-4′). The η-conversion axiom (TT1-4) follows immediately from λ-conversion (TT1-3) and functional extensionality (TT1-4′), as in [2, Thm. 2.12(2)]. Moreover, provable equality is reflexive, symmetric, and transitive by [2, Thm. 2.11]; the proof given there uses only λ-conversion and capture-free replacement. We now prove the derivability of (TT1-2).

We first derive congruence under λ-abstraction. Let $C_\delta = D_\delta$, and choose z_γ fresh for C_δ and D_δ. By substitution (TT1-1), we have $C_\delta[z_\gamma/x_\gamma] = D_\delta[z_\gamma/x_\gamma]$. Together with the two instances of λ-conversion (TT1-3), this gives $(\lambda x_\gamma C_\delta)z_\gamma = (\lambda x_\gamma D_\delta)z_\gamma$ by symmetry and transitivity of equality. Functional extensionality (TT1-4′) therefore yields

(1) $$C_\delta = D_\delta \;\vdash\; \lambda x_\gamma C_\delta = \lambda x_\gamma D_\delta.$$

Now let A'_β be obtained from A_β by replacing a specified occurrence of B_α by B'_α, with capture permitted as in (TT1-2). We show, by induction on the depth of the specified occurrence in A_β, that

(2) $$B_\alpha = B'_\alpha \;\vdash\; A_\beta = A'_\beta.$$

At depth zero, this is the premise itself. If A_β is an application $C_{\beta\gamma}D_\gamma$ and the specified occurrence lies in one of its two constituents, the induction hypothesis gives equality of the corresponding constituents, and (TT1-2′), applied to the reflexive equality $C_{\beta\gamma}D_\gamma = C_{\beta\gamma}D_\gamma$, gives (2); no new binder is crossed in this step, so the replacement is capture-free. If A_β is $\lambda x_\gamma C_\delta$, then $A'_\beta = \lambda x_\gamma C'_\delta$; the induction hypothesis gives $C_\delta = C'_\delta$, and (1) yields $A_\beta = A'_\beta$. This completes the proof of (2).

Finally, from the premises A_o and $B_\alpha = B'_\alpha$ of (TT1-2), derive $A_o = A'_o$ by (2); then obtain A'_o from the premise A_o by (TT1-2′), replacing the whole formula A_o by A'_o (this replacement is trivially capture-free). Thus (TT1-2) is derivable. □

The alternative presentation of Module 1 given in Proposition 3.8 is, up to the present notation and terminology, precisely the axiomatization of TT_0 in [2]. Consequently, Module 1 and TT_0 are deductively equivalent. Module 1 provides a slightly more convenient presentation for the present purposes, as its replacement rule requires no capture-freeness check and functional extensionality is represented by the η-conversion axiom rather than by an additional derivation rule. Nevertheless, the capture-free replacement rule is not specific to TT_0: it is also used, e.g., in HOL-style higher-order logic [5]. By contrast, Novák's rule (FT-R) explicitly follows the capture-permitting formulation used, e.g., by Andrews [1].

Remark 3.9. The claim in [2, Thm. 2.13] that the functional extensionality rule is interderivable with the η-conversion axiom over the remaining axioms and rules of TT_0 contains a gap: the proof in the direction from η-conversion to functional extensionality uses an instance of replacement that is not capture-free and is therefore not licensed by the replacement rule of TT_0. The converse direction, deriving η-conversion from functional extensionality [2, Thm. 2.12(2)], is valid. This gap does not affect TT_0 itself, since functional extensionality is one of its primitive rules. With the capture-permitting replacement rule of Module 1, however, the problematic step is legitimate. Consequently, over the remaining axioms and rules of Module 1, functional extensionality and η-conversion are indeed interderivable.

Separation without $\triangle$. Hierarchically, Module 3 presupposes the presence of Modules 1 and 2. Module 2a may be replaced by another fuzzy logic, and the interpretability proofs from Section 4 would then have to be adapted accordingly. However, for fuzzy logics without $\triangle$, i.e., with Module 2b omitted, an analogous modification encounters the further difficulty that the connective $\triangle$ is used in the axioms of separation (TT3-5) and weak extensionality (TT3-6).

In the absence of $\triangle$, a weaker form of the connection between the equality predicates $\approx$ and $=$ can be retained at the level of derivability by using the *separation rule* in place of both axioms (TT3-5) and (TT3-6) in Module 3:

(TT3-5′) $A_\alpha \approx B_\alpha \;/\; A_\alpha = B_\alpha$ separation rule

The separation rule ensures that fully valid fuzzy equality still corresponds to type-theoretic equality. Moreover, since fuzzy equality is provably reflexive at every type (see claim 3 of Lemma 4.11 below; the proof uses no $\triangle$), the separation rule together with the replacement rule (TT1-2) yields the rule form of weak extensionality as a derived rule:

(TT3-6′) $A_\alpha \approx B_\alpha \;/\; F_{\beta\alpha}A_\alpha \approx F_{\beta\alpha}B_\alpha.$ weak extensionality rule

Thus, in logics without $\triangle$, the separation rule retains, at the level of derivability, the behavior of fuzzy equality expressed by the separation and weak extensionality axioms of MFTT. However, the direct mutual-interpretability comparison with FTT carried out in Section 4 does not extend directly to modular fuzzy type theories without $\triangle$, since all variants of FTT defined by Novák make essential use of the $\triangle$ connective.

In the presence of some other rules and axioms, the separation rule (TT3-5′) can further be reduced to the following two base-type rules:

(TT3-5′$_\varepsilon$) $A_\varepsilon \approx B_\varepsilon \;/\; A_\varepsilon = B_\varepsilon$ type-ε separation rule

(TT3-5′$_o$) $A_o \approx B_o \;/\; A_o = B_o$ type-o separation rule

Proposition 3.10 (Base-type form of the separation rule)**.** *In the presence of Module 1 and the base-type separation rules* (TT3-5′$_\varepsilon$) *and* (TT3-5′$_o$)*, the unrestricted separation rule* (TT3-5′) *is derivable under either of the following additional assumptions:*

1. *the weak extensionality rule* (TT3-6′)*;*
2. *pointwise function equality* (TT3-7)*, specification* (TT2c-1)*, modus ponens* (TT2a-7)*, and any part of Module 2a sufficient for extracting the left-to-right implication from a biconditional.*

Proof. Assume (TT3-5′$_\varepsilon$) and (TT3-5′$_o$). We prove, by induction on the construction of the type γ, that the rule $A_\gamma \approx B_\gamma \;/\; A_\gamma = B_\gamma$ is derivable. The primitive cases $\gamma = \varepsilon$ and $\gamma = o$ are precisely (TT3-5′$_\varepsilon$) and (TT3-5′$_o$).

Now let $\gamma = \beta\alpha$, and assume the induction hypothesis for the codomain type β. Suppose $F_{\beta\alpha} \approx G_{\beta\alpha}$, and choose x_α not free in $F_{\beta\alpha}$ or $G_{\beta\alpha}$. First, we derive $F_{\beta\alpha}x_\alpha \approx G_{\beta\alpha}x_\alpha$.

Under assumption 1, put $H_{\beta(\beta\alpha)} \coloneqq \lambda h_{\beta\alpha}(h_{\beta\alpha}x_\alpha)$. By weak extensionality (TT3-6′), we have $H_{\beta(\beta\alpha)}F_{\beta\alpha} \approx H_{\beta(\beta\alpha)}G_{\beta\alpha}$. By λ-conversion (TT1-3) and replacement (TT1-2), this yields $F_{\beta\alpha}x_\alpha \approx G_{\beta\alpha}x_\alpha$.

Under assumption 2, the left-to-right implication of pointwise function equality (TT3-7) gives $(F_{\beta\alpha} \approx G_{\beta\alpha}) \to (\forall x_\alpha)(F_{\beta\alpha}x_\alpha \approx G_{\beta\alpha}x_\alpha)$. By modus ponens (TT2a-7) and specification (TT2c-1), we obtain $F_{\beta\alpha}x_\alpha \approx G_{\beta\alpha}x_\alpha$.

In either case, the induction hypothesis at type β yields $F_{\beta\alpha}x_\alpha = G_{\beta\alpha}x_\alpha$. Functional extensionality (TT1-4′) therefore gives $F_{\beta\alpha} = G_{\beta\alpha}$. This completes the induction. □

Part 1 of Proposition 3.10 makes no use of Module 2, so this reduction remains stable under changes to the underlying logic; part 2 gives a further economy when the relevant logical principles are available.

4 Equivalence of the modular axiomatics with Novák's FTT over IMTL$_\triangle$

The aim of this section is to compare Novák's FTT over IMTL$_\triangle$ with the modular axiomatics at the syntactic level.

Since the two systems are not formulated in a common language, the comparison is carried out by means of translations between their languages. We first define these translations and collect the derived principles needed in the proof. We then show that the translated axioms and rules of each system are derivable in the other. Finally, we examine the compositions of the two translations, clarifying in what sense the two systems remain equivalent despite their different treatment of equality.

Throughout this section, FTT denotes the IMTL$_\triangle$-based reference system from Definition 2.7. We write $\mathcal{L}_{\mathrm{FTT}}$ and $\mathcal{L}_{\mathrm{MFTT}}$ for the languages of FTT and MFTT, respectively. Bound variables are tacitly renamed when necessary to avoid capture. Accordingly, when translations, substitutions, or λ-reductions differ only by such renaming, we do not mention the resulting α-conversion explicitly.

4.1 Translations and notation

We compare FTT and MFTT by means of two syntactic translations, one from FTT to MFTT and one in the converse direction. Since the axiom lists of both systems are written using defined notation, the translations operate on the corresponding definitional extensions. After the definition of both translations we give several conventions introducing additional notation and terminology that will be used in the comparison of the two axiomatic systems.

Definition 4.1 (Translations between FTT and MFTT)**.** We define two type-preserving translations for the formulas of FTT and MFTT:

$$A_\alpha \longmapsto A^{\mathsf{M}}_\alpha, \qquad B_\alpha \longmapsto B^{\mathsf{F}}_\alpha.$$

The superscript M denotes translation from FTT into MFTT, and the superscript F denotes translation from MFTT into FTT. When a translation superscript is written directly on an object-language symbol or a metavariable, its scope is limited to that symbol or metavariable. Thus, e.g., AB^{M} is read as $A(B^{\mathsf{M}})$, not as $(AB)^{\mathsf{M}}$, and similarly for F.

The translations are defined on the definitional extensions of the two languages introduced in Definitions 2.5 and 3.1. Thus the clauses below apply to the notation as written, rather than to the result of first expanding every defined symbol into the primitive constants of the source language. We assume that the two languages share the same sets of variables and the same extralogical constants of each type. If one language is expanded by additional extralogical constants, the comparison is made after adding corresponding constants of the same types to the other language.

Both translations fix types, variables, and extralogical constants, and both commute with application and abstraction. They also commute with all propositional connectives whose notation is shared by the two languages. The translation from FTT to MFTT additionally commutes with the quantified binder notation of FTT. Below we list only the clauses where the translation differs.

- The translation from FTT to MFTT is given by the following clauses:

$$\begin{aligned} \boldsymbol{E}^{\mathsf{M}}_{o\alpha\alpha} &:= \lambda y_\alpha \lambda x_\alpha (\approx_{o\alpha\alpha} x_\alpha y_\alpha), & \boldsymbol{C}^{\mathsf{M}}_{ooo} &:= \wedge_{ooo}, \\ \equiv^{\mathsf{M}}_{\alpha} &:= \lambda y_\alpha \lambda x_\alpha (\approx_{o\alpha\alpha} x_\alpha y_\alpha), & \boldsymbol{D}^{\mathsf{M}}_{oo} &:= \triangle_{oo}. \end{aligned}$$

- The translation from MFTT to FTT uses the following clauses:

$$\begin{aligned} =^{\mathsf{F}}_{o\alpha\alpha} &:= \lambda x_\alpha \lambda y_\alpha (\equiv_\alpha y_\alpha x_\alpha), \\ \approx^{\mathsf{F}}_{o\alpha\alpha} &:= \lambda x_\alpha \lambda y_\alpha (\equiv_\alpha y_\alpha x_\alpha), \\ \leftrightarrow^{\mathsf{F}}_{ooo} &:= \lambda x_o \lambda y_o \big((x_o \to y_o) \wedge (y_o \to x_o)\big), \\ \forall^{\mathsf{F}}_{o(o\alpha)} &:= \lambda p_{o\alpha} (\forall x_\alpha)(p_{o\alpha} x_\alpha), \\ \exists^{\mathsf{F}}_{o(o\alpha)} &:= \lambda p_{o\alpha} (\exists x_\alpha)(p_{o\alpha} x_\alpha). \end{aligned}$$

Remark 4.2. In MFTT, we use the same curried argument orders $=_{o\alpha\alpha} A_\alpha B_\alpha$ and $\approx_{o\alpha\alpha} A_\alpha B_\alpha$ for the infix equality predicates $A_\alpha = B_\alpha$ and $A_\alpha \approx B_\alpha$ as for the other binary symbols. This differs from Novák's special convention for $A_\alpha \equiv B_\alpha$ as $(\equiv_\alpha B_\alpha) A_\alpha$. The translation clauses for the equality predicates in Definition 4.1 therefore include the exchange of arguments.

Remark 4.3. In MFTT, the binder quantification $(\forall x_\alpha) A_o$ is just a notational abbreviation for $\forall_{o(o\alpha)} \lambda x_\alpha A_o$. Consequently, its literal translation into FTT is

$$\big(\lambda p_{o\alpha} (\forall y_\alpha)(p_{o\alpha} y_\alpha)\big)(\lambda x_\alpha A^{\mathsf{F}}_o),$$

where y_α is chosen fresh. This λ-reduces to $(\forall x_\alpha) A^{\mathsf{F}}_o$. We shall use this immediate λ-reduct without further comment when convenient. The case of $\exists$ is analogous.

Convention 4.4 (Translation of rules)**.** If Γ is a set of FTT-formulas and Θ is a set of MFTT-formulas, we write

$$\Gamma^{\mathsf{M}} := \{\, A^{\mathsf{M}} \mid A \in \Gamma \,\}, \qquad \Theta^{\mathsf{F}} := \{\, B^{\mathsf{F}} \mid B \in \Theta \,\}.$$

Translations extend componentwise to inference rules; thus $(\Gamma \,/\, A)^{\mathsf{M}}$ is $\Gamma^{\mathsf{M}} \,/\, A^{\mathsf{M}}$ and similarly for F. For axiom and rule schemata, translations are understood instance-wise: each admissible source instance is translated separately.

4.2 Main comparison results

We now state the comparison results that will be proved in the following subsections. The first two results say that the two axiom systems interpret one another under the translations from Definition 4.1. The third result concerns the two round-trip compositions obtained by translating from FTT to MFTT and back, and from MFTT to FTT and back.

Theorem 4.5 (Interpretation of FTT in MFTT)**.** *Every axiom of* FTT *translates into a theorem of* MFTT*, and every inference rule of* FTT *translates into a derivable rule of* MFTT*. Consequently, for every set* $\Gamma \subseteq \mathsf{Form}_o(\mathcal{L}_{\mathrm{FTT}})$ *and every* $A_o \in \mathsf{Form}_o(\mathcal{L}_{\mathrm{FTT}})$*,*

$$\Gamma \vdash_{\mathrm{FTT}} A_o \quad \Longrightarrow \quad \Gamma^{\mathsf{M}} \vdash_{\mathrm{MFTT}} A_o^{\mathsf{M}}.$$

The proof is a direct verification of the translated axiom and rule schemata in MFTT; it will be given in Section 4.4, using MFTT principles derived in Section 4.3.

Theorem 4.6 (Interpretation of MFTT in FTT)**.** *Every axiom of* MFTT *translates into a theorem of* FTT*, and every inference rule of* MFTT *translates into a derivable rule of* FTT*. Consequently, for every set* $\Gamma \subseteq \mathsf{Form}_o(\mathcal{L}_{\mathrm{MFTT}})$ *and every* $B_o \in \mathsf{Form}_o(\mathcal{L}_{\mathrm{MFTT}})$*,*

$$\Gamma \vdash_{\mathrm{MFTT}} B_o \quad \Longrightarrow \quad \Gamma^{\mathsf{F}} \vdash_{\mathrm{FTT}} B_o^{\mathsf{F}}.$$

The proof of this direction will be given in Section 4.5, using Novák's derivability results from [8] and a few additional FTT principles derived in Section 4.3.

The third main theorem records the fact that the translations are inverse to one another up to provable equality, with the expected exception that on the MFTT-side, the round trip through the single-equality theory FTT collapses the two equality predicates $=$ and $\approx$.

Theorem 4.7 (Round-trip translations)**.** *For any* MFTT*-formula* B_α*, let* $B_\alpha^{\approx}$ *denote the formula obtained from* B_α *by replacing every occurrence of* $=$ *by* $\approx$*.*

Then, for all FTT*-formulas* A_α *and* MFTT*-formulas* B_α*, the following hold:*

1. $\vdash_{\mathrm{FTT}} (A_\alpha^{\mathsf{M}})^{\mathsf{F}} \equiv A_\alpha$.
2. $\vdash_{\mathrm{MFTT}} (B_\alpha^{\mathsf{F}})^{\mathsf{M}} = B_\alpha^{\approx}$.

As straightforward consequences, at type o the first claim of Theorem 4.7 yields interderivability of each FTT-formula with its double translation, and the second yields interderivability of $B_o^{\approx}$ with the double translation of B_o. On the MFTT side, the round-trip equality can equivalently be formulated using $\approx$; moreover, if B_α contains no occurrence of $=$, then $B_\alpha^{\approx} = B_\alpha$, so its double translation recovers B_α itself, up to either equality.

4.3 Derived principles used in the comparison

In this subsection we collect the auxiliary derived principles used in the proofs of the two translations. We first record the required FTT-principles that have already been proved by Novák in [8].

Lemma 4.8 (Derived principles of FTT from [8])**.** *The following principles are derivable in* FTT*.*

1. $B_o \vdash_{\mathrm{FTT}} B_o[A^1_{\alpha_1}/x^1_{\alpha_1}, \ldots, A^n_{\alpha_n}/x^n_{\alpha_n}]$*, provided each* $A^i_{\alpha_i}$ *is substitutable for* $x^i_{\alpha_i}$ *in* B_o*.*
2. *Every type-theoretic instance of a theorem schema or derived rule of* $\mathrm{IMTL}_{\triangle}\forall$ *is derivable in* FTT*.*

   (Here, a type-theoretic instance *means that schematic formula variables may be replaced by arbitrary* FTT*-formulas of type o, possibly containing subformulas of higher types, subject only to well-typedness and to the usual side conditions on free variables and substitutability.)*

3. $\vdash_{\mathrm{FTT}} A_\alpha \equiv A_\alpha$.

4. $\vdash_{\mathrm{FTT}} (A_\alpha \equiv B_\alpha)\,\&\,(B_\alpha \equiv C_\alpha) \to (A_\alpha \equiv C_\alpha)$.

5. $\vdash_{\mathrm{FTT}} (\forall x_\alpha)(F_{\beta\alpha}x_\alpha \equiv G_{\beta\alpha}x_\alpha) \equiv (F_{\beta\alpha} \equiv G_{\beta\alpha})$, *provided* x_α *is not free in* $F_{\beta\alpha}$ *or* $G_{\beta\alpha}$.

Proof. Claim 1 is the derived-rule formulation of [8, Thm. 6]. Claim 2 combines Novák's observations that predicate IMTL is obtained in FTT [8, Sect. 3.3 after Thm. 8] and that the required properties of $\triangle$ are obtained there as well [8, Sect. 3.5 after Thm. 16]; Novák derives these principles for arbitrary formulas of type o, which corresponds to the type-theoretic reading described in the statement. Finally, Claims 3–5 are [8, Thms. 2b, 12, and 11]. □

We will also need a few further FTT-principles, which we collect in the following lemma.

Lemma 4.9 (Further derived principles of FTT)**.** *The following principles are derivable in* FTT*.*

1. $\vdash_{\mathrm{FTT}} (A_o \equiv B_o) \leftrightarrow (A_o \leftrightarrow B_o)$.

2. $\vdash_{\mathrm{FTT}} (A_\alpha \equiv B_\alpha) \to (B_\alpha \equiv A_\alpha)$.

3. $\vdash_{\mathrm{FTT}} (F_{\beta\alpha} \equiv G_{\beta\alpha}) \leftrightarrow (\forall x_\alpha)(F_{\beta\alpha}x_\alpha \equiv G_{\beta\alpha}x_\alpha)$, *provided* x_α *is not free in* $F_{\beta\alpha}$ *or* $G_{\beta\alpha}$.

4. $\vdash_{\mathrm{FTT}} (\lambda x_\alpha F_{\beta\alpha}x_\alpha) \equiv F_{\beta\alpha}$, *provided* x_α *is not free in* $F_{\beta\alpha}$.

5. $A_\alpha \equiv B_\alpha \vdash_{\mathrm{FTT}} C_\gamma \equiv C_\gamma[B_\alpha /\!\!/ A_\alpha]$.

Proof.

1. Let $P_o \coloneqq A_o \equiv B_o$ and $Q_o \coloneqq (A_o \to B_o) \wedge (B_o \to A_o)$. By (FT5) and simultaneous substitution (claim 1 of Lemma 4.8), we have $\vdash_{\mathrm{FTT}} P_o \equiv Q_o$. A second instance of (FT5), now applied to P_o and Q_o, gives

   $$\vdash_{\mathrm{FTT}} (P_o \equiv Q_o) \equiv \big((P_o \to Q_o) \wedge (Q_o \to P_o)\big).$$

   Using the replacement rule (FT-R) with the preceding two formulas yields

   $$\vdash_{\mathrm{FTT}} (P_o \to Q_o) \wedge (Q_o \to P_o),$$

   which is precisely $\vdash_{\mathrm{FTT}} (A_o \equiv B_o) \leftrightarrow (A_o \leftrightarrow B_o)$.

2. We proceed by induction on the type α. For $\alpha = \varepsilon$, the claim is axiom (FT17). For $\alpha = o$, the claim follows from claim 1 and the symmetry of propositional biconditional by IMTL$_\triangle\forall$-reasoning (claim 2 of Lemma 4.8).

   Now let $\alpha = \beta\gamma$, and assume the claim for type β as the induction hypothesis. Choose x_γ not free in $F_{\beta\gamma}$ or $G_{\beta\gamma}$. By (FT2b), the induction hypothesis, and IMTL$_\triangle\forall$-reasoning,

   $$\vdash_{\mathrm{FTT}} (F_{\beta\gamma} \equiv G_{\beta\gamma}) \to (G_{\beta\gamma}x_\gamma \equiv F_{\beta\gamma}x_\gamma).$$

   Generalization, (FT16), and (FT2a), together with further IMTL$_\triangle\forall$-reasoning, then yield $\vdash_{\mathrm{FTT}} (F_{\beta\gamma} \equiv G_{\beta\gamma}) \to (G_{\beta\gamma} \equiv F_{\beta\gamma})$.

3. Put $P_o \coloneqq (F_{\beta\alpha} \equiv G_{\beta\alpha})$ and $Q_o \coloneqq (\forall x_\alpha)(F_{\beta\alpha}x_\alpha \equiv G_{\beta\alpha}x_\alpha)$. By claim 5 of Lemma 4.8, $\vdash_{\mathrm{FTT}} Q_o \equiv P_o$. By claim 2 of Lemma 4.9, we therefore obtain $\vdash_{\mathrm{FTT}} P_o \equiv Q_o$. Axiom (FT5) gives

$$\vdash_{\mathrm{FTT}} (P_o \equiv Q_o) \equiv \big((P_o \to Q_o) \wedge (Q_o \to P_o)\big).$$

Using the replacement rule (FT-R) with the preceding two formulas yields

$$\vdash_{\mathrm{FTT}} (P_o \to Q_o) \wedge (Q_o \to P_o),$$

i.e., $\vdash_{\mathrm{FTT}} (F_{\beta\alpha} \equiv G_{\beta\alpha}) \leftrightarrow (\forall x_\alpha)(F_{\beta\alpha}x_\alpha \equiv G_{\beta\alpha}x_\alpha)$.

4. By λ-conversion, $\vdash_{\mathrm{FTT}} (\lambda x_\alpha F_{\beta\alpha}x_\alpha)x_\alpha \equiv F_{\beta\alpha}x_\alpha$. Generalization on x_α, (FT2a), and modus ponens (Lemma 4.8, claim 2) then yield the claim.

5. By reflexivity (claim 3 of Lemma 4.8), we have $\vdash_{\mathrm{FTT}} C_\gamma \equiv C_\gamma$. Applying the replacement rule (FT-R) to this theorem and to the premise $A_\alpha \equiv B_\alpha$, replacing the specified occurrence of A_α in the second occurrence of C_γ, gives $C_\gamma \equiv C_\gamma[B_\alpha /\!\!/ A_\alpha]$. □

We now present the MFTT-derivable principles that will be needed in the main results. We first isolate the principles that depend only on the ground type-theoretic machinery of Module 1. Consequently, these principles are available in type theories over any many-valued logic, independently of the logical constants and logical principles added in later modules.

Lemma 4.10 (Derived principles of Module 1)**.** *The following principles are derivable already in Module 1 of* MFTT*, and hence also in the deductively equivalent theory* TT_0*.*

1. $\vdash_{\mathrm{MFTT}} A_\alpha = A_\alpha$.
2. $A_\alpha = B_\alpha \vdash_{\mathrm{MFTT}} B_\alpha = A_\alpha$.
3. $A_\alpha = B_\alpha \vdash_{\mathrm{MFTT}} C_\gamma = C_\gamma[B_\alpha /\!\!/ A_\alpha]$.

Proof. The following proofs use only the axioms and rules of Module 1.

1. Choose x_α not free in A_α. By λ-conversion (TT1-3), we have $(\lambda x_\alpha A_\alpha)A_\alpha = A_\alpha$. Use the replacement rule (TT1-2) on this formula itself, replacing $(\lambda x_\alpha A_\alpha)A_\alpha$ by A_α. This gives $A_\alpha = A_\alpha$.

2. Assume $A_\alpha = B_\alpha$. By claim 1, we have $A_\alpha = A_\alpha$. Apply the replacement rule (TT1-2) to the latter, replacing the left-hand occurrence of A_α by B_α; this gives $B_\alpha = A_\alpha$.

3. By claim 1, $C_\gamma = C_\gamma$ is provable. Applying the replacement rule (TT1-2) to this theorem and to the premise $A_\alpha = B_\alpha$ gives $C_\gamma = C_\gamma[B_\alpha /\!\!/ A_\alpha]$. □

Finally, we record the MFTT-principles depending on Modules 2 and 3 that will be needed for the interpretation of FTT in MFTT.

Lemma 4.11 (Further derived principles of MFTT)**.** *The following principles are* MFTT*-derivable:*

1. *All type-theoretic instances of theorems and derived rules of* $\mathrm{IMTL}_\triangle\forall$ *are derivable in* MFTT*, using Module 2 alone.*

2. $A_o \leftrightarrow B_o \vdash_{\mathrm{MFTT}} A_o \approx B_o$ *and* $A_o \approx B_o \vdash_{\mathrm{MFTT}} A_o \leftrightarrow B_o$.

3. $\vdash_{\mathrm{MFTT}} A_\alpha \approx A_\alpha$.

4. $A_\alpha = B_\alpha \vdash_{\mathrm{MFTT}} A_\alpha \approx B_\alpha$ *and* $A_\alpha \approx B_\alpha \vdash_{\mathrm{MFTT}} A_\alpha = B_\alpha$.

5. $\vdash_{\mathrm{MFTT}} \big(G_{oo}(\triangle A_o) \wedge G_{oo}(\neg\triangle A_o)\big) \leftrightarrow (\forall y_o) G_{oo}(\triangle y_o)$, *where* y_o *is not free in* G_{oo} *or* A_o.

6. $\vdash_{\mathrm{MFTT}} (A_o \approx \top) \leftrightarrow A_o$.

7. $\vdash_{\mathrm{MFTT}} \big((A_o \wedge B_o) \approx A_o\big) \leftrightarrow (A_o \to B_o)$.

8. $\vdash_{\mathrm{MFTT}} \big((\lambda x_\alpha A_o) \approx (\lambda x_\alpha \top)\big) \leftrightarrow (\forall x_\alpha) A_o$.

Proof.

1. Module 2 of MFTT consists precisely of the type-theoretic instances of the axioms and rules of predicate IMTL$_\triangle\forall$. Thus every Hilbert derivation in predicate IMTL$_\triangle\forall$ can be copied instance by instance inside MFTT, using only the axioms and rules of Module 2.

2. By (TT3-1), we have $(A_o \approx B_o) \leftrightarrow (A_o \leftrightarrow B_o)$. By the IMTL$_\triangle\forall$-reasoning licensed by claim 1, we may extract either implication from this biconditional and apply modus ponens. Hence both derivability claims follow.

3. We prove the claim by induction on the type α. For $\alpha = \varepsilon$, this is (TT3-2). For $\alpha = o$, the formula $A_o \leftrightarrow A_o$ is a theorem of predicate IMTL$_\triangle\forall$, hence is derivable in MFTT by claim 1, and $A_o \approx A_o$ follows by claim 2.

   Now let $\alpha = \beta\gamma$, and assume the claim for type β. Choose x_γ not free in $F_{\beta\gamma}$. By the induction hypothesis, $F_{\beta\gamma}x_\gamma \approx F_{\beta\gamma}x_\gamma$ is derivable, whence $(\forall x_\gamma)(F_{\beta\gamma}x_\gamma \approx F_{\beta\gamma}x_\gamma)$ is derivable by generalization (TT2c-4). Since (TT3-7) is a biconditional, the IMTL$_\triangle\forall$-reasoning covered by claim 1 gives its right-to-left implication

   $$(\forall x_\gamma)(F_{\beta\gamma}x_\gamma \approx F_{\beta\gamma}x_\gamma) \to (F_{\beta\gamma} \approx F_{\beta\gamma}),$$

   whence $F_{\beta\gamma} \approx F_{\beta\gamma}$ follows by modus ponens (TT2a-7).

4. By claim 3, $A_\alpha \approx A_\alpha$ is derivable. From the premise $A_\alpha = B_\alpha$, the replacement rule (TT1-2) applied to this formula yields $A_\alpha \approx B_\alpha$.

   Conversely, from $A_\alpha \approx B_\alpha$ derive $\triangle(A_\alpha \approx B_\alpha)$ by $\triangle$-necessitation (TT2b-6). The claim then follows from the separation axiom (TT3-5) by modus ponens (TT2a-7).

5. Put $P_o \coloneqq G_{oo}(\triangle A_o) \wedge G_{oo}(\neg\triangle A_o)$ and $Q_o \coloneqq (\forall y_o) G_{oo}(\triangle y_o)$. We prove $P_o \leftrightarrow Q_o$.

   First, $Q_o \to G_{oo}(\triangle A_o)$ follows by specification (TT2c-1). Specification with $\neg\triangle A_o$ in place of y_o gives $Q_o \to G_{oo}(\triangle\neg\triangle A_o)$. The formula $\triangle\neg\triangle A_o \leftrightarrow \neg\triangle A_o$ is a theorem of IMTL$_\triangle$, and therefore derivable by claim 1. By claim 2, $\triangle$-necessitation, and weak extensionality (TT3-6), we obtain $G_{oo}(\triangle\neg\triangle A_o) \to G_{oo}(\neg\triangle A_o)$. Thus $Q_o \to G_{oo}(\neg\triangle A_o)$. Together with the first implication, this gives $Q_o \to P_o$ by the usual IMTL$_\triangle\forall$ reasoning.

   Conversely, let y_o be fresh. The $\triangle$-exhaustion formula

   $$\triangle(\triangle y_o \approx \triangle A_o) \vee \triangle(\triangle y_o \approx \neg\triangle A_o)$$

is derivable from the corresponding theorem of IMTL$_\triangle\forall$ for the equivalence connective, using (TT3-1) and the Delta axioms. By weak extensionality (TT3-6), the first disjunct yields $G_{oo}(\triangle A_o) \to G_{oo}(\triangle y_o)$, and the second yields $G_{oo}(\neg\triangle A_o) \to G_{oo}(\triangle y_o)$, using claims 2 and 1 to pass from fuzzy equality of truth values to the corresponding implication. Since P_o entails both $G_{oo}(\triangle A_o)$ and $G_{oo}(\neg\triangle A_o)$, disjunction elimination in IMTL$_\triangle\forall$ gives $P_o \to G_{oo}(\triangle y_o)$. By generalization (TT2c-4) and axiom (TT2c-2), since y_o is not free in P_o, we obtain $P_o \to Q_o$. Hence $P_o \leftrightarrow Q_o$.

6. Use truth-value equality (TT3-1) and the IMTL$_\triangle\forall$ theorem $(A_o \leftrightarrow \top) \leftrightarrow A_o$.

7. Use (TT3-1) and the IMTL$_\triangle\forall$ theorem $\big((A_o \wedge B_o) \leftrightarrow A_o\big) \leftrightarrow (A_o \to B_o)$.

8. Apply pointwise function equality (TT3-7) to $\lambda x_\alpha A_o$ and $\lambda x_\alpha \top$. Since x_α is not free in these abstractions, we obtain

$$\vdash_{\mathrm{MFTT}} \big((\lambda x_\alpha A_o) \approx (\lambda x_\alpha \top)\big) \leftrightarrow (\forall x_\alpha)\big((\lambda x_\alpha A_o)x_\alpha \approx (\lambda x_\alpha \top)x_\alpha\big).$$

By λ-conversion (TT1-3) and replacement (TT1-2), the formula under the quantifier is provably equivalent to $A_o \approx \top$. Therefore, by IMTL$_\triangle\forall$-reasoning,

$$\vdash_{\mathrm{MFTT}} (\forall x_\alpha)\big((\lambda x_\alpha A_o)x_\alpha \approx (\lambda x_\alpha \top)x_\alpha\big) \leftrightarrow (\forall x_\alpha)(A_o \approx \top).$$

Since $(A_o \approx \top) \leftrightarrow A_o$ by claim 6, similar IMTL$_\triangle\forall$-reasoning yields

$$\vdash_{\mathrm{MFTT}} (\forall x_\alpha)(A_o \approx \top) \leftrightarrow (\forall x_\alpha)A_o.$$

Combining these equivalences gives the desired formula. □

4.4 From FTT to MFTT

We first verify the interpretation of Novák's FTT in the modular system MFTT. The only remaining preliminary point is coherence of the translation with expansion and contraction of FTT-definitions, recorded in the next lemmas.

Lemma 4.12 (Compatibility of the translation with FTT-definitions)**.** *Let P_δ and Q_δ be, respectively, the left- and right-hand sides of a defining or notational clause from Definition 2.5. Then $\vdash_{\mathrm{MFTT}} P_\delta^{\mathsf{M}} = Q_\delta^{\mathsf{M}}$.*

Proof. Of the symbols introduced in Definition 2.5, we spell out the cases of $\to$ and $\forall$; the remaining cases are proved in a similar way.

According to the defining clause for $\to$, we have

$$P_{ooo} \coloneqq \to, \qquad Q_{ooo} \coloneqq \lambda x_o \lambda y_o\big((x_o \wedge y_o) \equiv x_o\big).$$

Let u_o, v_o be fresh variables. By the translation clauses and λ-conversion (TT1-3),

$$\vdash_{\mathrm{MFTT}} P_{ooo}^{\mathsf{M}} u_o v_o = (u_o \to v_o), \qquad \vdash_{\mathrm{MFTT}} Q_{ooo}^{\mathsf{M}} u_o v_o = \big((u_o \wedge v_o) \approx u_o\big).$$

By claim 7 of Lemma 4.11, $\vdash_{\mathrm{MFTT}} (u_o \to v_o) \leftrightarrow \big((u_o \wedge v_o) \approx u_o\big)$. Hence, by claims 2 and 4 of the same Lemma, $\vdash_{\mathrm{MFTT}} P_{ooo}^{\mathsf{M}} u_o v_o = Q_{ooo}^{\mathsf{M}} u_o v_o$. Since u_o, v_o are fresh, applying the derived functional extensionality rule (TT1-4′), first to v_o and then to u_o, yields $\vdash_{\mathrm{MFTT}} P_{ooo}^{\mathsf{M}} = Q_{ooo}^{\mathsf{M}}$.

For the universal quantifier, the directly interchanged propositions are

$$P_o \coloneqq (\forall x_\alpha)A_o, \qquad Q_o \coloneqq \lambda x_\alpha A_o \equiv \lambda x_\alpha \top.$$

Their translations are $(\forall x_\alpha)A_o^{\mathsf{M}}$ and $\lambda x_\alpha A_o^{\mathsf{M}} \approx \lambda x_\alpha \top$. The biconditional between these two propositions is precisely claim 8 of Lemma 4.11. Hence, as above, claims 2 and 4 of the same Lemma yield $\vdash_{\mathrm{MFTT}} P_o^{\mathsf{M}} = Q_o^{\mathsf{M}}$.

Thus, in all cases, $\vdash_{\mathrm{MFTT}} P_\delta^{\mathsf{M}} = Q_\delta^{\mathsf{M}}$. □

Lemma 4.13 (Contextual coherence of the translation with FTT-definitions)**.** *Let A_γ, B_γ be* FTT*-formulas. Suppose that B_γ is obtained from A_γ by replacing one specified occurrence of one side of a defining or notational clause from Definition 2.5 by the other side of that clause. Then $\vdash_{\mathrm{MFTT}} A_\gamma^{\mathsf{M}} = B_\gamma^{\mathsf{M}}$.*

Consequently, if $\gamma = o$, then also $A_o^{\mathsf{M}} \vdash_{\mathrm{MFTT}} B_o^{\mathsf{M}}$ and $B_o^{\mathsf{M}} \vdash_{\mathrm{MFTT}} A_o^{\mathsf{M}}$.

Proof. Let P_δ, Q_δ be the expressions directly interchanged by the defining or notational clause. Lemma 4.12 gives $\vdash_{\mathrm{MFTT}} P_\delta^{\mathsf{M}} = Q_\delta^{\mathsf{M}}$ (for the converse direction of replacement, we use the symmetry of $=$ from Lemma 4.10, claim 2). By compositionality of the translation, B_γ^{M} is obtained from A_γ^{M} by replacing the corresponding occurrence of P_δ^{M} by Q_δ^{M}. Hence replacement (TT1-2), applied to the theorem $A_\gamma^{\mathsf{M}} = A_\gamma^{\mathsf{M}}$ (Lemma 4.10, claim 1), yields $\vdash_{\mathrm{MFTT}} A_\gamma^{\mathsf{M}} = B_\gamma^{\mathsf{M}}$.

Finally, if $\gamma = o$, then the two derivability claims follow by one further application of the replacement rule (TT1-2). □

We are now ready to prove the first theorem from Section 4.2, namely the interpretation of Novák's FTT in the modular system MFTT.

Proof of Theorem 4.5. We verify that the translations of the axioms of FTT are theorems of MFTT, and that the translations of the inference rules of FTT are derivable rules of MFTT. Preservation of provability then follows by induction on FTT-derivations. The ordinary axiom and rule cases are handled below. The additional case of expansion or contraction of FTT-defined notation is handled by Lemma 4.13: at type o, that Lemma gives mutual derivability of the two translated formulas, and at arbitrary types it gives type-theoretic equality, which is sufficient for replacement inside surrounding formulas by the replacement rule (TT1-2). Throughout the proof we use that the translation preserves types, variables, application, λ-abstraction, and the binding structure, and hence all relevant side conditions of axiom and rule schemata.

Each translated instance of the logical principles (FT7)–(FT13), (FT15)–(FT16), and (FT-N) is an instance of, or is immediately derivable from, the corresponding IMTL$_\triangle\forall$-principle, which is available in MFTT by claim 1 of Lemma 4.11. For the translated instances of logical principles that are stated as equalities of truth values rather than as implications, we additionally use claim 2 of the same Lemma.

Moreover, every translated instance of (FT1), (FT4), and (FT17) λ-reduces to an instance of (TT3-6), (TT3-4), and (TT3-3), respectively. The remaining cases are proved as follows, using claim 1 of Lemma 4.11 to justify IMTL$_\triangle\forall$-reasoning whenever required:

(FT2a) After λ-reducing the translated equality predicates, the translated axiom is the right-to-left implication extracted by IMTL$_\triangle\forall$-reasoning from (TT3-7).

(FT2b) After the same λ-reductions, the translated axiom follows from the left-to-right implication of (TT3-7) by specification (TT2c-1) and IMTL$_\triangle\forall$-reasoning.

(FT3) The translated axiom $(\lambda x_\alpha B_\beta^{\mathsf{M}})A_\alpha^{\mathsf{M}} \approx B_\beta^{\mathsf{M}}[A_\alpha^{\mathsf{M}}/x_\alpha]$ follows by claim 4 of Lemma 4.11 from the corresponding instance of λ-conversion (TT1-3).

(FT5) By the definition of propositional equivalence, the translated axiom is $(x_o \approx y_o) \approx (x_o \leftrightarrow y_o)$, which follows from (TT3-1) by claim 2 of Lemma 4.11.

(FT6) The translated axiom $(A_o^{\mathsf{M}} \approx \top) \approx A_o^{\mathsf{M}}$ follows from claims 6 and 2 of Lemma 4.11.

(FT14) After λ-reducing the translated equality predicate, the translated axiom follows directly from claims 5 and 2 of Lemma 4.11.

(FT-R) Assume $A_\alpha^{\mathsf{M}} \approx (A'_\alpha)^{\mathsf{M}}$ and B_o^{M}. By claim 4 of Lemma 4.11, the first premise yields $A_\alpha^{\mathsf{M}} = (A'_\alpha)^{\mathsf{M}}$. Since the translation preserves the specified subformula occurrence, replacement (TT1-2) applied to B_o^{M} yields the required $B_o^{\mathsf{M}}[(A'_\alpha)^{\mathsf{M}} /\!\!/ A_\alpha^{\mathsf{M}}]$. □

4.5 From MFTT to FTT

We now turn to the converse interpretation, from MFTT into Novák's FTT. Before checking the translated modules, we show the coherence of the translation with the definitions and notational clauses of the MFTT-language.

Lemma 4.14. *Let A_α and B_α be* MFTT*-formulas. Then the translations $(A_\alpha = B_\alpha)^{\mathsf{F}}$ and $(A_\alpha \approx B_\alpha)^{\mathsf{F}}$ both λ-reduce to $A_\alpha^{\mathsf{F}} \equiv B_\alpha^{\mathsf{F}}$.*

Proof. By Definition 3.1, the notation $A_\alpha = B_\alpha$ stands for $=_{o\alpha\alpha}A_\alpha B_\alpha$, and similarly for $\approx$. The translation clauses give both $=_{o\alpha\alpha}^{\mathsf{F}}$ and $\approx_{o\alpha\alpha}^{\mathsf{F}}$ as $\lambda x_\alpha \lambda y_\alpha(\equiv_\alpha y_\alpha x_\alpha)$. Hence both translations considered in this Lemma λ-reduce by (FT3) to $\equiv_\alpha B_\alpha^{\mathsf{F}} A_\alpha^{\mathsf{F}}$, which is $A_\alpha^{\mathsf{F}} \equiv B_\alpha^{\mathsf{F}}$ by Definition 2.5. □

Lemma 4.15 (Compatibility of the translation with MFTT-definitions)**.** *Let P_δ and Q_δ be, respectively, the left- and right-hand sides of a defining or notational clause from Definition 3.1. Then $\vdash_{\mathrm{FTT}} P_\delta^{\mathsf{F}} \equiv Q_\delta^{\mathsf{F}}$.*

Proof. Let P_δ, Q_δ be the two MFTT-formulas directly interchanged by the defining or notational clause. We first show that $\vdash_{\mathrm{FTT}} P_\delta^{\mathsf{F}} \equiv Q_\delta^{\mathsf{F}}$.

First consider the infix notation for equality. If the interchanged formulas are $C_\alpha = D_\alpha$ and $=_{o\alpha\alpha}C_\alpha D_\alpha$, then both translations λ-convert to $C_\alpha^{\mathsf{F}} \equiv D_\alpha^{\mathsf{F}}$ by Lemma 4.14. The case of $\approx$ is identical. For the infix notation of binary propositional connectives shared by both languages, there is nothing further to prove, since the same infix convention is used on the FTT side. For $\leftrightarrow$, both $\leftrightarrow A_o B_o$ and $A_o \leftrightarrow B_o$ translate, up to λ-conversion, to the formula $(A_o^{\mathsf{F}} \to B_o^{\mathsf{F}}) \wedge (B_o^{\mathsf{F}} \to A_o^{\mathsf{F}})$.

Next consider the clauses for defined symbols. For lattice disjunction, we have

$$P_{ooo} \coloneqq \vee, \qquad Q_{ooo} \coloneqq \lambda x_o \lambda y_o \neg(\neg x_o \wedge \neg y_o).$$

Let u_o, v_o be fresh variables. The translated applications $P_{ooo}^{\mathsf{F}} u_o v_o$ and $Q_{ooo}^{\mathsf{F}} u_o v_o$ λ-reduce to $u_o \vee v_o$ and $\neg(\neg u_o \wedge \neg v_o)$, respectively. Since these two formulas are equivalent in IMTL$_\triangle\forall$,

claim 2 of Lemma 4.8 and claim 1 of Lemma 4.9 give $\vdash_{\mathrm{FTT}} P^{\mathsf{F}}_{ooo} u_o v_o \equiv Q^{\mathsf{F}}_{ooo} u_o v_o$. Generalization and two applications of claim 5 of Lemma 4.8 then yield $\vdash_{\mathrm{FTT}} P^{\mathsf{F}}_{ooo} \equiv Q^{\mathsf{F}}_{ooo}$.

The clauses for the remaining propositional connectives are analogous: after saturation by fresh variables and λ-conversion, their translations reduce either to the same defining FTT-clause or to the standard IMTL$_\triangle\forall$-equivalence between the two propositional definitions.

Next consider the defining clause of the functional existential quantifier. Here,

$$P_{o(o\alpha)} \coloneqq \exists_{o(o\alpha)}, \qquad Q_{o(o\alpha)} \coloneqq \lambda p_{o\alpha} \neg\big(\forall_{o(o\alpha)} \lambda x_\alpha \neg(p_{o\alpha} x_\alpha)\big).$$

Let $q_{o\alpha}$ be fresh. Then $P^{\mathsf{F}}_{o(o\alpha)} q_{o\alpha}$ λ-converts to $(\exists x_\alpha)(q_{o\alpha} x_\alpha)$ and $Q^{\mathsf{F}}_{o(o\alpha)} q_{o\alpha}$ to $\neg(\forall x_\alpha)\neg(q_{o\alpha} x_\alpha)$. By the defining clause for $\exists$ in Definition 2.5, the two λ-converted formulas are the same. Thus, by reflexivity (Lemma 4.8, claim 3) and the preceding λ-conversions, $\vdash_{\mathrm{FTT}} P^{\mathsf{F}}_{o(o\alpha)} q_{o\alpha} \equiv Q^{\mathsf{F}}_{o(o\alpha)} q_{o\alpha}$. Generalization, claim 5 of Lemma 4.8, and (FT-R) then yield $\vdash_{\mathrm{FTT}} P^{\mathsf{F}}_{o(o\alpha)} \equiv Q^{\mathsf{F}}_{o(o\alpha)}$, as required.

Finally, quantified binder notation requires no separate case. As explained in Remark 4.3, the MFTT-notation $(\forall x_\alpha) C_o$ is translated via its expansion $\forall_{o(o\alpha)} \lambda x_\alpha C_o$, so compatibility with this notational abbreviation is built into the translation. The case of $(\exists x_\alpha) C_o$ is analogous.

Thus in all defining and notational cases, $\vdash_{\mathrm{FTT}} P^{\mathsf{F}}_\delta \equiv Q^{\mathsf{F}}_\delta$. □

Lemma 4.16 (Contextual coherence of the translation with MFTT-definitions)**.** *Let A_γ, B_γ be* MFTT*-formulas. Suppose that B_γ is obtained from A_γ by replacing one specified occurrence of one side of a defining or notational clause from Definition 3.1 by the other side of that clause. Then $\vdash_{\mathrm{FTT}} A^{\mathsf{F}}_\gamma \equiv B^{\mathsf{F}}_\gamma$.*

Consequently, if $\gamma = o$, then also $A^{\mathsf{F}}_o \vdash_{\mathrm{FTT}} B^{\mathsf{F}}_o$ and $B^{\mathsf{F}}_o \vdash_{\mathrm{FTT}} A^{\mathsf{F}}_o$.

Proof. The proof is analogous to that of Lemma 4.13, using Lemma 4.15, symmetry of $\equiv$, and the replacement rule (FT-R). □

With the coherence facts in place, we are ready to prove the second theorem of Section 4.2, namely the interpretation of MFTT into FTT.

Proof of Theorem 4.6. We verify the FTT-derivability of the translations of the axioms and rules from Modules 1–3. Preservation of provability then follows by induction on MFTT-derivations. The axiom and rule cases are handled below. The additional case of expansion or contraction of MFTT-defined notation is handled by Lemma 4.16, exactly as in the proof of Theorem 4.5. Throughout the proof we use that the translation into FTT preserves types, variables, application, λ-abstraction, binding structure, and free variables; consequently, all relevant side conditions of substitutability and freshness are preserved as well. For binder abbreviations of quantifiers, we use the immediate λ-reducts described in Remark 4.3.

(TT1) In Module 1, each translated instance of the substitution rule (TT1-1) is an instance of Novák's derived substitution rule (claim 1 of Lemma 4.8). Using Lemma 4.14, each translated instance of the replacement rule (TT1-2) λ-reduces to an instance of the replacement rule (FT-R), and each translated instance of the λ-conversion axiom (TT1-3) λ-reduces to an instance of (FT3). Finally, each translated instance of the η-conversion axiom (TT1-4) is an instance of claim 4 of Lemma 4.9.

(TT2) For Module 2, the claim is immediate from the embedding of predicate IMTL$_\triangle\forall$ in FTT by claim 2 of Lemma 4.8.

(TT3) Using Lemma 4.14 and immediate λ-reductions, the translated instances of (TT3-1), (TT3-2), (TT3-3), and (TT3-4) are instances, respectively, of claim 1 of Lemma 4.9, claim 3 of Lemma 4.8, the added symmetry axiom (FT17), and (FT4).

By Lemma 4.14, a translated instance of the separation axiom (TT3-5) λ-reduces to $\triangle(A^{\mathsf{F}}_\alpha \equiv B^{\mathsf{F}}_\alpha) \to (A^{\mathsf{F}}_\alpha \equiv B^{\mathsf{F}}_\alpha)$, a type-theoretic instance of the IMTL$_\triangle\forall$-axiom ($\triangle$3) covered by claim 2 of Lemma 4.8. Finally, the translated instances of (TT3-6) and (TT3-7) reduce to instances of (FT1) and of claim 3 of Lemma 4.9, respectively. □

4.6 Round-trip translations

It remains to justify the round-trip claims of Theorem 4.7. The proof requires some care both in tracking the defined notation through the two translations and in the treatment of the two equality predicates of MFTT under the translation to the single equality of FTT.

Proof of Theorem 4.7. We prove the two claims in order. Both proofs are by structural induction on formulas in the definitional extensions on which the translations are defined.

1. Let A_α be an FTT-formula. We prove, by induction on A_α, that $\vdash_{\mathrm{FTT}} (A^{\mathsf{M}}_\alpha)^{\mathsf{F}} \equiv A_\alpha$.

   The cases of variables and extralogical constants are immediate by reflexivity (claim 3 of Lemma 4.8). For the distinguished primitive constants of FTT, consider first $\boldsymbol{E}_{o\gamma\gamma}$, for an arbitrary type γ. By the translation clauses and the definition of $\equiv_\gamma$, $(\boldsymbol{E}^{\mathsf{M}}_{o\gamma\gamma})^{\mathsf{F}}$ is

   $$\lambda y_\gamma \lambda x_\gamma \big(\lambda u_\gamma \lambda v_\gamma \big(\lambda r_\gamma (\lambda s_\gamma \boldsymbol{E}_{o\gamma\gamma} s_\gamma) r_\gamma\big) v_\gamma u_\gamma\big) x_\gamma y_\gamma.$$

   Repeated λ-conversion, using claim 5 of Lemma 4.9 within the surrounding abstractions, gives $\vdash_{\mathrm{FTT}} (\boldsymbol{E}^{\mathsf{M}}_{o\gamma\gamma})^{\mathsf{F}} \equiv \lambda y_\gamma \lambda x_\gamma \boldsymbol{E}_{o\gamma\gamma} y_\gamma x_\gamma$. Two applications of η-conversion (claim 4 of the same Lemma) then yield $\vdash_{\mathrm{FTT}} (\boldsymbol{E}^{\mathsf{M}}_{o\gamma\gamma})^{\mathsf{F}} \equiv \boldsymbol{E}_{o\gamma\gamma}$.

   The cases of $\boldsymbol{C}_{ooo}$ and $\boldsymbol{D}_{oo}$ are analogous, using the definitions of $\wedge$ and $\triangle$. The defined equality symbol $\equiv_\gamma$ is handled in the same way. The shared propositional connectives are recovered directly by the double translation, since both translations commute with them by definition.

   The induction step for application is proved as follows. The double translation of $B_{\beta\gamma}C_\gamma$ is $(B^{\mathsf{M}}_{\beta\gamma})^{\mathsf{F}}(C^{\mathsf{M}}_\gamma)^{\mathsf{F}}$. Using the induction hypotheses and claim 5 of Lemma 4.9 twice, we obtain the required $\vdash_{\mathrm{FTT}} (B^{\mathsf{M}}_{\beta\gamma})^{\mathsf{F}}(C^{\mathsf{M}}_\gamma)^{\mathsf{F}} \equiv B_{\beta\gamma}C_\gamma$.

   For λ-abstraction, the induction hypothesis $\vdash_{\mathrm{FTT}} (D^{\mathsf{M}}_\delta)^{\mathsf{F}} \equiv D_\delta$ gives, by claim 5 of Lemma 4.9, the required $\vdash_{\mathrm{FTT}} \lambda x_\gamma (D^{\mathsf{M}}_\delta)^{\mathsf{F}} \equiv \lambda x_\gamma D_\delta$.

   For quantified formulas, consider first $(\forall x_\gamma) D_o$. Its MFTT-translation is $(\forall x_\gamma) D^{\mathsf{M}}_o$, i.e., the abbreviation for $\forall_{o(o\gamma)} \lambda x_\gamma D^{\mathsf{M}}_o$; thus its literal double translation is

   $$\big(\lambda p_{o\gamma} (\forall y_\gamma)(p_{o\gamma} y_\gamma)\big)\big(\lambda x_\gamma (D^{\mathsf{M}}_o)^{\mathsf{F}}\big), \tag{3}$$

   where y_γ is fresh. By λ-conversion, this is provably equal in FTT to $(\forall x_\gamma)(D^{\mathsf{M}}_o)^{\mathsf{F}}$. By the induction hypothesis, $\vdash_{\mathrm{FTT}} (D^{\mathsf{M}}_o)^{\mathsf{F}} \equiv D_o$. Applying claim 5 of Lemma 4.9 to the defining formula $\big(\lambda x_\gamma (D^{\mathsf{M}}_o)^{\mathsf{F}}\big) \equiv (\lambda x_\gamma \top)$ of $(\forall x_\gamma)(D^{\mathsf{M}}_o)^{\mathsf{F}}$ yields

   $$\vdash_{\mathrm{FTT}} (\forall x_\gamma)(D^{\mathsf{M}}_o)^{\mathsf{F}} \equiv (\forall x_\gamma) D_o.$$

On the other hand, λ-reduction of the literal translation (3) gives

$$\vdash_{\mathrm{FTT}} (((\forall x_\gamma) D_o)^{\mathsf{M}})^{\mathsf{F}} \equiv (\forall x_\gamma)(D_o^{\mathsf{M}})^{\mathsf{F}}.$$

Using (FT-R) with these two equalities, we obtain the required round-trip equality

$$\vdash_{\mathrm{FTT}} (((\forall x_\gamma) D_o)^{\mathsf{M}})^{\mathsf{F}} \equiv (\forall x_\gamma) D_o.$$

Finally, the case for existential quantification follows from the definition of $(\exists x_\gamma) D_o$ as $\neg(\forall x_\gamma)\neg D_o$ and the already treated cases of negation, application, and universal quantification.

2. Let B_α be an MFTT-formula. We prove, by induction on B_α, that $\vdash_{\mathrm{MFTT}} (B_\alpha^{\mathsf{F}})^{\mathsf{M}} = B_\alpha^{\approx}$.

   The cases of variables and extralogical constants are immediate by Lemma 4.10, claim 1.

   For the equality constants, the double translations of both $=_{o\gamma\gamma}$ and $\approx_{o\gamma\gamma}$ λ-convert, using (TT1-3), to $\lambda x_\gamma \lambda y_\gamma (x_\gamma \approx y_\gamma)$. This is the η-expansion of $\approx_{o\gamma\gamma}$, and so is provably equal to it by (TT1-4). Since $=_{o\gamma\gamma}^{\approx}$ and $\approx_{o\gamma\gamma}^{\approx}$ are both $\approx_{o\gamma\gamma}$, this proves the required base cases for the equality predicates.

   The remaining propositional symbols are recovered immediately by the double translation: the shared symbols are fixed, and $\leftrightarrow$ is mapped exactly to its MFTT-definiens.

   For the universal quantifier, $(\forall_{o(o\gamma)}^{\mathsf{F}})^{\mathsf{M}}$ is $\lambda p_{o\gamma} \forall_{o(o\gamma)} \lambda x_\gamma p_{o\gamma} x_\gamma$, where the quantified binder in the translation has been expanded. Two applications of η-conversion (TT1-4), using claim 3 of Lemma 4.10 in the surrounding context, yield $\vdash_{\mathrm{MFTT}} (\forall_{o(o\gamma)}^{\mathsf{F}})^{\mathsf{M}} = \forall_{o(o\gamma)}$. The case of $\exists_{o(o\gamma)}$ is analogous.

   For the induction step for application, the double translation of $F_{\beta\gamma} C_\gamma$ is $(F_{\beta\gamma}^{\mathsf{F}})^{\mathsf{M}} (C_\gamma^{\mathsf{F}})^{\mathsf{M}}$. Using the induction hypotheses and claim 3 of Lemma 4.10 twice, we obtain the required $\vdash_{\mathrm{MFTT}} ((F_{\beta\gamma} C_\gamma)^{\mathsf{F}})^{\mathsf{M}} = F_{\beta\gamma}^{\approx} C_\gamma^{\approx}$.

   For λ-abstraction, the induction hypothesis $\vdash_{\mathrm{MFTT}} (D_\delta^{\mathsf{F}})^{\mathsf{M}} = D_\delta^{\approx}$ gives, again by claim 3 of Lemma 4.10, the required $\vdash_{\mathrm{MFTT}} \lambda x_\gamma (D_\delta^{\mathsf{F}})^{\mathsf{M}} = \lambda x_\gamma D_\delta^{\approx}$.

   Finally, quantified formulas in binder notation, $(\forall x_\alpha) C_o$ and $(\exists x_\alpha) C_o$, are already covered by the application and λ-abstraction steps applied to their expansions. □

5 Necessity of the axiom of symmetry for fuzzy equality

The original printed axiomatization of fuzzy type theory over $\mathrm{IMTL}_\triangle$ in [8] is intended to be complete with respect to the class of models defined there. In these models, $\equiv_\alpha$ is interpreted at every type α as a fuzzy equality, hence in particular as a reflexive, symmetric, and &-transitive relation. Thus, at the semantic level, symmetry of equality is part of the intended notion of model.

The printed axiom list in [8], however, does not contain the element-type symmetry axiom $(A_\varepsilon \equiv B_\varepsilon) \to (B_\varepsilon \equiv A_\varepsilon)$. It does contain the element-type transitivity axiom (FT4), and reflexivity of $\equiv$ is derivable as a theorem [8, Thm. 2(b)]. Symmetry is also derivable at the truth-value type o, since axiom (FT5) identifies equality of truth values with biresiduation. Furthermore, once symmetry is available at a type β, it can be transferred to functional types $\beta\alpha$ by the pointwise characterization of functional equality [8, Thm. 11]. The remaining issue is therefore symmetry at the primitive element type ε.

This raises the question of whether the symmetry of fuzzy equality at type ε is derivable from the printed axiomatics. Novák proves a metatheoretic symmetry principle [8, Thm. 3(a)] saying that if $\vdash A_\alpha \equiv B_\alpha$, then $\vdash B_\alpha \equiv A_\alpha$. This does not, however, immediately yield the implication $\vdash (A_\alpha \equiv B_\alpha) \to (B_\alpha \equiv A_\alpha)$, nor the equality of truth values $\vdash (A_\alpha \equiv B_\alpha) \equiv (B_\alpha \equiv A_\alpha)$, which is nevertheless used in Novák's canonical-model construction [8, Sect. 4.3.2, Eq. (41)].

In the present paper, we make this symmetry principle explicit as axiom (FT17); since it is valid in Novák's intended semantics, its addition does not change the intended class of models. The non-redundancy of this axiom is shown in the next subsection.

5.1 Necessity of the symmetry axiom (FT17) in FTT over $\mathrm{IMTL}_\triangle$

In this subsection, we denote Novák's original printed axiomatization over $\mathrm{IMTL}_\triangle$ by FTT^-, reserving the designation FTT for the reference system of Definition 2.7, which includes the added type-ε symmetry axiom (FT17); see Remark 2.8. We show that FTT^- is incomplete with respect to the intended semantics of [8], by constructing a structure that validates all axioms of FTT^- and in which its inference rules preserve validity, while $\equiv_\varepsilon$ is not symmetric.

By an FTT^--*model* we mean a structure satisfying Novák's semantic conditions for FTT, except that symmetry of the element-type fuzzy equality is not required. The FTT^--model constructed below is obtained from the full frame semantics of [8, Sect. 2.3] by keeping the full function spaces, the interpretation of truth values, the pointwise definition of equality at functional types, and the weak extensionality requirement, but by allowing the element-type equality to be reflexive, &-transitive, and 1-faithful without being symmetric.

Let $Ł_3 = \{0, \frac{1}{2}, 1\}$ be the three-valued $\mathrm{IMTL}_\triangle$-algebra with the operations

$$\begin{aligned} a \wedge b &= \min(a,b), & a \vee b &= \max(a,b),\\ a \,\&\, b &= \max(0, a+b-1), & a \oplus b &= \min(a+b, 1),\\ a \to b &= \min(1, 1-a+b), & a \leftrightarrow b &= 1-|a-b|,\\ \neg a &= 1-a, & \triangle a &= 1-\operatorname{sgn}|1-a|. \end{aligned}$$

Let the domains of the base types be

$$M_o = Ł_3, \qquad M_\varepsilon = \{u, v\}.$$

Following [8], we write $[m =_\alpha m']^*$ for the truth degree assigned by the interpretation of the equality constant at type α in the modified structure constructed here; the superscript * is used throughout to distinguish this FTT^--model from Novák's intended models. At the truth-value type we put $[a =_o b]^* := a \leftrightarrow b$, for all $a, b \in Ł_3$. At the element type we define:

$$[u =_\varepsilon u]^* = [v =_\varepsilon v]^* = 1, \qquad [u =_\varepsilon v]^* = \tfrac{1}{2}, \qquad [v =_\varepsilon u]^* = 0.$$

This relation is reflexive, 1-faithful, and &-transitive, but it is not symmetric.

For complex types we take full function spaces, $M_{\beta\alpha} := M_\beta^{M_\alpha}$. The equality at a functional type is defined pointwise, as in [8]:

$$[f =_{\beta\alpha} g]^* := \bigwedge_{m \in M_\alpha} \big[f(m) =_\beta g(m)\big]^*.$$

Since the equality relation at both base types is 1-faithful, it follows by induction on types that $[m =_\alpha m']^* = 1$ iff $m = m'$. Consequently, every function in each full function space $M_{\beta\alpha}$ is weakly extensional: if $[m =_\alpha m']^* = 1$, then $m = m'$, and therefore $\big[f(m) =_\beta f(m')\big]^* = 1$, for every $f \in M_{\beta\alpha}$.

We define an interpretation $\mathcal{I}^*$ of the primitive constants as follows:

$$\begin{aligned}\mathcal{I}^*(\boldsymbol{E}_{o\alpha\alpha})(m')(m) &:= [m =_\alpha m']^*,\\ \mathcal{I}^*(\boldsymbol{C}_{ooo})(b)(a) &:= a \wedge b,\\ \mathcal{I}^*(\boldsymbol{D}_{oo})(a) &:= \triangle a.\end{aligned}$$

As usual, for each valuation p, the interpretation $\mathcal{I}^*$ extends recursively to all formulas; the resulting value of a formula A_α under p is denoted by $\mathcal{I}^*_p(A_\alpha) \in M_\alpha$. In particular, application, λ-abstraction, and quantification are interpreted by

$$\begin{aligned}\mathcal{I}^*_p(B_{\beta\alpha}A_\alpha) &:= \mathcal{I}^*_p(B_{\beta\alpha})\big(\mathcal{I}^*_p(A_\alpha)\big),\\ \mathcal{I}^*_p(\lambda x_\alpha B_\beta)(m) &:= \mathcal{I}^*_{p[x_\alpha := m]}(B_\beta),\\ \mathcal{I}^*_p\big((\forall x_\alpha)A_o\big) &:= \bigwedge_{m \in M_\alpha} \mathcal{I}^*_{p[x_\alpha := m]}(A_o).\end{aligned}$$

Consequently, by the definition of the equality notation $\equiv_\alpha$, we have

$$\mathcal{I}^*_p(A_\alpha \equiv B_\alpha) \;=\; \mathcal{I}^*_p(\boldsymbol{E}_{o\alpha\alpha})\big(\mathcal{I}^*_p(B_\alpha)\big)\big(\mathcal{I}^*_p(A_\alpha)\big) \;=\; \big[\mathcal{I}^*_p(A_\alpha) =_\alpha \mathcal{I}^*_p(B_\alpha)\big]^*. \tag{4}$$

Any additional extralogical constants, if present, may be interpreted arbitrarily as objects from the corresponding domains. Since the frame is full, the interpretation of every λ-abstract is again an object of the appropriate type. We denote the resulting structure by $\mathcal{M}^*$.

Proposition 5.1. *The structure $\mathcal{M}^*$ validates all of the axioms of* FTT^-*, and both inference rules of* FTT^- *preserve validity in $\mathcal{M}^*$. However, the fuzzy relation interpreting element-type equality in $\mathcal{M}^*$ is not symmetric, and axiom* (FT17) *fails in $\mathcal{M}^*$.*

Proof. The axioms (FT5)–(FT13) and (FT15) are valid in $\mathcal{M}^*$ because, at type o, the structure $\mathcal{M}^*$ interprets the connectives by the operations of the three-valued $\mathrm{IMTL}_\triangle$-algebra $\mathrm{Ł}_3$, and equality of truth values by biresiduation. Thus these axioms reduce to valid $\mathrm{IMTL}_\triangle$-identities in $\mathrm{Ł}_3$. Similarly, (FT-N) preserves validity, since $\triangle 1 = 1$ in $\mathrm{Ł}_3$. The remaining cases are verified as follows:

(FT1) Let $a := \mathcal{I}^*_p(x_\alpha)$, $b := \mathcal{I}^*_p(y_\alpha)$, and $F := \mathcal{I}^*_p(f_{\beta\alpha})$. If $\mathcal{I}^*_p\big(\triangle(x_\alpha \equiv y_\alpha)\big) = 0$, then the value of the implication in (FT1) is 1. If $\mathcal{I}^*_p\big(\triangle(x_\alpha \equiv y_\alpha)\big) = 1$, then $[a =_\alpha b]^* = 1$ by (4), and hence $a = b$ by 1-faithfulness. Therefore $F(a) = F(b)$, whence $\big[F(a) =_\beta F(b)\big]^* = 1$. Thus the consequent of (FT1) and the whole implication have value 1.

(FT2a) Let $F := \mathcal{I}^*_p(f_{\beta\alpha})$ and $G := \mathcal{I}^*_p(g_{\beta\alpha})$. By the interpretation of $\forall$ and (4),

$$\mathcal{I}^*_p\big((\forall x_\alpha)(f_{\beta\alpha}x_\alpha \equiv g_{\beta\alpha}x_\alpha)\big) \;=\; \bigwedge_{m \in M_\alpha} \big[F(m) =_\beta G(m)\big]^* \;=\; [F =_{\beta\alpha} G]^* \;=\; \mathcal{I}^*_p(f_{\beta\alpha} \equiv g_{\beta\alpha}).$$

Hence the antecedent and the consequent of (FT2a) have the same value, so the implication has value 1.

(FT2b) With F, G as above and $a \coloneqq \mathcal{I}^*_p(x_\alpha)$, we have

$$[F =_{\beta\alpha} G]^* \;=\; \bigwedge_{m \in M_\alpha} \big[F(m) =_\beta G(m)\big]^* \;\leq\; \big[F(a) =_\beta G(a)\big]^*.$$

Therefore $\mathcal{I}^*_p(f_{\beta\alpha} \equiv g_{\beta\alpha}) \leq \mathcal{I}^*_p(f_{\beta\alpha}x_\alpha \equiv g_{\beta\alpha}x_\alpha)$, and so the implication (FT2b) has value 1.

(FT3) By the clauses for application and λ-abstraction,

$$\mathcal{I}^*_p\big((\lambda x_\alpha B_\beta)A_\alpha\big) \;=\; \mathcal{I}^*_p(\lambda x_\alpha B_\beta)\big(\mathcal{I}^*_p(A_\alpha)\big) \;=\; \mathcal{I}^*_{p[x_\alpha \coloneqq \mathcal{I}^*_p(A_\alpha)]}(B_\beta) \;=\; \mathcal{I}^*_p\big(B_\beta[A_\alpha/x_\alpha]\big),$$

where the last equality is the usual substitution lemma for the Tarski interpretation of typed λ-terms (see [2, Lem. 2.26(3)] for the lemma in the many-valued setting).

(FT4) The relation $[=_\varepsilon]^*$ is &-transitive. Indeed, the only non-zero off-diagonal value is $\frac{1}{2}$, and in Ł$_3$ we have $\frac{1}{2}$ & $\frac{1}{2} = 0$. Thus, for all $a, b, c \in M_\varepsilon$,

$$[a =_\varepsilon b]^* \,\&\, [b =_\varepsilon c]^* \leq [a =_\varepsilon c]^*.$$

By residuation, this is equivalent to the validity of (FT4).

(FT14) For any valuation p, put $G \coloneqq \mathcal{I}^*_p(g_{oo})\colon \text{Ł}_3 \longrightarrow \text{Ł}_3$. Then both sides of (FT14) have value $G(0) \wedge G(1)$, since $\{\triangle a \mid a \in \text{Ł}_3\} = \{0, 1\}$.

(FT16) Let $a \coloneqq \mathcal{I}^*_p(A_o)$ and $b_m \coloneqq \mathcal{I}^*_{p[x_\alpha \coloneqq m]}(B_o)$ for each $m \in M_\alpha$. Since x_α is not free in A_o, the value a is independent of m. All type domains are finite, and implication in Ł$_3$ preserves finite meets in its consequent; hence

$$\bigwedge_{m \in M_\alpha} (a \to b_m) \;=\; a \to \bigwedge_{m \in M_\alpha} b_m.$$

(FT-R) Assume that $A_\alpha \equiv A'_\alpha$ is valid in $\mathcal{M}^*$. We first show, by induction on the depth of the specified occurrence of A_α in an arbitrary formula C_γ, that for every valuation p,

$$\mathcal{I}^*_p(C_\gamma) \;=\; \mathcal{I}^*_p\big(C_\gamma[A'_\alpha /\!\!/ A_\alpha]\big).$$

If the specified occurrence has depth 0, then C_γ is A_α, and hence $\gamma = \alpha$. Since $A_\alpha \equiv A'_\alpha$ is valid, we have $\mathcal{I}^*_p(A_\alpha \equiv A'_\alpha) = 1$ for every valuation p. By (4) and 1-faithfulness of $[=_\alpha]^*$, it follows that $\mathcal{I}^*_p(A_\alpha) = \mathcal{I}^*_p(A'_\alpha)$.

Suppose next that C_γ is $P_{\gamma\delta}Q_\delta$. If the specified occurrence lies in $P_{\gamma\delta}$, let $P'_{\gamma\delta}$ be the formula obtained from $P_{\gamma\delta}$ by the corresponding replacement. By the induction hypothesis, $\mathcal{I}^*_p(P_{\gamma\delta}) = \mathcal{I}^*_p(P'_{\gamma\delta})$ for every valuation p. Hence, by the Tarski clause for application,

$$\mathcal{I}^*_p(P_{\gamma\delta}Q_\delta) \;=\; \mathcal{I}^*_p(P_{\gamma\delta})\big(\mathcal{I}^*_p(Q_\delta)\big) \;=\; \mathcal{I}^*_p(P'_{\gamma\delta})\big(\mathcal{I}^*_p(Q_\delta)\big) \;=\; \mathcal{I}^*_p(P'_{\gamma\delta}Q_\delta).$$

If the specified occurrence lies in Q_δ, the argument is analogous.

Finally, suppose that $C_{\gamma\delta}$ is $\lambda x_\delta P_\gamma$, and let P'_γ be the formula obtained from P_γ by the replacement. For every valuation p and every $m \in M_\delta$, the induction hypothesis applied to the valuation $p[x_\delta \coloneqq m]$ gives

$$\mathcal{I}^*_{p[x_\delta \coloneqq m]}(P_\gamma) \;=\; \mathcal{I}^*_{p[x_\delta \coloneqq m]}(P'_\gamma).$$

By the Tarski clause for λ-abstraction,

$$\mathcal{I}^*_p(\lambda x_\delta P_\gamma)(m) \;=\; \mathcal{I}^*_{p[x_\delta \coloneqq m]}(P_\gamma) \;=\; \mathcal{I}^*_{p[x_\delta \coloneqq m]}(P'_\gamma) \;=\; \mathcal{I}^*_p(\lambda x_\delta P'_\gamma)(m).$$

Thus $\mathcal{I}^*_p(\lambda x_\delta P_\gamma) = \mathcal{I}^*_p(\lambda x_\delta P'_\gamma)$.

This completes the induction. Applying the result to $C_\gamma \coloneqq B_o$, we obtain, for every valuation p, that $\mathcal{I}^*_p(B_o) = \mathcal{I}^*_p\big(B_o[A'_\alpha /\!\!/ A_\alpha]\big)$. Therefore (FT-R) preserves validity in $\mathcal{M}^*$.

Finally, we show that (FT17) fails in $\mathcal{M}^*$. Indeed, for a valuation p such that $p(x_\varepsilon) = u$ and $p(y_\varepsilon) = v$, we have $\mathcal{I}^*_p(x_\varepsilon \equiv y_\varepsilon) = [u =_\varepsilon v]^* = \frac{1}{2}$ and $\mathcal{I}^*_p(y_\varepsilon \equiv x_\varepsilon) = [v =_\varepsilon u]^* = 0$. Consequently,

$$\mathcal{I}^*_p\big((x_\varepsilon \equiv y_\varepsilon) \to (y_\varepsilon \equiv x_\varepsilon)\big) \;=\; \big(\tfrac{1}{2} \to 0\big) \;=\; \tfrac{1}{2} \;<\; 1.$$

Thus $\mathcal{M}^*$ is an FTT$^-$-model, but (FT17) is not valid in $\mathcal{M}^*$. □

5.2 Equality in FTT based on EQ-algebras

By contrast, in Novák's later presentation of fuzzy type theory based on EQ-algebras [11], henceforth written as FTT(EQ), the symmetry of equality is derivable and need not be postulated separately. The reason is that the two frameworks use different conceptions of element-type equality.

In FTT over IMTL$_\triangle$, equality at type ε is an arbitrary fuzzy equivalence relation on the domain of urelements, subject to the appropriate reflexivity, symmetry, transitivity, and weak extensionality requirements. In FTT(EQ), however, fuzzy equality on urelements is tied to the primitive fuzzy-equality operation $\sim$ on truth values via a *generating function* $f\colon M_\varepsilon \longrightarrow M_o$, so that

$$[m = m'] \;\coloneqq\; f(m) \sim f(m')$$

defines a fuzzy equality on M_ε [11, Lem. 7]. Symmetry is then immediate, since the EQ-operation $\sim$ is symmetric [11, Lem. 1(a)]. This explains why the symmetry principle [11, Thm. 7(a)] can be proved syntactically in FTT(EQ) without adding a separate axiom corresponding to (FT17).

The price for this is that element-type equality in FTT(EQ) is substantially more restricted. In particular, it must be the pullback of the algebraic equality $\sim$ on truth values along a generating function. Not every fuzzy equality relation allowed in the IMTL$_\triangle$-based semantics of FTT has this form, even over the same algebra of truth values, as shown by the following example.

Example 5.2. Let Ł$_3 = \{0, \frac{1}{2}, 1\}$ be the three-valued IMTL$_\triangle$-algebra with the same operations as in Section 5.1. Modulo the change of signature, it can also be regarded as a separated EQ$_\triangle$-algebra by taking $a \sim b \coloneqq a \leftrightarrow b$.

Let the domain of urelements be $M_\varepsilon = \{a, b, c\}$. Define a fuzzy relation $R\colon M_\varepsilon^2 \longrightarrow \text{Ł}_3$ by

$$\begin{aligned} Raa = Rbb = Rcc &= 1 \\ Rab = Rba &= \tfrac{1}{2} \\ Rac = Rca = Rbc = Rcb &= 0 \end{aligned}$$

Then R is reflexive, symmetric, and 1-faithful. It is also &-transitive, since the only non-zero off-diagonal value is $\frac{1}{2}$, and $\frac{1}{2}$ & $\frac{1}{2} = 0$; hence Rxy & $Ryz \leq Rxz$ for all $x, y, z \in M_\varepsilon$. Note that this example is not contrived, since R is an instance of the Euclidean fuzzy equality $Rxy = \max(1 - |x - y|, 0)$ on $M_\varepsilon = \{a, b, c\}$, e.g., for $a = 0$, $b = 0.5$, and $c = 2$.

Consequently, R is a legitimate element-type equality in a standard FTT(IMTL$_\triangle$)-frame: take $M_o = \text{Ł}_3$, interpret equality on truth values by $\leftrightarrow$, and define equality at functional types pointwise as in [8, Sect. 2.3.5]. Since R is 1-faithful, every function in the full function spaces of the frame is weakly extensional.

However, R cannot be generated from any function $f\colon M_\varepsilon \longrightarrow \text{Ł}_3$ via the EQ-rule

$$Rxy \;=\; f(x) \sim f(y) \;=\; 1 - |f(x) - f(y)|.$$

Indeed, $Rab = \frac{1}{2}$ implies $|f(a) - f(b)| = \frac{1}{2}$, and $Rbc = 0$ implies $|f(b) - f(c)| = 1$, so $f(b)$ and $f(c)$ must be 0 and 1 in some order. It follows that $f(a) = \frac{1}{2}$, and therefore

$$1 - |f(a) - f(c)| \;=\; \tfrac{1}{2}.$$

Thus any f-generated equality would give $Rac = \frac{1}{2}$, contrary to the definition of R, where $Rac = 0$.

Example 5.2 can be readily generalized to any complete linear IMTL$_\triangle$-algebra with at least three truth degrees (i.e., not crisp).

Being generated from the truth-value operation $\sim$ imposes various constraints on element-type fuzzy equality in EQ-based FTT. One such constraint arises in FTT-models over those linear EQ$_\triangle$-algebras which, like IMTL$_\triangle$-algebras, have involutive negation. In these models, the existence of two fully distinct elements in M_ε imposes a global lower bound on the degrees of equality to any element lying at a given distance from one of the two fully distinct elements:

Proposition 5.3. *Let E be a linearly ordered involutive* EQ$_\triangle$*-algebra. Let equality on M_ε be generated by $f\colon M_\varepsilon \longrightarrow E$.*

Suppose that $a, b, c \in M_\varepsilon$ are such that $[a =_\varepsilon b] = 0$ and $[a =_\varepsilon c] = \alpha$. Then, for every $d \in M_\varepsilon$,

$$[c =_\varepsilon d] \;\geq\; \min(\alpha, \neg\alpha).$$

Before proving this proposition, we establish a lemma that will also be useful later:

Lemma 5.4. *Let E be a linearly ordered* EQ*-algebra. Then $u \wedge v \;\leq\; u \sim v$, for all $u, v \in E$.*

Proof. Recall that in every EQ-algebra, the derived implication is $a \to b \coloneqq (a \wedge b) \sim a$ and satisfies $b \leq a \to b$ [11, Lem. 1g]. Let $u, v \in E$. If $u \leq v$, then

$$u \;\leq\; v \to u \;=\; (u \wedge v) \sim v \;=\; u \sim v.$$

By the symmetry of $\sim$ [11, Lem. 1a], the same conclusion holds if $v \leq u$. Hence, in the linear order, $u \wedge v \leq u \sim v$. □

Proof of Proposition 5.3. Put $u \coloneqq f(a)$, $v \coloneqq f(b)$, and $w \coloneqq f(c)$. By assumption, $u \sim v = 0$ and $u \sim w = \alpha$. By Lemma 5.4, $u \wedge v = 0$. Since E is linearly ordered, this implies that either $u = 0$ or $v = 0$.

We first show that in both cases, $w \wedge \neg w = \alpha \wedge \neg\alpha$.

1. If $u = 0$, then $\alpha = u \sim w = 0 \sim w = \neg w$. Thus, since E is involutive, $w = \neg\alpha$. Hence $w \wedge \neg w = \alpha \wedge \neg\alpha$.

2. If $v = 0$, then $0 = u \sim v = u \sim 0 = \neg u$. Thus by involutivity, $u = \neg 0 = 1$. Hence $\alpha = u \sim w = 1 \sim w = w$ by goodness, as every involutive EQ-algebra is good [11, Thm. 1]. Consequently, $w \wedge \neg w = \alpha \wedge \neg\alpha$.

It remains to show that for every $d \in M_\varepsilon$, $[c =_\varepsilon d] \geq w \wedge \neg w$. Let $d \in M_\varepsilon$ and put $t = f(d)$. Since E is linearly ordered, either $w \leq t$ or $t \leq w$.

1. If $w \leq t$, then by Lemma 5.4 we have $[c =_\varepsilon d] = w \sim t \geq w \wedge t = w$.

2. If $t \leq w$, then $\neg w \leq \neg t$ by [11, Lem. 5c]. Hence, using [11, Lem. 5b] and Lemma 5.4, we obtain $[c =_\varepsilon d] = w \sim t = \neg w \sim \neg t \geq \neg w \wedge \neg t = \neg w$.

Therefore, in either case, $[c =_\varepsilon d] \geq w \wedge \neg w = \alpha \wedge \neg\alpha$. □

Thus, for example, in a model over the standard $\mathrm{MV}_\triangle$-algebra (regarded as an $\mathrm{EQ}_\triangle$-algebra), an urelement that is half equal to either of two fully distinct elements $a, b \in M_\varepsilon$ is at least half equal to every other urelement.

Another constraint on equality in EQ-based FTT is that the urelement domain M_ε cannot have more than two mutually fully distinct elements:

Proposition 5.5. *Let E be a linearly ordered $\mathrm{EQ}_\triangle$-algebra. Let equality on M_ε be generated by $f\colon M_\varepsilon \to E$. Then M_ε contains no 0-triangle, i.e., there are no $a, b, c \in M_\varepsilon$ such that*

$$[a =_\varepsilon b] = [b =_\varepsilon c] = [a =_\varepsilon c] = 0.$$

Proof. Suppose, towards a contradiction, that $a, b, c \in M_\varepsilon$ form a 0-triangle, and put $u \coloneqq f(a)$, $v \coloneqq f(b)$, and $w \coloneqq f(c)$. Then $u \sim v = v \sim w = u \sim w = 0$, so by Lemma 5.4, $u \wedge v = v \wedge w = u \wedge w = 0$. Since E is linearly ordered, this means that each of the pairs $\{u, v\}$, $\{v, w\}$, and $\{u, w\}$ contains 0. Therefore at least two of u, v, w must be equal to 0, but then their equality degree is $0 \sim 0 = 1$, a contradiction. □

The special case of Proposition 5.5 over the two-valued Boolean $\mathrm{EQ}_\triangle$-algebra shows that this restriction marks a significant model-theoretic difference between FTT(EQ) and classical type theory:

Corollary 5.6. *Let E be the two-valued Boolean $\mathrm{EQ}_\triangle$-algebra with carrier $\{0, 1\}$, and let equality on M_ε be generated by $f\colon M_\varepsilon \to E$. Then the generated equality has at most two equivalence classes. If this equality is required to be separated (i.e., realized by the identity), then M_ε can have at most two elements.*

Consequently, the two-valued semantics of FTT(EQ) does not recover arbitrary classical Henkin models of simple type theory. In classical type theory, the domain of urelements may be any non-empty set, and equality on it is ordinary identity. In the two-valued FTT(EQ) semantics, by contrast, separated equality requires the urelement domain to have at most two elements; on larger domains, distinct urelements necessarily lie in the same fiber of the generating function and are therefore identified. Thus FTT(EQ) is not a conservative fuzzy generalization of classical type theory in the sense of reducing, over the two-valued algebra, to the class of classical Henkin models with arbitrary urelement domains. In the same sense, FTT(EQ) does not generalize higher-order fuzzy logic, either as Novák's Church–Henkin fuzzy type theory over $\mathrm{IMTL}_{\triangle}$ [8] or as Russell-style fuzzy simple type theory [3], since both of these theories likewise allow arbitrarily large sets of pairwise fully distinct urelements.

The foregoing results show that FTT(EQ) is substantially more restrictive than FTT over $\mathrm{IMTL}_{\triangle}$ in its treatment of fuzzy equality. On the one hand, the generating function mechanism makes the EQ-based framework of [11] elegant and uniform. Although it admits at most two mutually fully distinct urelements, FTT(EQ) is sufficient—and perhaps advantageous—for the theory of trichotomous evaluative linguistic expressions [9], since that theory is based on a bipolar similarity scale between two extremal endpoints.[1] For applications requiring more than two mutually fully distinct urelements, however, this restriction becomes substantive; the original $\mathrm{IMTL}_{\triangle}$-based FTT imposes no such bound.

6 Conclusion

We have presented a modular axiomatization MFTT of Novák's $\mathrm{IMTL}_{\triangle}$-based fuzzy type theory, separating the type-theoretic machinery, the underlying logic, and the axioms governing fuzzy equality into distinct layers. This organization facilitates modifications to individual modules; it also suggests how variants over other many-valued logics may be constructed. Completeness proofs for such variants may draw on the Henkin-style methods for the ground many-valued type theory TT_0 and its extensions from [2]. The alternative rules considered in Section 3.2 indicate possible modifications of the fuzzy-equality module even when Module 2b, and hence $\triangle$, is absent.

We have compared MFTT with Novák's original FTT by means of syntactic translations in both directions. The translations preserve derivability, and their compositions recover the source formulas up to provable equality, with the expected collapse of the two equality predicates of MFTT when passing through the single-equality language of FTT. In this sense, MFTT provides a modular reconstruction of the deductive framework of FTT, while explicitly distinguishing between type-theoretic equality and fuzzy equality.

Along the way, we identified a small but substantive gap in the original axiomatization of FTT: the element-type symmetry principle, valid in Novák's intended semantics, is not derivable from the printed axiomatics, so we include it as an axiom in our reference system. Moreover, Remark 3.9 corrects a minor gap found in one alternative axiomatization of TT_0 [2, Thm. 2.13].

The final comparison concerns the status of the added symmetry axiom for fuzzy equality. The FTT^{-}-countermodel constructed in Section 5 shows that the addition is substantive: element-type symmetry can fail while all the remaining printed axioms hold. By contrast, in Novák's later EQ-based fuzzy type theory, the symmetry principle is derivable from the

[1] Thanks are due to M. Daňková for this observation.

generated-equality construction, which explains why no analogous axiom is needed there. However, the same construction substantially restricts the admissible fuzzy equalities: no three urelements in an FTT(EQ)-model can be mutually fully distinct. In particular, in two-valued FTT(EQ)-models with separated equality, the urelement domain is limited to at most two elements.

Taken together, these results clarify the relation between Novák's original $\mathrm{IMTL}_{\triangle}$-based FTT, its later EQ-based version, and the modular presentation developed here, while providing a basis for further variants and axiomatic extensions of the modular framework.

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