Automated proving of statements with composition-related notions

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1 Introduction

The systematic development of the formal theory of fuzzy sets and fuzzy relations has challenged many researchers since the very beginning, as reflected by a huge number of papers and book chapters. It might seem that nothing new can be done in this field. Nevertheless, interesting questions arise when studying compositions of fuzzy relations: Is it possible to represent the notions related to fuzzy sets and relations in a unified form? And if so, what are the benefits of such an approach?

In fact, many notions related to the theory of fuzzy sets and fuzzy relations can be expressed by means of logical connectives and the particular suitable choice of a composition enables to transform the given notion into its equivalent composition-based form. Hence, the answer for the first question is positive. The way of representation of composition-related notions has been briefly and informally sketched in Bělohlávek's book [4, Remark 6.16]. Indeed, the observation that finally leads to the apparatus exploiting the analogies between such notions systematically has appeared already in [7]. Recently, the soundness of this apparatus has been shown in [1] by means of formal interpretations.

The calculations with composition-related notions reduce to simple manipulation over a special language. Additionally, it is possible to obtain hundreds of (easy) theorems on fuzzy relations for free simply as consequences of several known properties of compositions. The unification of notions by means of compositions significantly simplifies proofs of properties of the composition-related notions and allows us to handle the above mentioned problems automatically by a computer; this presents one of the main benefits of this approach.

2 Fuzzy classes—basic notions

In [2], fuzzy class theory is developed as a theory in the multi-sorted first-order logic ŁΠ. However, it is obvious that the definitions of fuzzy class theory can be carried out in any fuzzy logic which extends, say, MTLΔ. For the sake of generality, we shall therefore prefer to work in fuzzy class theory over MTLΔ.

Let us recall that FCT is a theory with object variables (lowercase letters $x, y, \ldots$), class variables (uppercase letters $X, Y, \ldots$), variables for classes of classes (calligraphic letters $\mathcal{X}, \mathcal{Y}, \ldots$), etc. Tuples are denoted by $\langle x_1, \ldots, x_k \rangle$ or briefly $x_1 \ldots x_n$ (for all sorts of variables). The only primitive symbols of FCT are the binary membership predicates $\in$ between successive levels of the type hierarchy; instead of $x \in A$ we may write briefly Ax. For the axioms and details on multi-sorted first-order fuzzy logic see [2], where the necessary apparatus for the subsumption of sorts has been introduced.

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Strong (co-norm) disjunction will be denoted by $\veebar$, other connectives and quantifiers as usual. Empty conjunction is defined as 1. The following defined connectives will be useful:

$$\varphi = \psi \equiv_{\mathrm{df}} \Delta(\varphi \leftrightarrow \psi), \qquad \varphi \leq \psi \equiv_{\mathrm{df}} \Delta(\varphi \to \psi).$$

Definition 2.1 *Let $\varphi(p_1, \ldots, p_n)$ be a propositional formula.*

- *We define the n-ary class operation generated by φ as*

$$\mathrm{Op}_{\varphi}(X_1, \ldots, X_n) =_{\mathrm{df}} \{x \mid \varphi(x \in X_1, \ldots, x \in X_n)\}.$$

- *The n-ary uniform relation between $X_1, \ldots, X_n$ generated by φ is defined as*

$$\mathrm{Rel}^{\forall}_{\varphi}(X_1, \ldots, X_n) =_{\mathrm{df}} (\forall x)\varphi(x \in X_1, \ldots, x \in X_n).$$

- *The n-ary supremal relation between $X_1, \ldots, X_n$ generated by φ is defined as*

$$\mathrm{Rel}^{\exists}_{\varphi}(X_1, \ldots, X_n) =_{\mathrm{df}} (\exists x)\varphi(x \in X_1, \ldots, x \in X_n).$$

We will use the following important kinds of elementary class operations, properties, and relations:

Fuzzy class operations:

$$\begin{array}{rcll}
\emptyset & =_{\mathrm{df}} & \{x \mid 0\} & \text{empty class} \\
\mathrm{V} & =_{\mathrm{df}} & \{x \mid 1\} & \text{universal class} \\
\mathrm{V}^2 & =_{\mathrm{df}} & \{xy \mid 1\} & \text{total relation} \\
\bigcup \mathcal{A} & =_{\mathrm{df}} & \{x \mid (\exists A \in \mathcal{A})(x \in A)\} & \text{class union} \\
\bigcap \mathcal{A} & =_{\mathrm{df}} & \{x \mid (\forall A \in \mathcal{A})(x \in A)\} & \text{class intersection} \\
\mathrm{Pow}\, A & =_{\mathrm{df}} & \{X \mid X \subseteq A\} & \text{power class}
\end{array}$$

Fuzzy class properties and relations:

$$\begin{array}{rcll}
\mathrm{Hgt}(A) & \equiv_{\mathrm{df}} & (\exists x)Ax & \text{height} \\
A \subseteq B & \equiv_{\mathrm{df}} & (\forall x)(Ax \to Bx) & \text{inclusion} \\
A \sqsubseteq B & \equiv_{\mathrm{df}} & \Delta(A \subseteq B) & \text{strict inclusion} \\
A \parallel B & \equiv_{\mathrm{df}} & (\exists x)(Ax \mathbin{\&} Bx) & \text{compatibility}
\end{array}$$

We shall freely use all elementary theorems on these notions which follow from the metatheorems proved in [2], and thus can be checked by simple propositional calculations. In the same source, the wide overview of class operations, properties, and relations can be found.

3 Fuzzy classes and truth values as fuzzy relations

Fuzzy classes and truth values can be represented as fuzzy relations of a certain form, described below. This representation will allow us straightforwardly to apply the properties of various kinds of compositions of fuzzy relations to many more derived concepts which involve fuzzy classes and/or truth values.

The identification of fuzzy classes and truth values with certain fuzzy relations is carried out in a rigorous formal way by means of formal interpretations of fuzzy theories in FCT. For technical details on formal interpretations in FCT see [1].

Convention 3.1 *Let $\underline{0}$ be an arbitrary fixed element of V. (I.e., $\underline{0}$ is a constant denoting an atomic individual of the domain of discourse.) The fuzzy class $\{\underline{0}\}$ (i.e., the crisp singleton of $\underline{0}$) will be denoted by $\underline{1}$.*

Convention 3.2 *A fuzzy class $A \sqsubseteq \mathrm{V}$ will be identified with the fuzzy relation $A \times \underline{1} = \{\langle x, \underline{0}\rangle \mid x \in A\}$. When representing the fuzzy class A, the fuzzy relation $A \times \underline{1}$ will be written as $\boldsymbol{A}$ (the same letter in boldface).*

This identification is quite natural and well-known (the apparatus of FCT just extends it to infinite classes as well). If the universe of discourse is finite, consisting of elements $x_1, \ldots, x_n$, fuzzy relations can be represented by $(n \times n)$-matrices of truth values, $R = (Rx_i x_j)_{ij}$. Assume that $\underline{0}$ denotes the element x_1. The fuzzy class A is then identified with the relation

$$\boldsymbol{A} = \begin{pmatrix} A\underline{0} & 0 & \cdots & 0 \\ Ax_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Ax_n & 0 & \cdots & 0 \end{pmatrix}$$

which by the usual convention of linear algebra can be written as the (file) vector $n \times 1$, $\boldsymbol{A} = (A\underline{0}\ Ax_2\ \ldots\ Ax_n)^{\mathrm{T}}$.

A similar trick will allow us to represent truth values as certain relations. First observe that truth values can be internalized in FCT as subclasses of an arbitrary crisp singleton, e.g., $\underline{1}$, in the following way:

- The truth value of a formula φ will be represented by the class $\overline{\varphi} =_{\mathrm{df}} \{\underline{0} \mid \varphi\}$. Then by definition, $\overline{\varphi} \sqsubseteq \underline{1}$ and $\varphi \leftrightarrow (\underline{0} \in \overline{\varphi})$.
- Vice versa, every $\alpha \sqsubseteq \underline{1}$ represents the truth value of a formula—e.g., of $\underline{0} \in \alpha$, since $(\forall \alpha \sqsubseteq \underline{1})(\overline{\underline{0} \in \alpha} = \alpha)$.

The truth values are thus represented by subclasses of $\underline{1}$, where the truth value is the membership degree of $\underline{0}$ into the subclass. We shall therefore call the elements of $\operatorname{Ker} \operatorname{Pow} \underline{1}$ the *inner* (or *formal*) *truth values* and denote them by lowercase Greek letters $\alpha, \beta, \ldots$ The system $\operatorname{Ker} \operatorname{Pow} \underline{1}$ of formal truth values will for brevity's sake be denoted by L.

Now as the truth values are represented by special *fuzzy classes* (viz. subclasses of $\underline{1}$), they can be identified with certain fuzzy relations by Convention 3.2. Namely, an inner truth value $\alpha \sqsubseteq \underline{1}$ is identified with the fuzzy relation $\alpha \times \underline{1} = \{\langle \underline{0}, \underline{0}\rangle \mid \underline{0} \in \alpha\}$. By the same convention, when representing the truth value α, the fuzzy relation $\alpha \times \underline{1}$ can be denoted by boldface $\boldsymbol{\alpha}$.

Again, if the universe of discourse is finite and consists of the elements $\underline{0}, x_2, \ldots, x_n$, an inner truth value α is identified with the relation

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha\underline{0} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} \alpha\underline{0} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

which by usual conventions of linear algebra can be identified with the (1×1)-matrix (or *scalar*) $\big(\alpha\underline{0}\big)$. (Recall that $\alpha\underline{0}$, or $\underline{0} \in \alpha$, has the truth value that is represented by α.)

Convention 3.3 *We shall always assume that R, S, or T (possibly subscripted) denote fuzzy relations $\sqsubseteq \mathrm{V}^2$; A, B, or C (possibly subscripted) denote unary classes $\sqsubseteq \mathrm{V}$, and α, β, γ (possibly subscripted) denote inner truth values $\sqsubseteq \underline{1}$. (We can then abandon the distinction between A, α, etc. and $\boldsymbol{A}, \boldsymbol{\alpha}$, etc. in formulae.)*

4 Sup-T-compositions and derived notions

The usual definition of the composition of fuzzy relations R and S is as follows:

$$R \circ S \equiv_{\mathrm{df}} \{xy \mid (\exists z)(Rxz \,\&\, Szy)\}.$$

Notice that the defining formula is the same as the defining formula of the relational composition in classical mathematics, the fuzziness being introduced only by the semantics of the logical symbols $\exists$ and &. This makes it the "default" definition of fuzzy relational compositions according to the methodology of [3].

As already mentioned in the Introduction, the method of transferring the results on relational compositions to related notions like images or preimages has already been suggested in [4, Remark 6.16]. In our formal setting we can exploit the method systematically:

There are three variables in the definition of sup-T-composition and each of them can be replaced by the dummy value $\underline{0}$; this yields eight relational operations derived from sup-T-composition of fuzzy relations: they are summarized in Table 1.

	$\{xy \mid (\exists z)(Rxz \mathbin{\&} Szy)\}$	$=$	$R \circ S$	...	*composition*	$R \circ S$
$x = \underline{0}$	$\{\underline{0}y \mid (\exists z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \mathbin{\&} Rzy)\}$	$=$	$R^{\mathrm{T}} \circ \boldsymbol{A}$	...	*image*	$R''A$
$y = \underline{0}$	$\{x\underline{0} \mid (\exists z)(Rxz \mathbin{\&} \boldsymbol{A}z\underline{0})\}$	$=$	$R \circ \boldsymbol{A}$	...	*pre-image*	$R^{\leftarrow}A$
$z = \underline{0}$	$\{xy \mid (\exists \underline{0})(\boldsymbol{A}x\underline{0} \mathbin{\&} \boldsymbol{B}^{\mathrm{T}}\underline{0}y)\}$	$=$	$\boldsymbol{A} \circ \boldsymbol{B}^{\mathrm{T}}$	...	*Cartesian product*	$A \times B$
$x, y = \underline{0}$	$\{\underline{00} \mid (\exists z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \mathbin{\&} \boldsymbol{B}z\underline{0})\}$	$=$	$\boldsymbol{A}^{\mathrm{T}} \circ \boldsymbol{B}$	...	*compatibility*	$A \mathbin{\overline{\Vert}} B$
$x, z = \underline{0}$	$\{\underline{0}y \mid (\exists \underline{0})(\boldsymbol{\alpha}^{\mathrm{T}}\underline{00} \mathbin{\&} \boldsymbol{A}^{\mathrm{T}}\underline{0}y)\}$	$=$	$\boldsymbol{A} \circ \boldsymbol{\alpha}$	...	*α-resize*	αA
$y, z = \underline{0}$	$\{x\underline{0} \mid (\exists \underline{0})(\boldsymbol{A}x\underline{0} \mathbin{\&} \boldsymbol{\alpha}\underline{00})\}$	$=$	$\boldsymbol{A} \circ \boldsymbol{\alpha}$	...	*α-resize*	αA
$x, y, z = \underline{0}$	$\{\underline{00} \mid (\exists \underline{0})(\boldsymbol{\alpha}\underline{00} \mathbin{\&} \boldsymbol{\beta}\underline{00})\}$	$=$	$\boldsymbol{\alpha} \circ \boldsymbol{\beta}$	...	*conjunction*	$\alpha \mathbin{\overline{\&}} \beta$

Table 1: Operations derived from the sup-T-composition

Besides the operations listed in Table 1, further important relational operations are definable from compositions—e.g., by taking the universal class V for an argument in some of the derived notions. Some of such derived notions are listed in Table 2:

domain	$\mathrm{Dom}\, R$	$=$	$R^{\leftarrow}\mathrm{V}$	...	$R \circ \mathbf{V}$
range	$\mathrm{Rng}\, R$	$=$	$R''\mathrm{V}$	...	$R^{\mathrm{T}} \circ \mathbf{V}$
height	$\overline{\mathrm{Hgt}}\, A$	$=$	$A \mathbin{\overline{\Vert}} \mathrm{V} = \mathrm{V} \mathbin{\overline{\Vert}} A$	...	$\boldsymbol{A}^{\mathrm{T}} \circ \mathbf{V} = \mathbf{V}^{\mathrm{T}} \circ \boldsymbol{A}$

Table 2: Further operations derived from sup-T-compositions

The point of the reduction of the above notions to compositions is of course that the well-known properties of sup-T-compositions (see, e.g., [5]):

$$\begin{aligned}
\textit{Transposition}: &\quad (R \circ S)^{\mathrm{T}} = S^{\mathrm{T}} \circ R^{\mathrm{T}} \\
\textit{Associativity}: &\quad (R \circ S) \circ T = R \circ (S \circ T) \\
\textit{Monotony}: &\quad R_1 \subseteq R_2 \rightarrow R_1 \circ S \subseteq R_2 \circ S \\
\textit{Union}: &\quad \Big(\bigcup_{R \in \mathcal{A}} R\Big) \circ S = \bigcup_{R \in \mathcal{A}} (R \circ S) \\
\textit{Intersection}: &\quad \Big(\bigcap_{R \in \mathcal{A}} R\Big) \circ S \subseteq \bigcap_{R \in \mathcal{A}} (R \circ S)
\end{aligned}$$

automatically transfer to all of them. Thus we now get dozens of theorems on fuzzy relational operations entirely for free. Moreover, the associativity and transposition properties of sup-T-compositions plus properties of V, $^{\mathrm{T}}$ and $\times$ yield an enormous number of identities between expressions composed of the operations from Tables 1 and 2. As an example, let us show the following two identities

$$\begin{aligned}
(A \times B) \circ R &= A \circ B^{\mathrm{T}} \circ R = A \circ (R^{\mathrm{T}} \circ B)^{\mathrm{T}} &&= A \times (R''B), \\
A \mathbin{\Vert} \mathrm{Dom}\, R &= A^{\mathrm{T}} \circ R \circ \mathrm{V} = (R^{\mathrm{T}} \circ A)^{\mathrm{T}} \circ \mathrm{V} &&= \mathrm{Hgt}(R''A).
\end{aligned}$$

5 The equational calculus

The reduction of notions listed in Tables 1 and 2 to sup-T-compositions thus yields a simple method of proving identities between well-formed terms (i.e., terms that respect the required arities of arguments) composed of the operations $\circ, ^{\mathrm{T}}, '', ^{\leftarrow}, \times, \|, \mathrm{resize}, \&, \mathrm{Dom}, \mathrm{Rng}, \mathrm{Hgt}, \mathrm{V}$. As could be observed above, provable identities in this language can be derived equationally by only a few simple rules. These rules can therefore be viewed as axioms of an equational calculus for proving the identities between fuzzy relational operations. For the reference, we give it in a separate definition.

Definition 5.1 *The equational calculus* $\mathrm{Eqc}_\circ$ *for identities between well-formed terms in the typed language* $\mathcal{L}_\circ$ *consisting of the operations* $\circ, ^{\mathrm{T}}, '', ^{\leftarrow}, \times, \|, \mathit{resize}, \&, \mathrm{Dom}, \mathrm{Rng}, \mathrm{Hgt}, \mathrm{V}, \underline{1}$ *is given by the definitions of Tables 1 and 2 and the following rules:*

$$\begin{array}{llcllll} (a) & (X \circ Y)^{\mathrm{T}} & = & Y^{\mathrm{T}} \circ X^{\mathrm{T}} & (d) & X^{\mathrm{TT}} = X \\ (b) & \alpha^{\mathrm{T}} & = & \alpha & (e) & \mathrm{V}^{\mathrm{T}} \circ \mathrm{V} = \underline{1} \\ (c) & Z \circ \underline{1} & = & Z & & \end{array}$$

for α a scalar, Z a scalar or file-vector, X arbitrary, and Y of a compatible type for composing with X.

The rules of $\mathrm{Eqc}_\circ$ were chosen to be valid (i.e., provable in FCT) properties of the relational operations from $\mathcal{L}_\circ$. Thus $\mathrm{Eqc}_\circ$ is a *sound* calculus of fuzzy relational operations, i.e., all identities provable in $\mathrm{Eqc}_\circ$ are provable laws of FCT (modulo the representation of fuzzy classes and inner truth values by the corresponding fuzzy relations): for any well-formed terms τ_1, τ_2 over $\mathcal{L}_\circ$ (which are then also well-formed terms of FCT), if $\mathrm{Eqc}_\circ \vdash \tau_1 = \tau_2$, then $\mathrm{FCT} \vdash \tau_1 = \tau_2$. It is an open question whether the rules (a)–(e) are exhaustive, i.e., whether $\mathrm{Eqc}_\circ$ is also *complete* as regards the identities expressible in the language $\mathcal{L}_\circ$.

Remark: The calculations in $\mathrm{Eqc}_\circ$ are simple enough that they can be automated by a computer. A decision procedure for the derivability in the calculus can employ the fact that every term in the language of sup-T-operations can be translated to an expression consisting only of $\circ$, $^{\mathrm{T}}$, V, and variables, by the definitions from Tables 1 and 2. Every such expression can then be reduced to its "flat" form, i.e., a form in which $^{\mathrm{T}}$ is only applied to variables or V. Although the flat form of a term is never unique (as, e.g., $\alpha^{\mathrm{T}} = \alpha$ and $A^{\mathrm{T}} \circ \mathrm{V} = \mathrm{V}^{\mathrm{T}} \circ A$, and a flat term can always contain redundant sequences of $\mathrm{V}^{\mathrm{T}} \circ \mathrm{V}$), establishing equality of flat terms can be done effectively in a bounded number of steps.

6 Conclusions

The apparatus of fuzzy class theory employed here just extends the usual correspondence between fuzzy relations, sets, and truth values on the one hand and matrices, vectors, and scalars of truth values on the other hand, to arbitrary (not only finite) fuzzy relations and classes, and provides a uniform way of formal handling thereof. In this contribution, only the particular case study of sup-T-compositions has been elaborated. But the reduction of fuzzy classes and truth values to fuzzy relations allows us to extend the apparatus of various products (or generalized compositions e.g. [6]) of fuzzy relations to fuzzy classes and truth values, apply the results on compositions to a rich variety of derived notions, and get the proofs of their properties for free.

Additionally, the equational calculus $\mathrm{Eqc}_\circ$ consisting of the simple rules transforming between equivalent flat terms will be much more effective for automated proving of fuzzy relational identities expressible in $\mathcal{L}_\circ$ than automated proving of the same theorems directly in FCT (or even in semantical models of membership functions).

References

[1] Běhounek L., Relative interpretations over first-order fuzzy logics. To appear in *Doktorandský den,* Matfyzpress, Prague 2006.

[2] Běhounek L., Cintula P., Fuzzy class theory. *Fuzzy Sets and Systems* 154: 34–55, 2005.

[3] Běhounek L., Cintula P., From Fuzzy Logic to Fuzzy Mathematics: A Methodological Manifesto. *Fuzzy Sets and Systems* 157: 642–646, 2006.

[4] Bělohlávek R., *Fuzzy Relational Systems: Foundations and Principles.* Kluwer/Plenum, New York, 2002.

[5] De Baets B., Kerre E., Fuzzy relational compositions. *Fuzzy Sets and Systems* 60: 109–120, 1993.

[6] Bandler W., Kohout L.J., Fuzzy relational products and fuzzy implication operators. In *Int. Workshop on Fuzzy Reasoning and Applications,* Queen Mary College, University of London, 1978.

[7] Zadeh L.A., Similarity relations and fuzzy orderings. *Information Sciences* 3: 177–200, 1971.