# Features of Mathematical Theories in Formal Fuzzy Logic

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**Abstract.** A genuine fuzzy approach to fuzzy mathematics consists in constructing axiomatic theories over suitable systems of formal fuzzy logic. The features of formal fuzzy logics (esp. the invalidity of the law of contraction) entail certain differences in form between theories axiomatized in fuzzy logic and usual theories known from classical mathematics. This paper summarizes the most important differences and presents guidelines for constructing new theories, defining new notions, and proving new theorems in formal fuzzy mathematics.

**Key words:** Formal fuzzy logic, axiomatic theories, the law of contraction, fuzzy mathematics, graded properties

## 1 Introduction

As argued in [1], a genuine fuzzy approach to fuzzy mathematics consists in constructing axiomatic theories over suitable systems of formal fuzzy logic. There are numerous reasons supporting this thesis, let us name just a few: under this approach, there is a strong analogy with classical mathematics; most notions are naturally graded; the connection with real-valued analysis is loosened; a consistent methodology for introducing fuzzy counterparts of crisp notions is provided; hidden crispness can easily be avoided; etc.

The features of formal fuzzy logics (esp. the invalidity in general of the contraction law, see Sect. 2) enforce a specific approach to building axiomatic theories over such logics. Some of the usual practices of classical as well as traditional fuzzy mathematics cease to be useful and need to be adjusted when working in formal fuzzy logic. Examples of such traditional practices are the placement of preconditions in definitions rather than theorems, defining compound notions as conjunctions of several conditions, etc. Furthermore, the properties of the underlying logic entail certain differences in the form as well as strength of theorems

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that can be proved in theories over formal fuzzy logic as compared to theorems of traditional fuzzy mathematics. In particular, they allow studying graded notions and properties, mostly overlooked by traditional fuzzy mathematics (where usually just the predicate $\in$ is fuzzified).

This paper summarizes the most important differences between theories of classical mathematics or traditional fuzzy mathematics (e.g., [2–4]) on the one hand and those axiomatized in formal fuzzy logic (e.g., [5–7]) on the other hand, and presents guidelines for introducing new defined notions and formulating meaningful theorems in formal fuzzy mathematics.

## 2 Features of Formal Fuzzy Logics

Many systems of formal fuzzy logic emerged in the last decades. Here we survey their common features relevant to our needs.

Let us start at the propositional level. Formal fuzzy logics share the syntax of classical Boolean logic, only there are usually two different conjunctions—the *residuated* (strong) one & and the *minimum* (weak) one $\wedge$. Although there is a bunch of formal fuzzy logics described in the literature, the deductively well-behaved ones [9] contain some common basic propositional laws (axioms). The shared axioms form the logic MTL [10], the logic of left-continuous t-norms (i.e., the set of truth values is the interval $[0, 1]$, a left-continuous t-norm interprets &, and its residuum interprets implication). In order to enhance its expressive power, one usually adds one more propositional unary connective $\triangle$ with the standard semantics $\triangle x = 1$ if $x = 1$ and $\triangle x = 0$ otherwise. The logic MTL with the connective $\triangle$ will be denoted by $\mathrm{MTL}_\triangle$ further on.

It can be argued [9] that formal fuzzy logics suitable for axiomatizing mathematical theories extend the logic $\mathrm{MTL}_\triangle$; following [9], we shall call them *deductive* fuzzy logics. The most prominent examples of such logics are Łukasiewicz logic, Hájek's BL, involutive MTL, product logic, the logic ŁΠ, etc. (all of them with $\triangle$). The main distinction between classical logic and deductive fuzzy logics is the invalidity in general of the law of contraction $(\varphi \mathbin{\&} \varphi) \leftrightarrow \varphi$ in the latter. Non-contractivity has a huge impact on the axiomatic mathematical theories over deductive fuzzy logics: see Sect. 6–8 for details.

Propositional fuzzy logic is not expressive enough to support mathematical theories; at least first-order fuzzy logic is needed for fuzzy mathematics. For a recent survey of first-order fuzzy logics see [11]; for higher-order fuzzy logics see [12, 13, 8]. Unless stated otherwise, our background fuzzy logic is supposed to be the first-order logic $\mathrm{MTL}_\triangle$.[1]

[1] First-order $\mathrm{MTL}_\triangle$ retains the completeness w.r.t. semantics based on left-continuous t-norms (although this is, in general, not the case of stronger fuzzy logics like BL or Łukasiewicz). This allows us to transfer some results of traditional fuzzy mathematics proven for all left-continuous t-norms automatically into $\mathrm{MTL}_\triangle$. However, these results are usually much weaker than those achievable directly in the axiomatic theory (see Sect. 3 and 5 for more details).

Axiomatic mathematical theories are given by a set of formulae, called the axioms of the theory. The theorems of a theory are proved by formal deductions from its axioms by the deduction rules of the underlying formal fuzzy logic. For details on axiomatic theories over fuzzy logics see [14].

Further on, we shall adopt the following useful conventions for formulae of formal fuzzy logic:

*Convention 1.* In order to save some parentheses, we assume that $\rightarrow$ and $\leftrightarrow$ have less priority than other binary connectives, and that unary connectives have the highest priority. A chain of implications $\varphi_1 \rightarrow \varphi_2, \ldots, \varphi_{n-1} \rightarrow \varphi_n$ will be written as $\varphi_1 \longrightarrow \varphi_2 \longrightarrow \cdots \longrightarrow \varphi_n$, and similarly for the equivalence connective.

## 3 Graded Notions

Traditional fuzzy set theory fuzzifies (at least) the membership predicate: the membership degree $Ax$ of an element $x$ in a set $A$ can take intermediate values between 1 and 0. Fuzzy sets are identified with their membership functions $x \mapsto Ax$; their properties thus can be studied by means of usual methods of classical mathematics (which uses the laws of classical Boolean logic for reasoning), since membership functions are after all crisp objects of real-valued or lattice-valued analysis. Traditional properties of fuzzy sets are therefore *bivalent*: they either hold or not (e.g., a fuzzy relation either is or is not reflexive).

Only some properties of fuzzy sets are sometimes considered *graded* (i.e., with truth values in $[0, 1]$ or a lattice $L$) rather than bivalent (i.e., either true or false): most often the graded inclusion predicate $A \subseteq_{\mathrm{gr}} B$ defined as $\inf_x(Ax \Rightarrow Bx)$, where $\Rightarrow$ is a suitable fuzzy implication (compare it with the non-graded inclusion of fuzzy sets, defined by the condition $Ax \leq Bx$ for all $x$).

Formal fuzzy mathematics, on the other hand, uses formal fuzzy logic rather than classical Boolean logic for reasoning about fuzzy sets or other fuzzy notions, and therefore *all* formulae take truth values in $L$; thus *all* defined notions and *all* statements in general are graded and can be just partially true (unless they are defined as provably crisp).

Consequently, even such properties of fuzzy relations as reflexivity, which in traditional fuzzy mathematics is usually defined as bivalent (by requiring that $Rxx = 1$ for all $x$), are in formal fuzzy logic graded (defined as the truth value in $L$ of the formula $(\forall x)Rxx$, i.e., $\inf_x Rxx$). In principle, all properties in formal fuzzy logic are of a similar kind as the property of *height* of a fuzzy set, which even in traditional fuzzy mathematics naturally takes values in $L$.

Graded properties of fuzzy relations have for the first time been systematically studied in Gottwald's monograph [15], and more recently elaborated in Gottwald's [16] and Bělohlávek's [17]. Graded notions also have a long tradition in fuzzy topology, see e.g. [18]. The graded approach is important for several reasons. First, graded notions generalize the traditional (non-graded) ones, as the latter are definable (by means of $\triangle$) in terms of the former, but not vice versa. Second, graded notions are more informative—they allow inferring relevant information even when traditional notions are simply false (see Sect. 5).

Third, graded notions take the idea of fuzziness seriously, as there is no reason to assume that properties of fuzzy sets should only be crisp. Moreover, graded notions can easily be handled within the framework of formal fuzzy logic, so their gradedness does not present too much additional difficulty.

## 4 Natural Fuzzification of Classical Notions

One of the main motivations of formal fuzzy logic is the generalization of classical logic to non-crisp predicates: thus it is natural to fuzzify classical mathematical notions just by re-interpreting them in a suitable formal fuzzy logic. This methodology has been foreshadowed already in Höhle's 1987 paper [19, §5]:

> "It is the opinion of the author that from a mathematical viewpoint the important feature of fuzzy set theory is the replacement of the two-valued logic by a multiple-valued logic. [...I]t is now clear how we can find for every mathematical notion its 'fuzzy counterpart'. Since every mathematical notion can be written as a formula in a formal language, we have only to internalize, i.e. to interpret these expressions by the given multiple-valued logic."

Much later the principle was formalized in [12, §7], and proposed as an important guideline for formal fuzzy mathematics in [1].

Nevertheless, although an important guideline, the method cannot be applied mechanically, as some classically equivalent definitions may no longer be equivalent in the (weaker than classical) fuzzy logic. In some cases, one can select the most suitable version of the definition, by the criteria of fruitfulness, applicability, and the practice of traditional fuzzy mathematics. In other cases, a notion of classical mathematics splits into several meaningful fuzzy notions. This can be exemplified by the notion of equality of fuzzy sets. Besides the crisp identity $=$ of fuzzy sets, at least two graded notions of natural fuzzy equality are defined and used in the literature (e.g., the first one is used in [17] and the second one in [16]):

$$A \approx B \equiv_{\mathrm{df}} (\forall x)[(Ax \to Bx) \mathbin{\&} (Bx \to Ax)] \tag{1}$$

$$A \approxeq B \equiv_{\mathrm{df}} (\forall x)(Ax \to Bx) \mathbin{\&} (\forall x)(Bx \to Ax) \tag{2}$$

These notions are not equivalent (even in rather strong fuzzy logics, e.g., Łukasiewicz), as shown by the following counter-example:

*Example 1.* Let $A, B$ be interpreted in a model over the standard MV-algebra (see [14]) by the following assignment of truth values: $Aa = Bb = 1$ and $Ab = Ba = 0.5$ for some individuals $a$ and $b$, and $Ax = Bx = 0$ otherwise. Then the truth value of $A \approx B$ is 0.5, while the truth value of $A \approxeq B$ is 0.

Notice that in traditional fuzzy mathematics these two notions of graded equality coincide, since $\triangle(A \approx B) \longleftrightarrow \triangle(A \approxeq B) \longleftrightarrow A = B$ (see Prop. 2 in Sect. 8).

## 5 Theorems in the Form of Provable Implications

The general gradedness of all notions in formal fuzzy logic allows proving more general theorems that are not available for non-graded notions in traditional fuzzy mathematics. A typical non-graded theorem of traditional fuzzy mathematics has the following form:

> *If* some (non-graded) assumption is true (i.e., fully true),
> *then* some (non-graded) conclusion is (fully) true.

With graded notions we can formulate and prove much stronger theorems of the following form:

> *The more* some (graded) assumption is true (even if partially),
> *the more* some (graded) conclusion is true (i.e., at least as true as the assumption).

The latter can be expressed in formal fuzzy logic by means of implication $\varphi \to \psi$, where $\varphi$ is the formula which expresses the assumption and $\psi$ is the formula which expresses the conclusion. In deductive fuzzy logics, if $\varphi \to \psi$ is provable, then the truth value of $\psi$ is at least as large as the truth value of $\varphi$ in any model. Provable implications thus express exactly the graded theorems of the above form. Since the full truth of $\chi$ is expressed by $\triangle\chi$, the former non-graded theorem of traditional fuzzy mathematics is expressed by the formula $\triangle\varphi \to \triangle\psi$. The graded theorem $\varphi \to \psi$ is generally stronger than the non-graded theorem $\triangle\varphi \to \triangle\psi$, since the latter is an immediate consequence of the former in $\mathrm{MTL}_\triangle$, but not vice versa.

*Example 2.* If we set $Ixy = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ otherwise, then:} \end{cases}$

- Traditional fuzzy mathematics proves that if a fuzzy relation $R$ is reflexive (in the traditional sense), then $I$ is a fuzzy subset of $R$; i.e., if $Rxx = 1$ for each $x$, then $Ixy \le Rxy$ for each $x, y$.
- In formal fuzzy logic we can easily prove that the more a fuzzy relation $R$ is reflexive (in the graded sense), the more $I$ is a fuzzy subset of $R$; in symbols, $(\forall x)Rxx \to (\forall xy)(Ixy \to Rxy)$. Thus for any left-continuous t-norm $T$ we get $\inf_x Rxx \le \inf_{x,y} \overrightarrow{T}(Ixy, Rxy)$.

Notice that the latter result is indeed more general than the former one: if $R$ is 0.999-reflexive, the traditional theorem asserts nothing (as $R$ is *not* reflexive in the traditional sense), while the graded theorem ensures that $I$ is a fuzzy subset of $R$ at least to degree 0.999. (Much more complex examples of this kind can be found in [5].)

By the above considerations, it is preferable to prove theorems in the form of implication $\varphi \to \psi$, rather than traditional non-graded theorems, which in formal fuzzy logic can be formalized as $\triangle\varphi \to \triangle\psi$.

## 6 Exponents

As stressed in Sect. 2, the law of contraction $(\varphi \mathbin{\&} \varphi) \leftrightarrow \varphi$ is not generally valid in deductive fuzzy logics. Therefore, repeated occurrences of a premise $\varphi_i$ in a theorem of the form

$$\varphi_1 \mathbin{\&} \ldots \mathbin{\&} \varphi_n \to \psi \tag{3}$$

cannot be contracted into a single occurrence, as usual in classical mathematics. For convenience, the $k$ occurrences of $\varphi_i$ in (3) can be written as $\varphi_i^k$. Thus a typical form of a graded theorem is actually

$$\varphi_1^{k_1} \mathbin{\&} \ldots \mathbin{\&} \varphi_n^{k_n} \to \psi \tag{4}$$

Semantically, since the truth value of $\varphi \mathbin{\&} \varphi$ is in general smaller than $\varphi$ in usual fuzzy logics, the larger the exponent $k_i$ in (4) is, the truer $\varphi_i$ must be to ensure a large truth degree of the conclusion $\psi$. In other words, the conclusion of a theorem depends more on the truth degree of the premises with larger exponents than on those with smaller exponents.

Syntactically, the exponent $k_i$ in a theorem of the form (4) expresses how many times the premise $\varphi_i$ was used in an MTL-proof of $\psi$. This can be seen from the proof of the Local Deduction Theorem for propositional MTL (see [20]), or from the following proposition which justifies proving a conjunction by proving the conjuncts separately:

**Proposition 1.** *Propositional fuzzy logic* MTL *proves (see [10]):*

$$[(\varphi_1 \to \psi_1) \mathbin{\&} (\varphi_2 \to \psi_2)] \to [(\varphi_1 \mathbin{\&} \varphi_2) \to (\psi_1 \mathbin{\&} \psi_2)] \tag{5}$$

$$[(\varphi \to \psi_1) \mathbin{\&} (\varphi \to \psi_2)] \to [\varphi \to (\psi_1 \wedge \psi_2)] \tag{6}$$

Thus if we can prove

$$\varphi_1^{k_1} \mathbin{\&} \ldots \mathbin{\&} \varphi_n^{k_n} \to \psi_1 \quad \text{and} \quad \varphi_1^{l_1} \mathbin{\&} \ldots \mathbin{\&} \varphi_n^{l_n} \to \psi_2 \tag{7}$$

then we also have

$$\varphi_1^{k_1+l_1} \mathbin{\&} \ldots \mathbin{\&} \varphi_n^{k_n+l_n} \to \psi_1 \mathbin{\&} \psi_2 \tag{8}$$

$$\varphi_1^{\max(k_1,l_1)} \mathbin{\&} \ldots \mathbin{\&} \varphi_n^{\max(k_n,l_n)} \to \psi_1 \wedge \psi_2 \tag{9}$$

Notice two different ways of "counting the premises" based on whether we prove conjunction or min-conjunction of conclusions.

Since $\varphi^k \to \psi$ is weaker for larger $k$, one should actually find a counter-example against $\varphi^{k-1} \to \psi$ whenever one proves a theorem of the form $\varphi^k \to \psi$, to show that it cannot be strengthened. This may, however, be quite difficult for more complex theorems. Also if the exponents in a theorem grow too large, it may in some cases be preferable to weaken the theorem and use $\triangle\varphi$ as a premise instead of $\varphi^k$ (for $k \gg 0$).

## 7 Preconditions and Compound Notions

The fact that assumptions get variable exponents in theorems leads to two important guidelines for defining new notions in formal fuzzy mathematics.

In classical mathematics, definitions often have preconditions under which the defined notions are meaningful (e.g., "let $R$ be an ordering"). In formal fuzzy mathematics, such preconditions are in general fuzzy (the notion of ordering is graded). In proofs of graded theorems, such preconditions will be used various numbers of times, and so they will get various exponents.

A notion defined with a fuzzy precondition is therefore of little interest, since only such graded properties are provable about the notion that use the precondition at most *once*; more complex properties will need the precondition several times. Thus it is better to state the definition of the notion without the precondition, and add the precondition with the required multiplicity in a theorem of the form (4). Only such preconditions $\varphi$ can meaningfully be required in definitions that provably satisfy $(\varphi \mathbin{\&} \varphi) \leftrightarrow \varphi$ and so they do not acquire differing exponents in theorems. (In particular, *crisp* preconditions satisfy the latter and therefore can meaningfully be used in definitions.)

A similar effect of variable exponents can be seen in notions defined as conjunctions of two or more conditions. We exemplify the effect on the notions of fuzzy preordering and similarity.

*Example 3.* In traditional fuzzy mathematics we say that a fuzzy relation $R$ is a preordering iff $R$ is reflexive and transitive (where $R$ is transitive iff $Rxy * Ryz \leq Rxz$ for all $x, y, z$ and reflexive iff $Rxx = 1$ for all $x$); it is a similarity iff it is reflexive, transitive, and symmetric (where $R$ is symmetric iff $Rxy \leq Ryx$ for all $x, y$). In formal fuzzy logic, *graded* reflexivity, symmetry, and transitivity are defined by the following formulae:

$$\mathrm{Refl}\, R \equiv_{\mathrm{df}} (\forall x) Rxx \tag{10}$$

$$\mathrm{Sym}\, R \equiv_{\mathrm{df}} (\forall xy)(Rxy \to Ryx) \tag{11}$$

$$\mathrm{Trans}\, R \equiv_{\mathrm{df}} (\forall xyz)(Rxy \mathbin{\&} Ryz \to Rxz) \tag{12}$$

The traditional notions of preordering and similarity are then expressed by the formulae $\triangle \mathrm{Refl}\, R \mathbin{\&} \triangle \mathrm{Trans}\, R$ and $\triangle \mathrm{Refl}\, R \mathbin{\&} \triangle \mathrm{Sym}\, R \mathbin{\&} \triangle \mathrm{Trans}\, R$, respectively. The definition of *graded* preordering or similarity first needs to distinguish which conjunction is used between the conjuncts $\mathrm{Refl}\, R, \mathrm{Sym}\, R, \mathrm{Trans}\, R$. (Notice that in the traditional definition it is immaterial which one is used, since both conjunctions are 1-true under the same conditions.) The default choice is the strong conjunction &, since it allows using *all* conjuncts in proofs, while $\wedge$ only allows using any *one* of them (see [8]). Nevertheless, the definitions

$$\mathrm{Preord}\, R \equiv_{\mathrm{df}} \mathrm{Refl}\, R \mathbin{\&} \mathrm{Trans}\, R \tag{13}$$

$$\mathrm{Sim}\, R \equiv_{\mathrm{df}} \mathrm{Refl}\, R \mathbin{\&} \mathrm{Sym}\, R \mathbin{\&} \mathrm{Trans}\, R \tag{14}$$

still allow using each of the conjuncts just *once* in the proofs. However (cf. Sect. 6), the assumptions $\mathrm{Refl}\, R$, $\mathrm{Sym}\, R$, or $\mathrm{Trans}\, R$ are needed variable times in proofs of various theorems, and thus get variable exponents, independent of each other. Thus, rather than defining preorders and similarities by (13)–(14), it is more meaningful to define parameterized notions of $(r,t)$-preorders and $(r,s,t)$-similarities as follows:

$$\mathrm{Preord}^{r,t}\, R \equiv_{\mathrm{df}} \mathrm{Refl}^{r}\, R \;\&\; \mathrm{Trans}^{t}\, R \tag{15}$$

$$\mathrm{Sim}^{r,s,t}\, R \equiv_{\mathrm{df}} \mathrm{Refl}^{r}\, R \;\&\; \mathrm{Sym}^{s}\, R \;\&\; \mathrm{Trans}^{t}\, R \tag{16}$$

Typical graded theorems on fuzzy preorders or similarities then have the form $\mathrm{Preord}^{r,t}\, R \to \varphi$ resp. $\mathrm{Sim}^{r,s,t}\, R \to \varphi$ (for some $r,s,t$), and thus they are actually theorems on $(r,t)$-preorders and $(r,s,t)$-similarities. Recall from Sect. 6 that the parameters measure the strictness of requiring a large truth value of the respective conjunct; thus $(2,5)$-preorders are more sensitive to imperfections in transitivity than in reflexivity, while $(10,1)$-preorders are much more sensitive to flaws in reflexivity than transitivity.

## 8 Equivalences and Bounds

Many theorems of traditional fuzzy logic have the form of equivalence between two conditions, which in formal fuzzy logic is expressed by a formula of the form $\triangle\varphi \leftrightarrow \triangle\psi$. The graded version of such a theorem, $\varphi \leftrightarrow \psi$, is sometimes provable in formal fuzzy logic; if so, it expresses the fact that the truth degree of $\varphi$ equals the truth degree of $\psi$. (Observe that again the traditional non-graded version of the theorem, which expresses only the fact that $\varphi$ is 1-true iff $\psi$ is 1-true, follows immediately from the graded version.)

Often, however, the graded version of a theorem $\triangle\varphi \leftrightarrow \triangle\psi$ is more complicated than the simple equivalence $\varphi \leftrightarrow \psi$. It can be exemplified by the relationship between the two notions of graded equality (1)–(2) (for a proof, see [5]):

**Proposition 2.** *The following theorems are provable in first order* MTL*:*

1. $A \approx^2 B \longrightarrow A \approxeq B \longrightarrow A \approx B$
2. $\triangle(A \approx B) \longleftrightarrow \triangle(A \approxeq B) \longleftrightarrow A = B$

Observe that the first statement says that the truth value of $A \approxeq B$ is bounded by the truth values of $A \approx^2 B$ (a lower bound) and $A \approx B$ (an upper bound). In traditional non-graded fuzzy mathematics both notions coincide, since they are 1-true under the same conditions, as shown by the second statement of Prop. 2.

The situation that a theorem $\triangle\varphi \leftrightarrow \triangle\psi$ has a graded version of the form $\varphi^n \longrightarrow \psi^m \longrightarrow \varphi^k$ for some $n \geq m \geq k$ occurs regularly under some conditions:

**Theorem 1.** *Let $\varphi$ and $\psi$ be formulae of the first-order logic* MTL *(i.e., they contain no $\triangle$) such that $\triangle\varphi \leftrightarrow \triangle\psi$ is provable in a theory $T$ over (first-order)* $\mathrm{MTL}_\triangle$*. Then there exist $n,m$ such that $\varphi^n \to \psi$ and $\psi^m \to \varphi$ are provable in $T$.*

*Proof.* Follows directly from the $\triangle$–Deduction Theorem and Local Deduction Theorem for the first-order logic $\mathrm{MTL}_\triangle$ resp. MTL (see [11]). If $\varphi,\psi$ are not closed formulae (to which the Deduction Theorems apply), first replace free variables by new constant symbols, which is harmless for provability in $T$. □

**Corollary 1.** *Under the conditions of Th. 1, we get the following mutual estimates for the truth degrees of $\varphi$ and $\psi$ (for $m, n$ from Th. 1):*

$$\varphi^{m\cdot n} \longrightarrow \psi^m \longrightarrow \varphi \tag{17}$$

$$\psi^{m\cdot n} \longrightarrow \varphi^n \longrightarrow \psi \tag{18}$$

It is worth noting that graded theorems of this form have occurred in the traditional fuzzy literature, see e.g. [21, L.16].

We conclude this section by an illustrative example which can be viewed as a graded generalization of (a certain variant of) the well-known Valverde representation theorem for preorders (see [22] for its non-graded version).

**Proposition 3.** [23] *The following graded characterizations are provable in first-order* MTL*:*

$$\mathrm{Refl}\, R \leftrightarrow (\forall xy)[(\forall z)(Rzx \to Rzy) \to Rxy] \tag{19}$$

$$\mathrm{Trans}\, R \leftrightarrow (\forall xy)[Rxy \to (\forall z)(Rzx \to Rzy)] \tag{20}$$

Recall from [10] that the following implications are provable in first-order MTL:

$$((\forall u)(\psi \mathbin{\&} \chi))^2 \longrightarrow (\forall u)\psi \mathbin{\&} (\forall u)\chi \longrightarrow (\forall u)(\psi \mathbin{\&} \chi), \tag{21}$$

and it cannot be improved as the converse implication $(\forall u)(\psi \mathbin{\&} \chi) \to (\forall u)\psi \mathbin{\&} (\forall u)\chi$ does not generally hold in fuzzy logics. As $\mathrm{Preord}^{r,t}\, R \equiv_{\mathrm{df}} \mathrm{Refl}^r\, R \mathbin{\&} \mathrm{Trans}^t\, R$, we obtain just the following graded variant of Valverde representation:

**Corollary 2.** [5] *Define $\varphi(R)$ as $(\forall xy)[Rxy \leftrightarrow (\forall z)(Rzx \to Rzy)]$. Then*

$$\varphi^2(R) \longrightarrow \mathrm{Preord}^{1,1}\, R \longrightarrow \varphi(R), \tag{22}$$

*i.e., $\varphi^2(R)$ and $\varphi(R)$ give respectively the lower and upper bounds for the truth value of $\mathrm{Preord}^{1,1}\, R$. Considering only 1-truth of both conditions, we get a non-graded characterization $\triangle\, \mathrm{Preord}\, R \leftrightarrow \triangle\, \varphi(R)$.*

## 9 Conclusion

As can be seen from the previous sections, in that part of fuzzy mathematics that can be formalized in formal fuzzy logic the apparatus of the latter allows deriving more general (graded) theorems than traditional methods. In order to utilize the strength of the apparatus to the full extent, however, the guidelines sketched in this paper have to be observed, namely:

- Defining new notions graded (§3), by formulae analogical to definitions in classical mathematics (§4); parameterizing definitions of compound notions by (variable) exponents and giving preconditions with variable exponents in theorems rather than definitions (§7)
- Proving theorems in the form of fuzzy implication (§5) rather than crisp consequence of fully true premises, using the laws of formal fuzzy logic (§2) and counting the exponents of premises properly (§6)

This leads to stronger, even though sometimes more complicated (§8) theorems than traditional methods. Failing to respect these unusual features when building graded fuzzy theories would unnecessarily weaken the theorems obtained.

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