Relations in Fuzzy Class Theory: Initial Steps

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Abstract

This paper studies fuzzy relations in the graded framework of Fuzzy Class Theory (FCT). This includes (i) rephrasing existing work on graded properties of binary fuzzy relations in the framework of Fuzzy Class Theory and (ii) generalizing existing crisp results on fuzzy relations to the graded framework. Our particular aim is to demonstrate that Fuzzy Class Theory is a powerful and easy-to-use instrument for handling fuzzified properties of fuzzy relations. This paper does not rephrase the whole theory of (fuzzy) relations; instead, it provides an illustrative introduction showing some representative results, with a strong emphasis on fuzzy preorders and fuzzy equivalence relations.

Key words: Fuzzy Class Theory, fuzzy relation, fuzzy preorder, fuzzy equivalence relation, similarity, graded properties.
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1 Introduction

Fuzzy relations are of fundamental importance in almost all sub-fields of fuzzy logic and fuzzy set theory, including particularly fuzzy preference modeling, fuzzy mathematics, fuzzy inference, and many more. In the most general setting, fuzzy relations are mappings from the Cartesian product of non-empty domains $U_1 \times \cdots \times U_n$ (usually with $n \geq 2$) to the unit interval or a more general lattice of truth values L (see e.g. [38, 41, 42, 47, 51]). Clearly the motivation behind fuzzy relations is to allow more flexibility by admitting intermediate degrees of relationship [12, 36, 57, 59, 60, 69].

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An important class are the so-called *binary* fuzzy relations that are used to express graded relationships between two objects coming from the same domain. Technically, they are defined as $U \times U \to L$ mappings, where U is some non-empty set and L is again the lattice of truth values we consider. There are many important sub-classes, such as, fuzzy preorders [18, 67, 69], fuzzy orders [12, 15, 47, 69], and fuzzy equivalence relations [12,20,50,51,66,67,69]. Interestingly, however, the traditional characterizing properties of these important types of fuzzy relations, such as, reflexivity, symmetry, transitivity, and so forth, are defined in a strictly crisp way, i.e., as properties that either hold fully or do not hold at all. One may be tempted to argue that it is somewhat peculiar to fuzzify relations by allowing intermediate degrees of relationships, but, at the same time, to still enforce strictly crisp properties on fuzzy relations. This particularly implies that all results are effective only if some assumptions are fulfilled, but say nothing at all if the assumptions are not fulfilled (even if they are *almost* fulfilled).

To illustrate our point, let us shortly consider a toy example. It is common in the theory of fuzzy relations to call a fuzzy relation $R\colon U \times U \to [0,1]$ *reflexive* if $R(x,x) = 1$ holds for all $x \in U$. From the reflexivity of a fuzzy relation R, we can infer

$$R \sqsubseteq R \circ_* R,$$

where $\sqsubseteq$ is the traditional crisp inclusion of fuzzy sets or relations [68]

$$R_1 \sqsubseteq R_2 \text{ if and only if } R_1(x,y) \leq R_2(x,y) \text{ for all } x,y \in U,$$

and $R \circ_* R$ is the composition of R with itself (with respect to some triangular norm $*$), i.e.,

$$(R \circ_* R)(x,y) = \sup\{R(x,z) * R(z,y) \mid z \in U\}.$$

What, however, happens if a given fuzzy relation R is not reflexive, but *almost* reflexive? Let us consider $U = \{1,2,3\}$ and the fuzzy relation (in convenient matrix notation)

$$R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & a \end{pmatrix},$$

where $a \in [0,1]$. Using the Łukasiewicz t-norm $x *_{Ł} y = \max(0, x+y-1)$, routine calculations show that

$$R \circ_{Ł} R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & a' \end{pmatrix},$$

where $a' = \max(0, 2a-1)$. So we confirm that only if $a = 1$, we also have $a' = 1$, and only in this case $R \sqsubseteq R \circ_{Ł} R$ holds. What is also apparent, however, is the fact that, the closer the value a is to 1, the less R exceeds $R \circ_{Ł} R$. Actually, in this example, this degree is

$$a - a' = a - \max(0, 2a-1) = \min(a, 1-a).$$

For example, if $a = 0.99$, we obtain $a' = 0.98$, and R exceeds $R \circ_{Ł} R$ only by 0.01. So we see that, even if some assumptions are not fully satisfied, we may obtain some meaningful results. The classical theory of fuzzy relations, however, does not offer any concepts for handling this kind of "gradedness". We only know that the classical result is not applicable, since R is not reflexive.

It was actually S. Gottwald who first attempted to eliminate this eyesore by introducing what he called "graded properties of fuzzy relations" [39–41]. Let us shortly recall these ideas in the light of the above example. For instance, Gottwald defined the *degree of reflexivity* of a fuzzy relation R as

$$\mathrm{Refl}(R) = \inf\{R(x,x) \mid x \in U\}$$

and the *degree of inclusion* with respect to a left-continuous t-norm $*$ as

$$R_1 \subseteq_* R_2 = \inf\{R_1(x,y) \Rightarrow_* R_2(x,y) \mid x,y \in U\},$$

where $(x \Rightarrow_* y) = \sup\{u \in [0,1] \mid x * u \leq y\}$ is the *residual implication* of $*$. Then it is straightforward to prove the following result

$$\mathrm{Refl}(R) \leq (R \subseteq_* R \circ_* R) \tag{1.1}$$

which perfectly confirms the results that we obtained for the above example, as we have $\mathrm{Refl}(R) = a$ and (by $x \Rightarrow_{Ł} y = \min(1, 1 - x + y)$)

$$(R \subseteq_{Ł} R \circ_{Ł} R) = \min(1, 1 - a + a') = \min(1, 1 - a + 2a - 1) = a.$$

Even though these ideas seem obvious and meaningful, Gottwald's approach unfortunately found only little resonance (exceptions are, for instance, [12, 48]). What may be the reasons? In our humble opinion, the following facts may have contributed to the reluctance of the research community to adopt and advance Gottwald's ideas: although Gottwald's syntax is geared to classical mathematics for better readability, he is not using a full-fledged axiomatic framework and is not strictly separating syntax from semantics. As in our example above, he has to refer to the operations used (t-norms, etc.) explicitly. For this reason, proofs are complicated and difficult. Finally, the results that he obtains are already quite difficult to prove, but still too basic to provide solid argumentation in favor of a full-fledged graded theory of fuzzy relations.

This paper aims at reviving and advancing Gottwald's highly valuable ideas. To overcome the difficulties sketched above, we take a slightly different approach. We use the formal axiomatic framework of Fuzzy Class Theory (FCT), introduced in [4]. Fuzzy Class Theory is a powerful and expressive, yet easy-to-read and easy-to-handle, framework for fuzzy mathematics in which it is just natural to consider properties of fuzzy relations in a graded manner. In Fuzzy Class Theory, most notions are inspired by (and derived from) the corresponding notions of classical mathematics [5]; the syntax of Fuzzy Class Theory is close to the syntax of classical mathematical theories; and the proofs in Fuzzy Class Theory resemble the classical proofs of the corresponding classical theorems. Therefore, in FCT it is technically easier to handle graded properties of fuzzy relations than in Gottwald's previous works. Thus we are able to access deeper results than what was possible in Gottwald's framework.

This paper is organized as follows. In Section 2, we first highlight how to read results in FCT, as the language of Fuzzy Class Theory may be unusual for some readers. Section 3 is concerned with basic graded properties of fuzzy relations, which mainly means rephrasing existing results on graded properties of fuzzy relations in the frame of Fuzzy Class Theory. Section 4 deals with images under fuzzy relations in the graded framework, including closures and opening operators, whereas Section 5 deals with bounds, maxima, and suprema. Section 6 generalizes the classical representation theorems due to Valverde [67] to the graded framework. In Section 7, we finally generalize the well-known links between fuzzy equivalence relations and fuzzy partitions to the graded framework. Throughout the whole paper, we will highlight links between the graded approach presented here and the existing results available in the literature. Where possible and meaningful, we provide concrete non-trivial examples.

The aim of this paper is to demonstrate that Fuzzy Class Theory is a powerful and easy-to-use instrument for handling fuzzified properties of fuzzy relations. As this paper has the appellative sub-title "Initial Steps", we do not aim at rephrasing the whole theory of fuzzy relations (or the whole existing theory of crisp relations, which is even much larger). Instead, we attempt to provide a kind of illustrative kick-off by picking out some representative results, with a strong emphasis on the most important classes of binary fuzzy relations—fuzzy preorders and fuzzy equivalence relations.

2 Preliminaries

Fuzzy Class Theory aims at axiomatizing the notion of fuzzy set. A brief overview of FCT can be found in Appendix B, where also all standard defined predicates of the theory, freely used in the following sections, are introduced. For a detailed account of the theory we refer the reader to the original paper [4] or a freely available primer [6]. In the present section we only give a brief dictionary explaining how formulae of FCT can be translated to a more traditional language of fuzzy set theory, and highlight some peculiar features of FCT that play a role in formal reasoning about the graded properties of fuzzy relations.

2.1 A Brief Dictionary

We aim this paper at researchers in the theory and applications of fuzzy relations to attract their interest in graded theories of fuzzy relations. In the traditional theory of fuzzy relations, it is not usual to separate formal syntax from semantics as it is the case in FCT. So it may be difficult for some readers who are new to FCT to follow the results. Therefore, we would like to provide the readers with a dictionary that improves understanding of the results in this paper and that demonstrates how the results would translate to the traditional language of fuzzy relations.

FCT strictly distinguishes between its syntax and semantics. This feature has two important consequences:

- To keep the distinction (and also for certain metamathematical reasons, see [6, Subsection 1.1]), the objects of the formal theory are called *fuzzy classes* rather than fuzzy sets. The name *fuzzy set* is reserved for membership functions in the *models* of the theory (see Appendix B). Nevertheless (in virtue of the soundness of FCT with respect to its models), the theorems of FCT about fuzzy classes are always valid for fuzzy *sets* and fuzzy relations. Thus, whenever we speak of classes, the reader can always safely substitute usual fuzzy sets for our "classes".
- FCT screens off direct references to truth values; truth degrees belong to the *semantics* of FCT, rather than to its syntax (this ensures that FCT renders fuzzy sets as a primitive notion instead of modeling them by membership functions). Thus, there are *no variables for truth degrees* in the language of FCT. The degree to which an element x belongs to a fuzzy class A is expressed simply by the atomic formula $x \in A$ (which can alternatively be written in a more traditional way as Ax).

The algebraic structure behind the semantics of FCT are $\mathrm{MTL}_\triangle$-chains [33]. All results in this paper hold for all $\mathrm{MTL}_\triangle$-chains. As noted in Appendix A, if the domain of truth values is the unit interval $[0,1]$, $\mathrm{MTL}_\triangle$-chains are characterized as algebras

$$([0,1], *, \Rightarrow, \min, \max, 0, 1, \triangle),$$

where $*$ is a left-continuous t-norm, $\Rightarrow$ is its residual implication, and $\triangle$ is a unary operation defined as

$$\triangle x = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This means that we can translate the results to the language of fuzzy relations in the following way, where we may specify an arbitrary universe of discourse U, a left-continuous t-norm $*$, its residuum $\Rightarrow$:

FCT	Fuzzy Relations
object variable x	element $x \in U$
(fuzzy) class A	fuzzy set $A \in \mathcal{F}(U)$
(fuzzy) class of (fuzzy) classes $\mathcal{A}$	fuzzy set $\mathcal{A} \in \mathcal{F}(\mathcal{F}(U))$
unary predicate	fuzzy subset of U, $\mathcal{F}(U)$, $\mathcal{F}(\mathcal{F}(U))$, etc.
n-ary predicate	n-ary fuzzy relation on U^n, $(\mathcal{F}(U))^n$, etc.
strong conjunction &	t-norm $*$
implication $\rightarrow$	residual implication $\Rightarrow$
weak conjunction $\wedge$	minimum
weak disjunction $\vee$	maximum
negation $\neg$	the function $\neg x = (x \Rightarrow 0)$
equivalence $\leftrightarrow$	bi-residuum, i.e., $\min(x \Rightarrow y, y \Rightarrow x)$
universal quantifier $\forall$	infimum
existential quantifier $\exists$	supremum
predicate $=$	crisp identity
predicate $\in$	evaluation of membership function
class term $\{x \mid \varphi(x)\}$	fuzzy set defined as $Ax = \varphi(x)$, for all $x \in U$

Let us now shortly consider some examples of definitions and results. For instance, the truth degree of $A \subseteq B$ (defined in FCT by the formula $(\forall x)(x \in A \rightarrow x \in B)$, see Definition B.5) is in an MTL$_\triangle$-chain computed as

$$\inf\{Ax \Rightarrow Bx \mid x \in U\}$$

which is a well-known concept of fuzzy inclusion (see [1, 12, 15, 40] and many more). The degree of reflexivity $\mathrm{Refl}(R)$, defined in Section 3 as $(\forall x)Rxx$, is nothing else but

$$\inf\{Rxx \mid x \in U\}.$$

As another example (cf. Definition B.4), it is easy to see that $\mathrm{Ker}(A)$ for some fuzzy set A exactly gives the crisp set of all values $x \in U$ for which $Ax = 1$ holds. Analogously (see Definition B.5), $\mathrm{Norm}(A)$ evaluates to 1 if and only if there exists an $x \in U$ such that $Ax = 1$ holds and to 0 otherwise.

The question remains how the theorems in the following sections can be read in a graded way (although they do not necessarily look graded at first glance). In traditional (fuzzy) logic, a theorem is read as follows:

> *If* some (non-graded) assumption is true (i.e., fully true, since non-graded),
> *then* some (non-graded) conclusion is (fully) true.

If we can prove an implication in FCT, by soundness, this implication always holds to degree 1. Now take into account that, in all MTL$_\triangle$-chains (comprising all standard MTL$_\triangle$-chains), the following correspondence holds:

$$(x \Rightarrow y) = 1 \text{ if and only if } x \leq y.$$

So an implication that we can prove in FCT can be read as follows:

The more some (graded) assumption is true (even if partially),
the more some (graded) conclusion is true (i.e., at least as true as the assumption).

In other words, the truth degree of an assumption is a lower bound for the truth degree of the conclusion in provable implications.

Thus, for instance, the assertion (R13) of Theorem 3.5 easily translates into our motivating example (1.1).

Remark 2.1 To motivate and illustrate the results in this paper, we will use a significant number of examples. In order to make them compact and readable, we will, *in examples*, deviate from our principle to keep formulae separate from their semantics. Instead of mentioning models over some logics, we will simply say that we use some standard logic, for instance, standard Łukasiewicz logic (standing for the standard $\mathrm{MTL}_\triangle$-chain induced by the Łukasiewicz t-norm; analogously for other logics). In examples, we shall furthermore not distinguish between predicate symbols and the fuzzy sets or relations that model them. Instead of saying that a certain model of a fuzzy predicate R fulfills reflexivity to a degree of 0.8, we will simply write $\mathrm{Refl}(R) = 0.8$. This is not the cleanest way of writing it, but it is short and expressive, and it should always be clear to the reader what is meant.

2.2 Some Precautions

It can be observed that the defining formulae of most notions in FCT are exactly the same as the definitions of these properties for crisp relations in classical mathematics. This correlates with the motivation of fuzzy logic as generalization of classical logic to non-crisp predicates: classical mathematical notions are then fuzzified in a natural way just by interpreting the classical definitions in fuzzy logic. This methodology has been foreshadowed in [44, Section 5] by Höhle, much later formalized in [4, Section 7], and suggested as a general principle for formal fuzzy mathematics in [5].

Nevertheless, although such a translation of notions of classical mathematics into FCT is an important guideline, the method cannot be applied mechanically, as some classically equivalent definitions may no longer be equivalent in the logic $\mathrm{MTL}_\triangle$. In some cases, the most suitable version of the definition can be chosen; in other cases, a notion of classical mathematics splits into several meaningful notions in FCT. This can be exemplified by the notion of equality of fuzzy classes:

Besides the primitive crisp identity $=$ of fuzzy classes, at least two graded notions of natural fuzzy equality, $\approx$ and $\approxeq$, can be defined (see Definition B.5). Both of these notions have already appeared in the fuzzy literature. For instance Gottwald

[41] uses $\approxeq$ while Bělohlávek [12] uses $\approx$ for graded equality of fuzzy classes. The two notions are not equivalent in FCT, as the following counter-example demonstrates.

Example 2.2 Let us consider a two-element set $U = \{x, y\}$ and standard Łukasiewicz logic. Let us consider two fuzzy sets $A, B \in \mathcal{F}(U)$ defined as $Ax = By = 1$ and $Ay = Bx = 0.5$. Then the truth value of $A \approx B$ is 0.5, while the truth value of $A \approxeq B$ is 0.

Only the following relationships hold between these notions.

Theorem 2.3 *The following theorems are provable in* FCT*:*

(L1) $A \approx B \leftrightarrow (A \subseteq B \wedge B \subseteq A)$
(L2) $A \approx^2 B \longrightarrow A \approxeq B \longrightarrow A \approx B$
(L3) $\triangle(A \approx B) \longleftrightarrow \triangle(A \approxeq B) \longleftrightarrow A = B$

Proof. We give the proof of this lemma in full detail; proofs in the following sections will usually be more compressed and easy steps will be omitted.

(L1) By Definition B.5 and the rule of distribution of $\forall$ over $\wedge$ (which is provable in $\mathrm{MTL}_\triangle$), we have

$$\begin{aligned} A \approx B &\longleftrightarrow (\forall x)(Ax \leftrightarrow Bx) \longleftrightarrow (\forall x)((Ax \to Bx) \wedge (Bx \to Ax)) \\ &\longleftrightarrow (\forall x)(Ax \to Bx) \wedge (\forall x)(Bx \to Ax) \longleftrightarrow A \subseteq B \wedge B \subseteq A. \end{aligned}$$

(L2) We have the following:

$$\begin{aligned} A \approx^2 B &\longleftrightarrow (\forall x)(Ax \leftrightarrow Bx) \,\&\, (\forall x)(Ax \leftrightarrow Bx) \\ &\longrightarrow (\forall x)(Ax \to Bx) \,\&\, (\forall x)(Bx \to Ax) \\ &\longleftrightarrow A \subseteq B \,\&\, B \subseteq A \longleftrightarrow A \approxeq B \end{aligned}$$

Moreover, $A \subseteq B \,\&\, B \subseteq A \longrightarrow A \subseteq B \wedge B \subseteq A \longleftrightarrow A \approx B$ by (L1).

(L3) The first equivalence follows from (L2) by the rule of $\triangle$-necessitation (see Appendix A) and distribution of $\triangle$ over $\to$ and &, which is provable in propositional $\mathrm{MTL}_\triangle$. The second equivalence can be proved as

$$\triangle(A \approx B) \longleftrightarrow \triangle(\forall x)(Ax \leftrightarrow Bx) \longleftrightarrow (\forall x)\triangle(Ax \leftrightarrow Bx) \longleftrightarrow A = B$$

by the axiom of extensionality (see Definition B.1). □

Let us add some comments on the meaning of the previous theorem. By definition, the "strong" bi-inclusion $A \approxeq B$ is $A \subseteq B \,\&\, B \subseteq A$; compare it with "weak" bi-inclusion $A \approx B$, which by (L1) just uses weak conjunction $\wedge$ instead of &. Indeed, by the second implication of (L2), $\approxeq$ is stronger than $\approx$. Notice further that (L2) in fact says that the truth value of $A \approxeq B$ is bounded by the truth values of $A \approx^2 B$ (a lower bound) and $A \approx B$ (an upper bound). In traditional non-graded fuzzy mathematics both notions coincide, since they are fully true under the same conditions, as shown by (L3); however, under the graded approach they differ, since in graded fuzzy mathematics we do not require them to be true to degree 1. This relationship between two related, but non-equivalent notions is quite common in graded fuzzy mathematics and will be met several times in this paper. [1]

Finally, it should be pointed out that, unlike in classical Boolean logic, in fuzzy logic it does make a difference how many times an assumption is used to prove a certain conclusion. For instance, if we have to use an assumption φ twice to prove a conclusion ψ, this means

$$\varphi \to (\varphi \to \psi).$$

So finally, by the axiom (A5a) of $\mathrm{MTL}_\triangle$ (see Appendix A), we have proved $\varphi^2 \to \psi$, but it need not be possible to prove $\varphi \to \psi$. Such situations will occur frequently in this paper. For instance, Example 2.2 shows that $A \approx B \to A \approxeq B$ indeed does not hold in FCT, even though $A \approx^2 B \to A \approxeq B$ is provable by (L2).

The warnings listed above may appear as eyesores that somehow spoil the beauty and quality of FCT. Our opinion is, however, that exactly the opposite is the case. Otherwise, this paper could only reproduce and slightly generalize crisp results with analogous proofs, without creating really new results. However, due to the above features, FCT indeed allows to derive new, previously unknown results.

3 Basic Properties of Fuzzy Relations

As announced above, the first item on the agenda of this paper is to embed existing results on so-called graded properties of fuzzy relations into the framework of FCT. Such properties were introduced first by S. Gottwald in 1991 [39]. Later on, he extended this research in his 1993 book [40]; his more recent book [41] contains an up-to-date review of the topic. Properties of fuzzy relations are studied in the graded manner also in Bělohlávek's book [12]. The idea of graded properties of fuzzy relations had also been followed by Jacas and Recasens [48]. In this section, we closely follow the structure and philosophy of [41, Section 18.6].

Definition 3.1 In FCT, we define basic properties of fuzzy relations as follows:

[1] It occurs regularly under certain conditions in graded generalizations of non-graded theorems, see [7].

$$
\begin{array}{rcll}
\mathrm{Refl}(R) & \equiv_{\mathrm{df}} & (\forall x)Rxx & \text{reflexivity} \\
\mathrm{Irrefl}(R) & \equiv_{\mathrm{df}} & (\forall x)\neg Rxx & \text{irreflexivity} \\
\mathrm{Sym}(R) & \equiv_{\mathrm{df}} & (\forall x,y)(Rxy \to Ryx) & \text{symmetry} \\
\mathrm{Trans}(R) & \equiv_{\mathrm{df}} & (\forall x,y,z)(Rxy \,\&\, Ryz \to Rxz) & \text{transitivity} \\
\mathrm{AntiSym}_E(R) & \equiv_{\mathrm{df}} & (\forall x,y)(Rxy \,\&\, Ryx \to Exy) & E\text{-antisymmetry} \\
\mathrm{ASym}(R) & \equiv_{\mathrm{df}} & (\forall x,y)\neg(Rxy \,\&\, Ryx) & \text{asymmetry}
\end{array}
$$

Note that we slightly deviate from Gottwald in the definition of antisymmetry, which we generalize by defining it with respect to some relation E (usually a similarity). In other words, we follow the ideas of so-called similarity-based orderings which have turned out to be more suitable concepts of fuzzy orderings [15, 47]. Let us adopt the convention that the index E is dropped if $E = \mathrm{Id}$ (then it coincides with the concept of antisymmetry that Gottwald uses).

Obviously, all above properties except $\mathrm{AntiSym}_E$ remain unchanged if we replace R with its inverse relation R^{-1}. Hence, we can infer the following trivial correspondences:

$$
\begin{array}{ccc}
\mathrm{Refl}(R^{-1}) \leftrightarrow \mathrm{Refl}(R) & & \mathrm{Irrefl}(R^{-1}) \leftrightarrow \mathrm{Irrefl}(R) \\
\mathrm{Sym}(R^{-1}) \leftrightarrow \mathrm{Sym}(R) & & \mathrm{Asym}(R^{-1}) \leftrightarrow \mathrm{Asym}(R) \\
\mathrm{Trans}(R^{-1}) \leftrightarrow \mathrm{Trans}(R) & &
\end{array}
$$

Example 3.2 Let us start with a simple example to illustrate the concepts introduced above. Consider the domain $U = \{1, \dots, 6\}$ and the following fuzzy relation (for convenience, in matrix notation):

$$
P_1 = \begin{pmatrix}
1.0 & 1.0 & 0.5 & 0.4 & 0.3 & 0.0 \\
0.8 & 1.0 & 0.4 & 0.4 & 0.3 & 0.0 \\
0.7 & 0.9 & 1.0 & 0.8 & 0.7 & 0.4 \\
0.9 & 1.0 & 0.7 & 1.0 & 0.9 & 0.6 \\
0.6 & 0.8 & 0.8 & 0.7 & 1.0 & 0.7 \\
0.3 & 0.5 & 0.6 & 0.4 & 0.7 & 1.0
\end{pmatrix}
$$

It is easy to see that P_1 is a fuzzy preorder with respect to the Łukasiewicz t-norm $\max(x+y-1,0)$, hence, taking standard Łukasiewicz logic, we obtain $\mathrm{Refl}(P_1) = 1$ and $\mathrm{Trans}(P_1) = 1$. In this setting, one can easily infer $\mathrm{Sym}(P_1) = 0.4$ (note that for a finite fuzzy relation R, in standard Łukasiewicz logic, $\mathrm{Sym}(R)$ is nothing else but the largest difference between two values Rxy and Ryx) as well as $\mathrm{Irrefl}(P_1) = 0$ and $\mathrm{Asym}(P_1) = 0$.

Now let us see what happens if we add some disturbances to P_1. We added normally distributed pseudo-random numbers to the above table (with zero mean and a standard deviation of 0.05) and truncated these values to the unit interval. Finally, we rounded the values to two digits and obtained the following fuzzy relation:

$$P_2 = \begin{pmatrix} 1.00 & 1.00 & 0.56 & 0.40 & 0.30 & 0.00 \\ 0.87 & 1.00 & 0.33 & 0.44 & 0.26 & 0.02 \\ 0.67 & 0.92 & 0.93 & 0.87 & 0.70 & 0.39 \\ 0.93 & 1.00 & 0.64 & 1.00 & 0.97 & 0.67 \\ 0.52 & 0.79 & 0.82 & 0.71 & 1.00 & 0.59 \\ 0.27 & 0.50 & 0.61 & 0.41 & 0.72 & 1.00 \end{pmatrix}$$

Then simple computations give the following results: $\text{Refl}(P_2) = 0.93$, $\text{Irrefl}(P_2) = 0$, $\text{Sym}(P_2) = 0.41$, $\text{Trans}(P_2) = 0.85$, and $\text{Asym}(P_2) = 0$.

Example 3.3 Now consider $U = \mathbb{R}$ and let us define the following parametrized class of fuzzy relations (with $a, c > 0$):

$$E_{a,c}xy = \min(1, \max(0, a - \frac{1}{c}|x - y|))$$

It is well known that, for $a = 1$, we obtain fuzzy equivalence relations with respect to the Łukasiewicz t-norm [25,27,66,67], hence, using standard Łukasiewicz logic again, $\text{Refl}(E_{1,c}) = 1$, $\text{Sym}(E_{1,c}) = 1$, and $\text{Trans}(E_{1,c}) = 1$ for all $c > 0$. On the contrary, it is well-known and easy to see that, for $a < 1$, reflexivity in the non-graded manner cannot be maintained. Actually, we obtain

$$\text{Refl}(E_{a,c}) = \min(1, a).$$

for all $a, c > 0$. Similarly, it is a well-known fact that, for $a > 1$, transitivity in the non-graded sense is violated. This is a fact that, in some sense, has its roots in the Poincaré paradox [61, 62]. Note that relations like $E_{a,c}$ (for $a \geq 1$) appear prominently in De Cock and Kerre's framework of *resemblance relations* [28]. Regarding graded transitivity, we obtain the following:

$$\text{Trans}(E_{a,c}) = \min(1, \max(0, 2 - a))$$

Observe that $\text{Trans}(E_{a,c})$ does not depend on c either. This is not surprising, however, because the parameter c only corresponds to a re-scaling of the domain. Finally, let us mention the following results (for all $a, c > 0$):

$$\begin{aligned} \text{Irrefl}(E_{a,c}) &= \max(0, 1 - a) \\ \text{Sym}(E_{a,c}) &= 1 \\ \text{Asym}(E_{a,c}) &= \min(1, \max(0, 2 - 2a)) \end{aligned}$$

We can conclude that the larger a, the more reflexive, but less irreflexive, asymmetric, and transitive, $E_{a,c}$ is. Figure 1 shows two examples.

The following lemma provides us with some results that will be helpful in the following. Note that it is actually a corollary of more general theorems appearing in

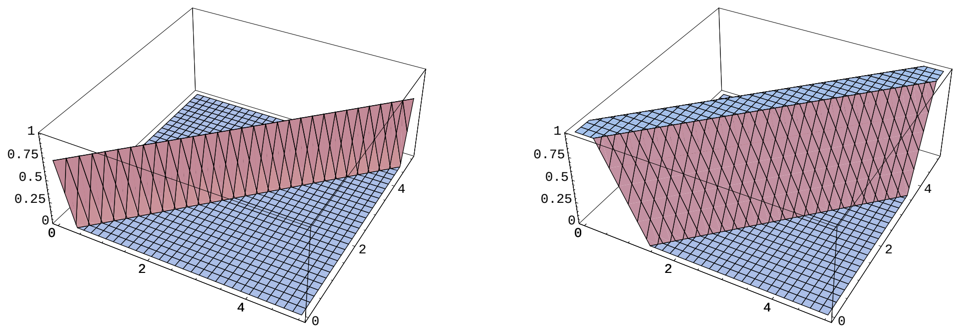

Fig. 1. The fuzzy relations $E_{0.7,2}$ (left) and $E_{1.4,1}$ (right). From Example 3.3, we can infer that $\mathrm{Refl}(E_{0.7,2}) = 0.7$, $\mathrm{Trans}(E_{0.7,2}) = \mathrm{Refl}(E_{1.4,1}) = 1$, and $\mathrm{Trans}(E_{1.4,1}) = 0.6$.

an upcoming paper [22]; here we give a direct proof. Some weaker variants can be obtained from [12, Lemma 4.8].

Lemma 3.4 In FCT, we can prove the following:

(R1) $R \subseteq S \to (\mathrm{Refl}(R) \to \mathrm{Refl}(S))$
(R2) $S \subseteq R \to (\mathrm{Irrefl}(R) \to \mathrm{Irrefl}(S))$
(R3) $R \approxeq S \to (\mathrm{Sym}(R) \to \mathrm{Sym}(S))$
(R4) $R \subseteq S \,\&\, S \subseteq^2 R \to (\mathrm{Trans}(R) \to \mathrm{Trans}(S))$
(R5) $S \subseteq^2 R \to (\mathrm{AntiSym}(R) \to \mathrm{AntiSym}(S))$
(R6) $S \subseteq^2 R \to (\mathrm{ASym}(R) \to \mathrm{ASym}(S))$

Proof. Here we prove just (R4), the others are analogous. Obviously $S \subseteq^2 R \to (Sxy \,\&\, Syz \to Rxy \,\&\, Ryz)$. So by $\mathrm{Trans}(R)$ we get $S \subseteq^2 R \to (Sxy \,\&\, Syz \to Rxz)$, and $R \subseteq S$ completes the proof. □

The following proposition provides us with a few basic results. Most of them are obvious translations of results that can be found in [41, Proposition 18.6.1], where (R11) has been extended to the more general concept of antisymmetry with respect to a fuzzy relation E (as noted above, this is in line with the similarity-based approach to fuzzy orderings [15,47]) and (R13) is new in the graded framework (yet well-known in the non-graded theory of fuzzy relations).

Theorem 3.5 *The following theorem are provable in FCT:*

(R7) $\mathrm{Refl}(R) \leftrightarrow \mathrm{Id} \subseteq R$
(R8) $\mathrm{Irrefl}(R) \leftrightarrow \mathrm{Id} \cap R \approx \emptyset$
(R9) $\mathrm{Trans}(R) \leftrightarrow R \circ R \subseteq R$
(R10) $\mathrm{Sym}(R) \leftrightarrow R^{-1} \subseteq R$
(R11) $\mathrm{AntiSym}_E(R) \leftrightarrow R \cap R^{-1} \subseteq E$
(R12) $\mathrm{Asym}(R) \leftrightarrow R \cap R^{-1} \approx \emptyset$
(R13) $\mathrm{Refl}(R) \to R \subseteq R \circ R$

Proof. We omit the obvious and concentrate on the following non-trivial issues:

(R9) Obviously, $\langle x,y\rangle \in (R \circ R) \leftrightarrow (\exists z)(Rxz \,\&\, Rzy)$. Then, by $\mathrm{Trans}(R)$ we get $(\exists z)(Rxy)$, which is just Rxy. Now let us prove the converse direction: for any x, y, we have that $(\exists z)(Rxz \,\&\, Rzy) \rightarrow Rxy$. Then the rule of quantifier shift completes the proof.

(R10) Starting from $R^{-1}xy$, i.e., Ryx, by $\mathrm{Sym}(R)$ we get Rxy. The other direction is trivial.

(R13) $Rxx \,\&\, Rxy \rightarrow (\exists z)(Rxz \,\&\, Rzy)$. Thus $Rxx \rightarrow (Rxy \rightarrow (\exists z)(Rxz \,\&\, Rzy))$. □

The following theorem collects several results that can be found in [41] as well (Propositions 18.6.1–18.6.5).

Theorem 3.6 *The following theorems are provable in FCT:*

(R14) $\mathrm{Refl}(R \sqcup \mathrm{Id})$

(R15) $\mathrm{Irrefl}(R \setminus \mathrm{Id})$

(R16) $\mathrm{Trans}(R) \rightarrow \mathrm{Trans}(R \sqcup \mathrm{Id})$

(R17) $\mathrm{Trans}(R \setminus \mathrm{Id}) \rightarrow \mathrm{Trans}(R)$

(R18) $\mathrm{Trans}(R) \,\&\, \mathrm{AntiSym}(R) \rightarrow \mathrm{Trans}(R \setminus \mathrm{Id})$

(R19) $\mathrm{AntiSym}(R) \rightarrow \mathrm{ASym}(R \setminus \mathrm{Id})$

(R20) $\mathrm{ASym}(R \setminus \mathrm{Id}) \leftrightarrow \mathrm{AntiSym}(R \setminus \mathrm{Id})$

(R21) $\mathrm{ASym}(R) \rightarrow \mathrm{AntiSym}(R \sqcup \mathrm{Id})$

(R22) $\mathrm{Trans}(R) \,\&\, \mathrm{Irrefl}(R) \rightarrow \mathrm{ASym}(R)$

(R23) $\mathrm{Trans}(R) \,\&\, \mathrm{Trans}(Q) \rightarrow \mathrm{Trans}(R \cap Q)$

Proof. For brevity, we again omit trivial and obvious parts.

(R15) $\langle x,x\rangle \in (R \setminus \mathrm{Id}) \longleftrightarrow Rxx \,\&\, x \neq x \longleftrightarrow 0$.

(R16) Observe that for $x \neq y$ we have $\langle x,y\rangle \in (R \sqcup \mathrm{Id}) \leftrightarrow Rxy$. We start from $\langle x,y\rangle \in (R \sqcup \mathrm{Id})$ and $\langle y,z\rangle \in (R \sqcup \mathrm{Id})$ and distinguish four cases: if $x = y$ and $y = z$ then $x = z$ and so $\langle x,z\rangle \in (R \sqcup \mathrm{Id})$. If $x = y$ and $y \neq z$, then we have Rxz, thus obviously $\langle x,z\rangle \in (R \sqcup \mathrm{Id})$. The case $x \neq y$ and $y = z$ is analogous. The last case is just the transitivity of R.

(R17) We start from $Rxy \,\&\, Ryz$. If $x \neq y \,\&\, y \neq z$ we get Rxz using $\mathrm{Trans}(R \setminus \mathrm{Id})$. The cases that either $x = y$ or $y = z$ are trivial.

(R18) Observe that if $x \neq y$ we have $\langle x,y\rangle \in (R \setminus \mathrm{Id}) \leftrightarrow Rxy$. Start from $\langle x,y\rangle \in (R \setminus \mathrm{Id})$ and $\langle y,z\rangle \in (R \setminus \mathrm{Id})$. Again we distinguish four cases: the only non-trivial one is $x \neq y$ and $y \neq z$. Thus we have Rxy and Ryz, observe that from $\mathrm{AntiSym}(R)$ we get that $z \neq x$ (because $z = x$ would give $x = y$).

(R19) $(\langle x,y\rangle \in (R \setminus \mathrm{Id})) \,\&\, (\langle y,x\rangle \in (R \setminus \mathrm{Id})) \longleftrightarrow (Rxy \,\&\, Ryx \,\&\, x \neq y) \longrightarrow (x = y \,\&\, x \neq y) \longleftrightarrow 0$ (in the second step we used $\mathrm{AntiSym}(R)$).

(R22) From $\mathrm{Trans}(R)$ we get $Rxy \,\&\, Ryx \rightarrow Rxx$, which leads to $\neg Rxx \rightarrow \neg(Rxy \,\&\, Ryx)$. As we have $\neg Rxx$ from $\mathrm{Irrefl}(R)$, the proof is done.

(R23) From $Rxy \& Ryz \to Rxz$ and $Qxy \& Qyz \to Qxz$ we immediately get $Rxy \& Ryz \& Qxy \& Qyz \to Rxz \& Qxz$ which is the same as $(R \cap Q)xy \& (R \cap Q)yz \to (R \cap Q)xz$. □

Example 3.7 Consider standard Łukasiewicz logic and the following family of fuzzy relations (with $a \in [0,1]$ and $U = \mathbb{R}$):

$$L_a xy = \min(1, \max(0, a - x + y))$$

Easy computations show that the fuzzy relations L_a are transitive for all $a \in [0,1]$ (i.e. $\text{Trans}(L_a) = 1$). Obviously, L_1 is also reflexive, so it is a fuzzy preorder [15], and L_0 is irreflexive, hence a typical fuzzy strict order [19, 36, 60]. Generally, we obtain $\text{Refl}(L_a) = a$ and $\text{Irrefl}(L_a) = 1 - a$. Therefore, we can conclude by (R22) that $\text{Asym}(L_a) \geq 1 - a$ for all $a \in [0,1]$. This is only a lower bound, however. It is possible to show that

$$\text{Asym}(L_a) = \min(1, \max(0, 2 - 2a))$$

holds (compare with Example 3.3). This demonstrates that under transitivity, irreflexivity is indeed a stronger requirement than asymmetry. In the non-graded framework, this is an essential fact for simplifying the definition of strict fuzzy orders [19].

Now we turn our attention to the property of extensionality of a fuzzy class with respect to a fuzzy relation. Previously, extensionality was defined as a crisp property that a given fuzzy set either had or had not [18, 50–52]. In FCT, we can generalize extensionality to the graded framework effortlessly. (See [3] for the changed role of extensionality in the fully graded theory of fuzzy relations.)

Definition 3.8 In FCT, we define the (degree of) extensionality of a fuzzy class A with respect to a fuzzy relation E as

$$\text{Ext}_E(A) \quad \equiv_{\text{df}} \quad (\forall x, y)(Exy \,\&\, x \in A \to y \in A).$$

In the non-graded framework, it is well-known that inf-intersections and sup-unions of families of extensional fuzzy sets are also extensional [18, 50–52]. The following theorem states that a similar result holds in the graded framework.

Theorem 3.9 *The following theorems are provable in FCT:*

(E1) $(\mathcal{J} \subseteq \mathcal{J} \cap \mathcal{J}) \,\&\, (\forall A \in \mathcal{J})\, \text{Ext}_E(A) \to \text{Ext}_E(\bigcap \mathcal{J})$
(E2) $(\mathcal{J} \subseteq \mathcal{J} \cap \mathcal{J}) \,\&\, (\forall A \in \mathcal{J})\, \text{Ext}_E(A) \to \text{Ext}_E(\bigcup \mathcal{J})$.

Proof. By Lemma B.8 (L14) and (L15) we have $(\forall A \in \mathcal{J})\, \text{Ext}_E(A) \longrightarrow (Exy \to (\forall A \in \mathcal{J})(x \in A \to y \in A)) \longrightarrow (Exy \to (\forall A \in \mathcal{J})(x \in A) \to (\forall A \in \mathcal{J} \cap \mathcal{J})(y \in A))$.

Now from $\mathcal{J} \subseteq \mathcal{J} \cap \mathcal{J}$ we get $A \in \mathcal{J} \to A \in \mathcal{J} \cap \mathcal{J}$, and as $(\forall A \in \mathcal{J})(x \in A)$ is exactly $x \in \bigcap \mathcal{J}$, the proof of (E1) is done. The proof of (E2) is analogous, only we use (L16) instead of (L15). □

Remark 3.10 It is easy to see that the condition $\mathcal{J} \subseteq \mathcal{J} \cap \mathcal{J}$ in (C28) is satisfied to degree 1 in models if and only if $A \in \mathcal{J}$ only acquires truth values that are idempotent with respect to conjunction. In particular, it is always true for crisp classes $\mathcal{J}$, and in Gödel logic for all classes. In standard Łukasiewicz logic, the condition expresses the closeness of $\mathcal{J}$ to crispness (it gets large truth values if and only if $A \in \mathcal{J}$ has only truth values that are close to 0 or to 1). Thus, in Ł, the theorem expresses the fact that the property of extensionality is "almost closed" under intersections and unions of "almost crisp" families of classes. In standard product logic, the situation is similar, but the condition is much stricter in smaller truth values: it gets a large truth value if and only if $A \in \mathcal{J}$ is either equal to 0, or close to 1.

The condition of the form $X \subseteq X \cap X$ is encountered quite often in graded fuzzy mathematics (cf. for instance (C28) of Theorem 5.18 below) and we could call it the *(graded)* 2*-contractiveness* of X. It can be generalized to n-contractiveness $X \subseteq X \cap \overset{n}{\ldots} \cap X$, with similar, but stricter, semantical meaning for $n > 2$.

In particular, Theorem 3.9 includes the case of crisp two-element families of fuzzy classes.

Corollary 3.11 *The following theorems are provable in* FCT*:*

(E3) $\quad \mathrm{Ext}_E(A) \wedge \mathrm{Ext}_E(B) \to \mathrm{Ext}_E(A \sqcap B)$

(E4) $\quad \mathrm{Ext}_E(A) \wedge \mathrm{Ext}_E(B) \to \mathrm{Ext}_E(A \sqcup B)$

Example 3.12 Let us consider $U = \mathbb{R}$, standard Łukasiewicz logic, $E_{1,1}$ from Example 3.3 and the two fuzzy sets

$$Ax = \min(\tfrac{1}{2}, \max(0, -2(x-1))) \text{ and } Bx = \min(\tfrac{2}{3}, \max(0, 2(x-2))).$$

Then we obtain $\mathrm{Ext}_{E_{1,1}}(A) = \frac{3}{4}$ and $\mathrm{Ext}_{E_{1,1}}(B) = \frac{2}{3}$. The two fuzzy sets A and B are disjoint, i.e. $A \sqcap B = \emptyset$, hence, $\mathrm{Ext}_{E_{1,1}}(A \sqcap B) = \mathrm{Ext}_{E_{1,1}}(\emptyset) = 1$. This fact underlines that (E3) and (E4) provide us with lower bounds for the extensionality of intersections/unions, but these bounds need not always be very helpful.

In classical mathematics, special properties of relations are rarely studied completely independently of each other. Instead, these properties most often occur in some combinations in the definitions of special classes of relations—with (pre)orders and equivalence relations being two most fundamental examples. The same is true in the theory of fuzzy relations, where fuzzy (pre)orders and fuzzy equivalence relations are the most important classes. Compound properties of this kind are defined as conjunctions of some of the simple properties of Definition 3.1. In the non-graded case, the properties are crisp, so the conjunction we need is the classical Boolean conjunction. In FCT, however, all properties are graded, so it indeed matters which conjunction we take. Thus, besides the (more usual) combinations by strong conjunction & (corresponding to the t-norm in the standard case), we also define their weak variants combined by weak conjunction (corresponding to the minimum). In this paper, we restrict ourselves to investigation of basic properties of fuzzy preorders and similarities.[2]

Definition 3.13 In FCT we define the following compound properties of fuzzy relations:

$$
\begin{array}{rcll}
\mathrm{Preord}(R) & \equiv_{\mathrm{df}} & \mathrm{Refl}(R) \mathbin{\&} \mathrm{Trans}(R) & \text{(strong) preorder} \\
\mathrm{wPreord}(R) & \equiv_{\mathrm{df}} & \mathrm{Refl}(R) \wedge \mathrm{Trans}(R) & \text{weak preorder} \\
\mathrm{Sim}(R) & \equiv_{\mathrm{df}} & \mathrm{Refl}(R) \mathbin{\&} \mathrm{Sym}(R) \mathbin{\&} \mathrm{Trans}(R) & \text{(strong) similarity} \\
\mathrm{wSim}(R) & \equiv_{\mathrm{df}} & \mathrm{Refl}(R) \wedge \mathrm{Sym}(R) \wedge \mathrm{Trans}(R) & \text{weak similarity}
\end{array}
$$

Example 3.14 Let us shortly revisit Example 3.2. We can conclude the following:

$$
\begin{array}{rlrl}
\mathrm{Preord}(P_1) = 1 & & \mathrm{Preord}(P_2) = 0.78 \\
\mathrm{wPreord}(P_1) = 1 & & \mathrm{wPreord}(P_2) = 0.85 \\
\mathrm{Sim}(P_1) = 0.4 & & \mathrm{Sim}(P_2) = 0.19 \\
\mathrm{wSim}(P_1) = 0.4 & & \mathrm{wSim}(P_2) = 0.41
\end{array}
$$

The values in the second column once more demonstrate why it is justified to speak of strong and weak properties—the stronger (i.e. smaller) the conjunction, the harder a property can be fulfilled.

For the class of fuzzy relations defined in Example 3.3, we obtain the interesting result

$$\mathrm{Preord}(E_{a,c}) = \mathrm{wPreord}(E_{a,c}) = \max(0, 1 - |1 - a|),$$

from which we can infer that $\mathrm{Preord}(E_{a,c}) = \mathrm{wPreord}(E_{a,c}) = 1$ if and only if $a = 1$. Note that $\mathrm{Sym}(E_{a,c}) = 1$, so $\mathrm{Sim}(E_{a,c}) = \mathrm{Preord}(E_{a,c})$ and $\mathrm{wSim}(E_{a,c}) = \mathrm{wPreord}(E_{a,c})$ which implies that $\mathrm{Sim}(E_{a,c}) = \mathrm{wSim}(E_{a,c}) = 1$ if and only if $a = 1$.

For the class of fuzzy relations introduced in Example 3.7, we trivially obtain the following result: $\mathrm{Preord}(L_a) = \mathrm{wPreord}(L_a) = a$ and $\mathrm{Sim}(L_a) = \mathrm{wSim}(L_a) = 0$.

Obviously $\mathrm{Preord}(R) \to \mathrm{wPreord}(R)$ and $\mathrm{Sim}(R) \to \mathrm{wSim}(R)$. From Lemma 3.4 we further obtain:

Lemma 3.15 FCT proves:

(R24) $R \approxeq^2 S \to (\mathrm{Preord}(R) \to \mathrm{Preord}(S))$
(R25) $R \subseteq S \mathbin{\&} S \subseteq^2 R \to (\mathrm{wPreord}(R) \to \mathrm{wPreord}(S))$

---

[2] In line with Zadeh's original work [69], we use the term *similarity (relation)* synonymously for fuzzy equivalence (relation).

(R26) $R \approxeq^3 S \to (\mathrm{Sim}(R) \to \mathrm{Sim}(S))$
(R27) $R \subseteq S \,\&\, S \subseteq^2 R \to (\mathrm{wSim}(R) \to \mathrm{wSim}(S))$

4 Images and Dual Images

In this section, we address images of fuzzy relations in the framework of FCT. Such operations are of central importance in fuzzy inference [9, 70], in the theory of fuzzy relational equations [23, 65], and in the study of properties of fuzzy relations, too [12, 18]. These concepts are also strongly linked with fuzzy mathematical morphology [14, 16, 55, 56]. [3]

Definition 4.1 In FCT, we define the following operations:

$$\begin{aligned} R^{\uparrow}A &=_{\mathrm{df}} \{y \mid (\exists x)(x \in A \,\&\, Rxy)\} \\ R^{\downarrow}A &=_{\mathrm{df}} \{x \mid (\forall y)(Rxy \to y \in A)\} \end{aligned}$$

Let us shortly clear up the terminology. In the literature, the image operator $R^{\uparrow} A$ is called *full image*, *direct image*, *conditioned fuzzy set*, or simply *image* of A under/with respect to R, while $R^{\downarrow}A$ appears under the names *superdirect image* and α-*operation*; its systematic name in [8] (submitted to this issue) is *subproduct preimage*. We will simply call both operators *images*. Where necessary, we refer to $^{\downarrow}$ explicitly as *dual image*. [4]

Example 4.2 Let us consider $U = \mathbb{R}$ and the fuzzy set

$$Ax = \min(1, \max(0, \tfrac{1}{10}(x - 175))).$$

Straightforward computations then show the following (with the fuzzy relation $E_{1.5,10}$ defined as in Example 3.3):

$$\begin{aligned} (E_{1.5,10}{}^{\uparrow}A)x &= \min(1, \max(0, \tfrac{1}{10}(x - 170))) \\ (E_{1.5,10}{}^{\downarrow}A)x &= \min(1, \max(0, \tfrac{1}{10}(x - 180))) \end{aligned}$$

Figure 2 shows a plot of these three fuzzy sets. Note that De Cock and Kerre use the two image operators in conjunction with their resemblance relations [28] to define linguistic hedges like for instance *roughly* and *very* [29]. If we consider A as a model of *tall* (in the context of European men), we can interpret $E_{1.5,10}{}^{\uparrow}A$ as a model of *roughly tall* and $E_{1.5,10}{}^{\downarrow}A$ as a model of *very tall* according to De Cock's and Kerre's argumentation.

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[3] Note that the references in this paragraph are just pointers to some important works, but do not cover all the relevant literature.

[4] The relationship between the operations $^{\uparrow}$ and $^{\downarrow}$ is in fact an instance of Morsi's duality [54] combined with the inversion duality (i.e., the duality between R and R^{-1}).

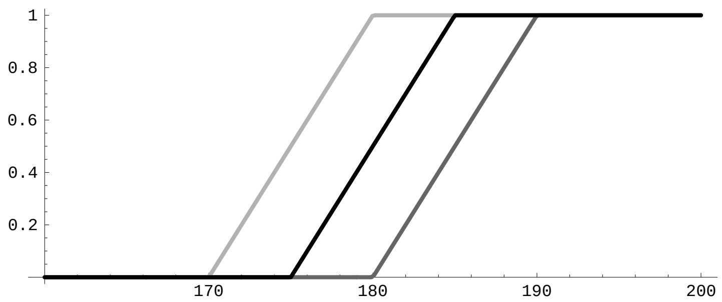

Fig. 2. The fuzzy set A (middle, solid black) and the result that is obtained when applying image operators: $E_{1.5,10}{}^{\downarrow}A$ (left, light gray) and $E_{1.5,10}{}^{\uparrow}A$ (right, medium gray).

The next theorem clarifies some basic properties of images under fuzzy relations. Their non-graded versions are well-known and easy to prove (see e.g. [18, 40, 41]). The graded theorems (I6)–(I14) are also corollaries of more general theorems in the paper [8] submitted to this issue.

Theorem 4.3 *The following properties of images are provable in* FCT*:*

(I1) $R^{\uparrow}\emptyset = \emptyset$
(I2) $R^{\uparrow}\mathrm{V} = \{y \mid (\exists x)(Rxy)\} = \mathrm{Rng}(R)$
(I3) $R^{\uparrow}\{z\} = \{y \mid Rzy\}$
(I4) $R^{\downarrow}(\emptyset) = \{x \mid (\forall y)(\neg Rxy)\}$
(I5) $R^{\downarrow}\mathrm{V} = \mathrm{V}$
(I6) $R^{\uparrow}(A \sqcup B) = R^{\uparrow}A \sqcup R^{\uparrow}B$
(I7) $R^{\downarrow}(A \sqcap B) = R^{\downarrow}A \sqcap R^{\downarrow}B$
(I8) $R^{\uparrow}(A \sqcap B) \subseteq R^{\uparrow}A \sqcap R^{\uparrow}B$
(I9) $R^{\downarrow}(A \sqcup B) \supseteq R^{\downarrow}A \sqcup R^{\downarrow}B$
(I10) $A \subseteq B \rightarrow R^{\uparrow}A \subseteq R^{\uparrow}B$
(I11) $A \subseteq B \rightarrow R^{\downarrow}A \subseteq R^{\downarrow}B$
(I12) $R \subseteq S \rightarrow R^{\uparrow}A \subseteq S^{\uparrow}A$
(I13) $R \subseteq S \rightarrow S^{\downarrow}A \subseteq R^{\downarrow}A$
(I14) $R^{\uparrow}A \subseteq B \leftrightarrow A \subseteq R^{\downarrow}B$

Proof.

(I1)–(I5) are trivial to prove.
(I6)–(I9) are simple consequences of Lemma B.8 (L10)–(L13).
(I10) From $(Ax \rightarrow Bx) \rightarrow (Ax \,\&\, Rxy \rightarrow Bx \,\&\, Rxy)$ we get $A \subseteq B \rightarrow ((\exists x)(Ax \,\&\, Rxy) \rightarrow (\exists x)(Bx \,\&\, Rxy))$.
(I11) We know $(Ay \rightarrow By) \rightarrow ((Rxy \rightarrow Ay) \rightarrow (Rxy \rightarrow By))$, whence the required statement follows by generalization and quantifier shifts.
(I12) $(Rxy \rightarrow Sxy) \rightarrow (Rxy \,\&\, Ax \rightarrow Sxy \,\&\, Ax)$. Thus $R \subseteq S \rightarrow (x \in R^{\uparrow}A \rightarrow x \in S^{\uparrow}A)$.
(I13) $(Rxy \rightarrow Sxy) \rightarrow ((Sxy \rightarrow Ay) \rightarrow (Rxy \rightarrow Ay))$, then use generalization and

quantifier shifts.

(I14) Left to right: $(\forall y)((\exists x)(Ax \& Rxy) \to By) \longrightarrow (\forall y)(Ax \to (Rxy \to By)) \longleftrightarrow (Ax \to (\forall y)(Rxy \to By))$. Right to left: $(\forall x)(Ax \to (\forall y)(Rxy \to By)) \longrightarrow (\forall x)(Ax \& Rxy \to By)) \longleftrightarrow ((\exists x)(Ax \& Rxy) \to By)$. □

The previous theorem addressed the monotonicity of images of fuzzy relations and how these images interact with intersections and unions with respect to the weak conjunction and disjunction, respectively. The question remains how images of fuzzy relations interact with intersections with respect to the strong conjunction. The following theorem gives an answer (for its non-graded version, see [40, Proposition 2.16] or [41, Proposition 18.4.1]).

Theorem 4.4 *The following properties of relations are provable in* FCT*:*

(I15) $(R \cap R)^{\uparrow}(A \cap B) \subseteq (R^{\uparrow}A) \cap (R^{\uparrow}B)$

(I16) $(R^{\downarrow}A) \cap (R^{\downarrow}B) \subseteq (R \cap R)^{\downarrow}(A \cap B)$

Proof.

(I15) $(\exists x)(Rxy \,\&\, Rxy \,\&\, Ax \,\&\, Bx) \longrightarrow (\exists x)(Rxy \,\&\, Ax) \,\&\, (\exists x)(Rxy \,\&\, Bx)$

(I15) $(\forall y)(Rxy \to y \in A) \,\&\, (\forall y)(Rxy \to y \in B) \longrightarrow (\forall y)((Rxy \to y \in A) \,\&\, (Rxy \to y \in B)) \longrightarrow (\forall y)(Rxy \,\&\, Rxy \to y \in A \,\&\, y \in B)$ □

Remark 4.5 Theorem 4.4 intentionally cites only the first two of three assertions of [41, Proposition 18.4.1] (and, correspondingly, [40, Proposition 2.16]). If we translate the third assertion to our terminology, we obtain

$$(R^{\uparrow_{\mathrm{G}}}A) \cup (R^{\uparrow_{\mathrm{G}}}B) \subseteq (R \cup R)^{\uparrow_{\mathrm{G}}}(A \cup B),$$

where $R^{\uparrow_{\mathrm{G}}}A$ stands for the image with respect to the weak conjunction, i.e.,

$$R^{\uparrow_{\mathrm{G}}}A =_{\mathrm{df}} \{y \mid (\exists x)(x \in A \wedge Rxy)\}.$$

First of all, this assertion relies on a certain concept of strong disjunction (a t-conorm in the standard case) which we cannot define in $\mathrm{MTL}_{\triangle}$ (we can do so only in FCT over stronger logics with involutive negation like $\mathrm{IMTL}_{\triangle}$ or ŁΠ). Secondly, we would like to point out that this result actually does not hold. Let us consider the case $U = \{1,2\}$, standard Łukasiewicz logic (with the Łukasiewicz t-conorm $\min(1, x+y)$ as strong disjunction), and the following fuzzy relation and fuzzy sets (membership degrees in matrix/vector notation):

$$R = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.4 \end{pmatrix} \quad A = \begin{pmatrix} 0.5 \\ 0.6 \end{pmatrix} \quad B = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix}$$

Then easy computations show the following:

$$R^{\uparrow_G}A = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \text{ and } R^{\uparrow_G}B = \begin{pmatrix} 0.4 \\ 0.4 \end{pmatrix} \text{ which implies } (R^{\uparrow_G}A) \cup (R^{\uparrow_G}B) = \begin{pmatrix} 0.9 \\ 0.9 \end{pmatrix}.$$

On the other hand, we obtain

$$R \cup R = \begin{pmatrix} 1.0 & 0.8 \\ 1.0 & 0.8 \end{pmatrix} \text{ and } A \cup B = \begin{pmatrix} 0.8 \\ 1.0 \end{pmatrix} \text{ yielding } (R \cup R)^{\uparrow_G}(A \cup B) = \begin{pmatrix} 0.8 \\ 0.8 \end{pmatrix}.$$

So we have got a counter-example. Note that the converse inclusion does not hold either, as can be seen from the following counter-example (with analogous computations like above):

$$R' = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 0.5 \end{pmatrix} \quad A' = \begin{pmatrix} 0.7 \\ 0.6 \end{pmatrix} \quad B' = \begin{pmatrix} 0.0 \\ 0.4 \end{pmatrix}$$

Since the assertions do not hold in the non-graded case, it makes no sense to try to generalize them to a graded version.

Now let us turn our attention to how image operations interact with the common special properties of fuzzy relations and the concept of extensionality.

Theorem 4.6 *The following properties of* $^{\uparrow}$ *are provable in* FCT*:*

(I17) $\mathrm{Refl}(R) \leftrightarrow (\forall A)(A \subseteq R^{\uparrow}A)$
(I18) $\mathrm{Trans}(R) \leftrightarrow (\forall A)(R^{\uparrow}(R^{\uparrow}A) \subseteq R^{\uparrow}A)$
(I19) $\mathrm{Preord}(R) \to R^{\uparrow}(R^{\uparrow}A) \approxeq R^{\uparrow}A$
(I20) $\mathrm{wPreord}(R) \to R^{\uparrow}(R^{\uparrow}A) \approx R^{\uparrow}A$
(I21) $\mathrm{Trans}(R) \leftrightarrow (\forall A)(\mathrm{Ext}_R(R^{\uparrow}A))$
(I22) $A \subseteq B \,\&\, \mathrm{Ext}_R(B) \to R^{\uparrow}A \subseteq B$
(I23) $\mathrm{Refl}(R) \,\&\, \mathrm{Ext}_R(A) \to R^{\uparrow}A \approxeq A$
(I24) $\mathrm{Refl}(R) \wedge \mathrm{Ext}_R(A) \to R^{\uparrow}A \approx A$
(I25) $R^{\uparrow}A \subseteq A \leftrightarrow \mathrm{Ext}_R(A)$
(I26) $\mathrm{Refl}(R) \to (\mathrm{Ext}_R(A) \leftrightarrow (A \approx R^{\uparrow}A)$
(I27) $\mathrm{Refl}(R) \to (\mathrm{Ext}_R(A) \leftrightarrow (A \approxeq R^{\uparrow}A))$

Proof.

(I17) Left to right: obviously $Rxx \,\&\, Ax \to (\exists y)(Rxy \,\&\, Ax)$ and generalize as usual. Right to left: $(\forall A)(A \subseteq R^{\uparrow}A) \longrightarrow \{z\} \subseteq R^{\uparrow}\{z\} \longrightarrow (z = z \to Rzz)$; we used (I3) in the last step.

(I18) Left to right: From $(Rxz \to Rxy) \to (Ax \& Rxz \to Ax \& Rxy)$ we get $(\forall x)(Rxz \to Rxy) \to (z \in R^{\uparrow}A \to y \in R^{\uparrow}A)$. Next we get $(Rzy \to (\forall x)(Rxz \to Ryx)) \to (Rzy \to (z \in R^{\uparrow}A \to y \in R^{\uparrow}A))$. Thus $(\forall x)(Rzy \to (Rxz \to Rxy)) \to (Rzy \,\&\, z \in R^{\uparrow}A \to y \in R^{\uparrow}A))$. Right to left: $(\forall A)(R^{\uparrow}(R^{\uparrow}A) \subseteq R^{\uparrow}A) \longrightarrow (R^{\uparrow}(R^{\uparrow}$

$\{z\}) \subseteq R^{\uparrow}\{z\}) \longleftrightarrow (R^{\uparrow}\{y \mid Rzy\}\} \subseteq \{y \mid Rzy\} \longrightarrow ((\exists x)(Rzx \,\&\, Rxy) \to Rzy)$, and quantifier shifts complete the proof.

(I19) and (I20) are direct consequences of (I17) and (I18).

(I21) From $(\forall x)(Ryz \to (Rxy \to Ryz))$ we get $(Ryz \to ((\exists x)(Rxy \& Ax) \to (\exists x)(Rxz \& Ax))$. The converse direction: $(\forall A)(\mathrm{Ext}_R(R^{\uparrow}A)) \longrightarrow \mathrm{Ext}_R(R^{\uparrow}\{z\}) \longrightarrow (Rzx \,\&\, Rxy \to Rzy)$.

(I22) From $A \subseteq B$ we get $Ax \,\&\, Rxy \to Bx \,\&\, Rxy$ and from $\mathrm{Ext}_R(B)$ we get $Bx \,\&\, Rxy \to By$. Thus we have $Ax \,\&\, Rxy \to By$.

(I23) and (I24) follow directly from (I22) by (I17).

(I25) $((\exists x)(Rxy \,\&\, Ax) \to Ay) \leftrightarrow (\forall x)(Rxy \,\&\, Ax \to Ay)$.

(I26) and (I27) then follow trivially. □

Theorem 4.7 *The following properties of* $^{\downarrow}$ *are provable in* FCT*:*

(I28) $\mathrm{Refl}(R) \to R^{\downarrow}A \subseteq A$

(I29) $\mathrm{Trans}(R) \to R^{\downarrow}A \subseteq R^{\downarrow}(R^{\downarrow}A)$

(I30) $\mathrm{Preord}(R) \to R^{\downarrow}(R^{\downarrow}A) \approxeq R^{\downarrow}A$

(I31) $\mathrm{wPreord}(R) \to R^{\downarrow}(R^{\downarrow}A) \approx R^{\downarrow}A$

(I32) $\mathrm{Trans}(R) \to \mathrm{Ext}_R(R^{\downarrow}A)$

(I33) $B \subseteq A \,\&\, \mathrm{Ext}_R(B) \to B \subseteq R^{\downarrow}A$

(I34) $\mathrm{Refl}(R) \,\&\, \mathrm{Ext}_R(B) \to R^{\downarrow}A \approxeq A$

(I35) $\mathrm{Refl}(R) \wedge \mathrm{Ext}_R(B) \to R^{\downarrow}A \approx A$

(I36) $A \subseteq R^{\downarrow}A \leftrightarrow \mathrm{Ext}_R(A)$

(I37) $\mathrm{Refl}(R) \to (\mathrm{Ext}_R(A) \leftrightarrow (A \approx R^{\downarrow}A))$

(I38) $\mathrm{Refl}(R) \to (\mathrm{Ext}_R(A) \leftrightarrow (A \approxeq R^{\downarrow}A))$

Proof.

(I28) $(\forall y)(Rxy \to Ay) \to (Rxx \to Ax)$, thus $Rxx \to (x \in R^{\downarrow}A \to Ax)$. Generalization and quantifier shifts complete the proof.

(I29) From $(Rzy \to Rxy) \to ((Rxy \to Ay) \to (Rzy \to Ay))$ we get $(\forall x)(Rzy \to Rxy) \to (x \in R^{\downarrow}A \to z \in R^{\downarrow}A)$. Next we get $(Rxz \to (\forall x)(Rzy \to Rxy)) \to (Rxz \to (x \in R^{\downarrow}A \to z \in R^{\downarrow}A))$. Thus $(\forall y)(Rxz \to (Rzy \to Rxy)) \to (x \in R^{\downarrow}A \to (Rxz \to z \in R^{\downarrow}A))$.

(I30) and (I31) are direct consequences of (I28) and (I29).

(I32) From $(\forall y)(Rzx \to (Rxy \to Rzy))$ we get $(Rzx \to ((\forall y)(Rzy \to Ay) \to (\forall y)(Rxy \to Ay))$.

(I33) From $B \subseteq A$ we get $(Rxy \to By) \to (Rxy \to Ax)$ and from $\mathrm{Ext}_R(B)$ we get $Bx \to (Rxy \to By)$. Thus we have $Bx \to (Rxy \to Ay)$.

(I34) and (I35) follow directly from (I33) using (I28).

(I36) Left to right: $(Ax \to (\forall y)(Rxy \to Ay)) \to (\forall y)(Rxy \,\&\, Ax \to Ay)$. The converse direction follows from (I32).

(I37) and (I38) then follow trivially. □

Inspired by the concepts of fuzzy mathematical morphology [14, 24, 55], Bodenhofer has introduced a general concept of opening and closure operators with respect to arbitrary fuzzy relations [**?**]. Now we generalize this concept to the graded framework.

Definition 4.8 We define the operations of *opening* and *closure* of A in R as

$$R^{\circ}A =_{\mathrm{df}} R^{\uparrow}(R^{\downarrow}A)$$
$$R^{\bullet}A =_{\mathrm{df}} R^{\downarrow}(R^{\uparrow}A)$$

Furthermore, we define two properties of fuzzy classes, *R-openness* and *R-closedness*:

$$\mathrm{Open}_R(A) \equiv_{\mathrm{df}} R^{\circ}A \approx A$$
$$\mathrm{Closed}_R(A) \equiv_{\mathrm{df}} R^{\bullet}A \approx A$$

The following lemma provides us with several properties of opening and closure operators. In particular, the question arises why R-openness and R-closedness were defined using $\approx$ rather than $\approxeq$. A clear answer to this question is given by (I40) and (I41) which state that it actually does not matter whether we use $\approx$ or $\approxeq$ in the definition of openness and closedness.

Theorem 4.9 *The following properties of relations are provable in FCT:*

(I39) $R^{\circ}A \subseteq A \subseteq R^{\bullet}A$
(I40) $\mathrm{Open}_R(A) \leftrightarrow R^{\circ}A \approxeq A$
(I41) $\mathrm{Closed}_R(A) \leftrightarrow R^{\bullet}A \approxeq A$
(I42) $A \subseteq B \rightarrow R^{\circ}A \subseteq R^{\circ}B$
(I43) $A \subseteq B \rightarrow R^{\bullet}A \subseteq R^{\bullet}B$
(I44) $\mathrm{Open}_R(A) \leftrightarrow (\exists B)(A \approxeq R^{\uparrow}B)$
(I45) $\mathrm{Closed}_R(A) \leftrightarrow (\exists B)(A \approxeq R^{\downarrow}B)$
(I46) $\mathrm{Open}_R(R^{\circ}A)$
(I47) $\mathrm{Closed}_R(R^{\bullet}A)$

Proof.

(I39) First, we can show $y \in R^{\uparrow}(R^{\downarrow}A) \longleftrightarrow (\exists x)(Rxy \,\&\, (\forall z)(Rxz \rightarrow Az)) \longrightarrow (\exists x)(Rxy \,\&\, (Rxy \rightarrow Ay)) \longrightarrow (\exists x)Ay \longleftrightarrow Ay$. Secondly, we have $Ax \longrightarrow (Rxy \rightarrow Rxy \,\&\, Ax) \longrightarrow (Rxy \rightarrow (\exists x)(Rxy \,\&\, Ax))$. Thus $Ax \rightarrow (\forall y)(Rxy \rightarrow y \in R^{\uparrow}A)$.

(I40) and (I41) are then direct consequences of (I39).

(I42) and (I43) are direct consequences consequence of (I10) and (I11).

(I44) The left-to-right direction is trivial (take $B = R^{\downarrow}A$). The converse direction: By (I14) and (I10), $R^{\uparrow}B \subseteq A \longleftrightarrow B \subseteq R^{\downarrow}A \longrightarrow R^{\uparrow}B \subseteq R^{\uparrow}(R^{\downarrow}A)$. Thus

$A \approxeq R^{\uparrow}B \longleftrightarrow A \subseteq R^{\uparrow}B \subseteq A \longrightarrow A \subseteq R^{\uparrow}B \subseteq R^{\uparrow}(R^{\downarrow}A) = R^{\circ}A$. Since by (I39) always $R^{\circ}A \subseteq A$, the proof is done.

(I45) Analogous to the proof of (I44).

(I46) and (I47) are direct consequences of (I44) and (I45), respectively. □

Note that, from (I44)–(I47), we can easily deduce the following corollaries:

(I48) $R^{\circ}(R^{\circ}A) = R^{\circ}A$

(I49) $R^{\bullet}(R^{\bullet}A) = R^{\bullet}A$

Thus, we can conclude that the two operators $^{\circ}$ and $^{\bullet}$ fulfill the most essential properties we need to require from opening and closure operators (as stated in [?] to motivate the definition of the two operators). Unlike [?], in classical mathematics (e.g. in topology), it is more usual to start from an axiomatic framework of openness and closedness (or opening and closure operators, respectively). Such general frameworks have been introduced in the fuzzy setting by Bělohlávek and Funioková [10, 11, 13]. They require that opening operators always give subsets, that closure operators always yield supersets, that both operators are monotonic with respect to the graded inclusion and that both operators are idempotent. Therefore, we can conclude that our two operators perfectly fit into the axiomatic framework of Bělohlávek and Funioková.

In many classical axiomatic frameworks (including topological ones), it is also common to represent opening and closure operators as unions of all open subsets and intersections of all closed supersets, respectively. This is well-known in the non-graded framework; the following theorem provides a generalization to the graded case.

Theorem 4.10 *The following properties of relations are provable in FCT:*

(I50) $R^{\circ}A = \bigcup\{B \mid \mathrm{Open}_R(B) \;\&\; B \subseteq A\} = \bigcup\{B \mid \triangle(\mathrm{Open}_R(B) \;\&\; B \subseteq A)\}$

(I51) $R^{\bullet}A = \bigcap\{B \mid \mathrm{Closed}_R(B) \;\&\; A \subseteq B\} = \bigcap\{B \mid \triangle(\mathrm{Closed}_R(B) \;\&\; A \subseteq B)\}$

Proof. To prove (I50), let us denote $\bigcup\{B \mid \mathrm{Open}_R(B) \;\&\; B \subseteq A\}$ as C. Then $y \in C \leftrightarrow (\exists B)(y \in B \;\&\; \mathrm{Open}_R(B) \;\&\; B \subseteq A)$. Since $R^{\circ}A \subseteq A$ and $\mathrm{Open}_R(R^{\circ}A)$ we get that $y \in R^{\circ}A \to y \in C$. To prove the converse direction we use Lemma B.8 (L6). We fix B and show that $\mathrm{Open}_R(B) \;\&\; B \subseteq A$ implies $B \subseteq R^{\circ}A$. From $\mathrm{Open}_R(B)$ we get that $R^{\circ}B \approx B$ and from $B \subseteq A$ we get $R^{\circ}B \subseteq R^{\circ}A$. Thus $B \subseteq R^{\circ}A$. The proof of the second equality is almost straightforward. (I51) can be proved analogously. □

From the two representations (I50) and (I51), we can deduce immediately how opening and closure operators interact with unions and intersections (with respect to weak conjunction).

Corollary 4.11 *The following properties of relations are provable in* FCT*:*

(I52) $R^{\circ}(A \sqcup B) = R^{\circ}A \sqcup R^{\circ}B$
(I53) $R^{\bullet}(A \sqcap B) = R^{\bullet}A \sqcap R^{\bullet}B$
(I54) $\mathrm{Open}_R(A) \,\&\, \mathrm{Open}_R(B) \to \mathrm{Open}_R(A \sqcup B)$
(I55) $\mathrm{Closed}_R(A) \,\&\, \mathrm{Closed}_R(B) \to \mathrm{Closed}_R(A \sqcap B)$

Proof.

(I52) Use (I50) and (L10) of Lemma B.8.
(I53) Use (I51) and (L11).
(I54) and (I55) are then direct consequences of (I52) and (I53), respectively. □

As shown in [?], under the presence of reflexivity and/or transitivity, the results concerning opening and closure operators can be strengthened. We will see in the following that, by this way, results for images of fuzzy preorders are obtained that are well-known in the non-graded framework [18].

Theorem 4.12 *The following properties of relations are provable in FCT:*

(I56) $\mathrm{Preord}(R) \to (R^{\bullet}A \approxeq R^{\uparrow}A)$
(I57) $\mathrm{wPreord}(R) \to (R^{\bullet}A \approx R^{\uparrow}A)$
(I58) $\mathrm{Preord}(R) \to (R^{\circ}A \approxeq R^{\downarrow}A)$
(I59) $\mathrm{wPreord}(R) \to (R^{\circ}A \approx R^{\downarrow}A)$
(I60) $\mathrm{Trans}(R) \to (\mathrm{Open}_R(A) \to \mathrm{Ext}_R(A))$
(I61) $\mathrm{Trans}(R) \to (\mathrm{Closed}_R(A) \to \mathrm{Ext}_R(A))$
(I62) $\mathrm{Refl}(R) \to (\mathrm{Ext}_R(A) \to \mathrm{Open}_R(A))$
(I63) $\mathrm{Refl}(R) \to (\mathrm{Ext}_R(A) \to \mathrm{Closed}_R(A))$
(I64) $\mathrm{wPreord}(R) \to (\mathrm{Ext}_R(A) \leftrightarrow \mathrm{Open}_R(A))$
(I65) $\mathrm{wPreord}(R) \to (\mathrm{Ext}_R(A) \leftrightarrow \mathrm{Closed}_R(A))$
(I66) $\mathrm{Preord}(R) \to (\mathrm{Open}_R(A) \leftrightarrow \mathrm{Closed}_R(A))$

Proof.

(I56) and (I57): From (I32) we know $\mathrm{Trans}(R) \to \mathrm{Ext}_R(R^{\downarrow}A)$. Then (I56) follows from (I23) and (I57) follows from (I24).
(I58) and (I59): From (I21) we know $\mathrm{Trans}(R) \to \mathrm{Ext}_R(R^{\uparrow}A)$. Then (I58) follows from (I34) and (I59) follows from (I35).
(I60) We start from $\mathrm{Open}_R(A)$ (i.e. $A \approx R^{\circ}A$) and $\mathrm{Trans}(R)$. Using (I32) we get $\mathrm{Ext}_R(R^{\downarrow}A)$, thus by (I22), $R^{\uparrow}(R^{\downarrow}A) \subseteq R^{\downarrow}A$. So we obtain $A \subseteq R^{\downarrow}A$. Now we use (I36) and get $\mathrm{Ext}_R(A)$.
(I61) Analogously to (I60), by (I21), (I33), and (I25).

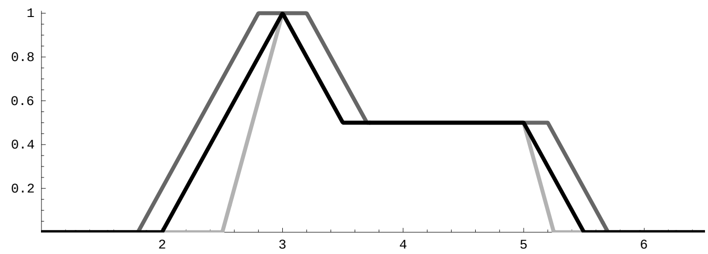

Fig. 3. The fuzzy set A from Example 4.13 (light gray), $E_{1.2,1}{}^{\uparrow}A$ (medium gray), and $E_{1.2,1}{}^{\bullet}A$ (solid black).

(I62) We start from $\mathrm{Refl}(R)$ and $\mathrm{Ext}_R(A)$. From $\mathrm{Refl}(R)$, by (I17), we get $R^{\downarrow}A \subseteq R^{\uparrow}(R^{\downarrow}A)$. From $\mathrm{Ext}_R(A)$, we obtain by (I33) that $A \subseteq R^{\downarrow}A$. So finally, we can conclude $A \subseteq R^{\circ}A$ which, with (I39), proves $\mathrm{Open}_R(A)$.

(I63) Analogously to (I62), using (I28), (I22), and the second inclusion of (I39).

(I64)–(I66) then follow trivially. □

Example 4.13 Let us consider standard Łukasiewicz logic and the following fuzzy set (with $U = \mathbb{R}$):

$$Ax = \begin{cases} 2x-5 & \text{if } x \in [2.5,3] \\ 4-x & \text{if } x \in]3,3.5] \\ 0.5 & \text{if } x \in]3.5,5] \\ 10.5-2x & \text{if } x \in]5,5.25] \\ 0 & \text{otherwise} \end{cases}$$

Further we consider the fuzzy relation $E_{1.2,1}$ from Example 3.3 for which we know $\mathrm{Refl}(E_{1.2,1}) = 1$ and $\mathrm{Trans}(E_{1.2,1}) = \mathrm{Preord}(E_{1.2,1}) = \mathrm{wPreord}(E_{1.2,1}) = 0.8$. Figure 3 shows plots of A, $E_{1.2,1}{}^{\uparrow}A$, and $E_{1.2,1}{}^{\bullet}A$. Basic computations show that $\mathrm{Closed}_{E_{1.2,1}}(A) = 0.5$. Moreover, we have that $(A \approx E_{1.2,1}{}^{\uparrow}A) = 0.3$. From (I26) we can infer, therefore, that $\mathrm{Ext}_{E_{1.2,1}}(A) = 0.3$. It also holds that $(E_{1.2,1}{}^{\bullet}A \approx E_{1.2,1}{}^{\uparrow}A) = (E_{1.2,1}{}^{\bullet}A \approxeq E_{1.2,1}{}^{\uparrow}A) = 0.8$. Figure 4 shows plots of A, $E_{1.2,1}{}^{\downarrow}A$ and $E_{1.2,1}{}^{\circ}A$. We can show that $\mathrm{Open}_{E_{1.2,1}}(A) = 0.5$ and $(A \approx E_{1.2,1}{}^{\downarrow}A) = 0.3$. Thus, we can infer $\mathrm{Ext}_{E_{1.2,1}}(A) = 0.3$ also via (I37). Further we can show that $(E_{1.2,1}{}^{\circ}A \approx E_{1.2,1}{}^{\downarrow}A) = (E_{1.2,1}{}^{\circ}A \approxeq E_{1.2,1}{}^{\downarrow}A) = 0.8$. If we take into account that $(0.5 \to 0.3) = (0.3 \leftrightarrow 0.5) = 0.8$, these numbers demonstrate that, in this special case, the estimations provided by Theorem 4.12 are tight.

Finally, we can formulate representations of images under fuzzy preorders. Note that the first four assertions (I67)–(I70) of the following theorem are "fuzzy representations", i.e. they do not determine the truth degree of $R^{\uparrow}A$ or $R^{\downarrow}A$ itself. We can only infer from the degree to which R is a (weak) preorder to which degree the image is guaranteed to resemble to the intersection (resp. union). The "real" (non-graded) representations (I71)–(I72), known from [?, 18], are their special cases for R being a preorder to degree 1.

Corollary 4.14 *The following properties of relations are provable in* FCT*:*

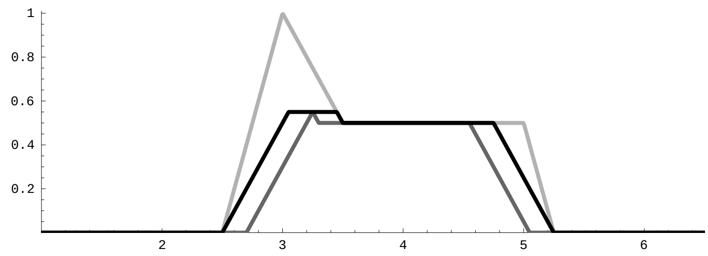

Fig. 4. The fuzzy set A from Example 4.13 (light gray), $E_{1.2,1}{}^{\downarrow}A$ (medium gray), and $E_{1.2,1}{}^{\circ}A$ (solid black).

(I67) $\quad \mathrm{Preord}(R) \to R^{\uparrow}A \approxeq \bigcap\{X \mid A \subseteq X \,\&\, \mathrm{Ext}_R(X)\}$
(I68) $\quad \mathrm{Preord}(R) \to R^{\downarrow}A \approxeq \bigcup\{X \mid X \subseteq A \,\&\, \mathrm{Ext}_R(X)\}$
(I69) $\quad \mathrm{wPreord}(R) \to R^{\uparrow}A \approxeq \bigcap\{X \mid A \subseteq X \wedge \mathrm{Ext}_R(X)\}$
(I70) $\quad \mathrm{wPreord}(R) \to R^{\downarrow}A \approxeq \bigcup\{X \mid X \subseteq A \wedge \mathrm{Ext}_R(X)\}$
(I71) $\quad \triangle\mathrm{Preord}(R) \to R^{\uparrow}A = \bigcap\{X \mid \triangle(A \subseteq X \,\&\, \mathrm{Ext}_R(X))\}$
(I72) $\quad \triangle\mathrm{Preord}(R) \to R^{\downarrow}A = \bigcup\{X \mid \triangle(X \subseteq A \,\&\, \mathrm{Ext}_R(X))\}$

Proof. (I67) For any X such that $A \subseteq X \,\&\, \mathrm{Ext}_R(X)$ we can infer $R^{\uparrow}A \subseteq X$ from (I22). Hence, the first inclusion $R^{\uparrow}A \subseteq \bigcap\{X \mid A \subseteq X \,\&\, \mathrm{Ext}_R(X)\}$ follows by (L7) of Lemma B.8. Conversely, (I18) and (I21) imply $\mathrm{Preord}(R) \to \mathrm{Ext}_R(R^{\uparrow}A) \,\&\, A \subseteq R^{\uparrow}A$. Then (L8) completes the proof.

The proofs of (I68)–(I70) are analogous. The assertions (I71) and (I72) follow from (I67) and (I68), respectively, if we take basic properties of $\triangle$ into account. □

Remark 4.15 At the beginning of this section, we mentioned the close relationship of images, closures and openings with concepts in fuzzy mathematical morphology. In (crisp) mathematical morphology, images are considered as crisp subsets of an Abelian group $(U, +, \mathbf{0})$ (more commonly, a linear vector space structure is assumed). Given a set A (the *image*) and a set B (the so-called *structuring element*), the four standard operations (on the image A with respect to the structuring element B) can be defined as follows:

$$A \oplus B =_{\mathrm{df}} \{y \mid (\exists x)(Ax \,\&\, B(y-x))\} \qquad \text{(dilation)}$$
$$A \ominus B =_{\mathrm{df}} \{x \mid (\forall y)(B(y-x) \to Ay)\} \qquad \text{(erosion)}$$
$$A \bullet B =_{\mathrm{df}} (A \oplus B) \ominus B \qquad \text{(closure)}$$
$$A \circ B =_{\mathrm{df}} (A \ominus B) \oplus B \qquad \text{(opening)}$$

The language in the definitions above has been chosen intentionally to comply fully with the language of FCT. Thus, if we consider gray level images as $U \to L$ mappings (with the standard case $L = [0,1]$ being the natural choice), we can generalize the four morphological operations to gray level images and gray level structuring elements simply by the above formulae. In the standard case $L = [0,1]$, the well-known t-norm based fuzzy mathematical morphology is obtained [?, 14, 55, 56].

This is not at all new, but it demonstrates that the expressive power of FCT allows rather effortless generalizations—the obvious secret is the commonality of its syntax with classical Boolean logic. As demonstrated in [?], the operations of fuzzy mathematical morphology can be embedded in the concepts of this section in the following way:

(1) If we define a fuzzy relation R as $Rxy = B(y - x)$ for a given structuring element B, then the following four equalities hold:

$$A \oplus B = R^{\uparrow} A \tag{4.1}$$

$$A \ominus B = R^{\downarrow} A \tag{4.2}$$

$$A \bullet B = R^{\bullet} A \tag{4.3}$$

$$A \circ B = R^{\circ} A \tag{4.4}$$

(2) If R is a shift-invariant fuzzy relation, i.e. if

$$(\forall x, y, z)(Rxy \leftrightarrow R(x+z)(y+z))$$

holds, then the equalities (4.1)–(4.4) hold if we define the structuring element B as $Bx = R\mathbf{0}x$.

This relationship particularly implies that we can transfer all results of this section to fuzzy mathematical morphology without any restriction. For the non-graded case, most of these results are already known [?, 24], but it is worth to mention that, hereby, we have generalized fuzzy mathematical morphology to the graded framework almost effortlessly. It may be questionable whether a graded framework of fuzzy mathematical morphology is useful in image processing practice, but it is certainly interesting from a theoretical perspective.

5 Bounds, Maxima, and Suprema

The aim of this section is to study the lattice-like structure induced by a fuzzy relation. We follow the philosophy of Demirci's approach [31, 32]. Note that this is not a classical axiomatic approach to lattices; instead, lattice-theoretical notions are defined on the basis of a given fuzzy relation, where Demirci assumes that fuzzy relation under consideration is a similarity-based fuzzy ordering [15, 47]. As in the previous sections, we do not restrict ourselves to a particular class of fuzzy relations in advance, but we infer gradual results from the degrees to which the relation fulfills some properties (in particular, reflexivity and transitivity).

Throughout this section, assume that R denotes a binary fuzzy relation that is arbitrary, but fixed.

Definition 5.1 The properties of being an *upper* or *lower class* in X with respect to R are defined as follows:

$$\begin{array}{rcl}\mathrm{Upper}_R^X(A) & \equiv_{\mathrm{df}} & (\forall x \in X)(\forall y \in X)[Rxy \to (Ax \to Ay)] \\ \mathrm{Lower}_R^X(A) & \equiv_{\mathrm{df}} & (\forall x \in X)(\forall y \in X)[Rxy \to (Ay \to Ax)]\end{array}$$

Let us further make the conventions $\mathrm{Upper}_R(A) \equiv_{\mathrm{df}} \mathrm{Upper}_R^{\mathrm{V}}(A)$ and $\mathrm{Lower}_R(A) \equiv_{\mathrm{df}} \mathrm{Lower}_R^{\mathrm{V}}(A)$. Further, to ease notation, we omit the lower index R unless we require special properties of R or unless a relation different from the default choice R is used.

Remark 5.2 Note that $\mathrm{Upper}_R(A)$ is in fact nothing else but $\mathrm{Ext}_R(A)$ and that $\mathrm{Lower}_R(A)$ is just $\mathrm{Ext}_{R^{-1}}(A)$. We make this terminological distinction in order to increase readability and to make explicit that we have some preorder-related notions in mind.

Remark 5.3 There is an "inversion duality" between the pairs of notions defined in this section, consisting in the observation that the second notion of each pair is just the first one applied to the inverse relation. Thus, $\mathrm{Lower}_R^X(A) \leftrightarrow \mathrm{Upper}_{R^{-1}}^X(A)$ in Definition 5.1 above, $R^{\triangledown}A = (R^{-1})^{\triangle}A$ in Definition 5.7 below, $\mathrm{Min}_R(A) = \mathrm{Max}_{R^{-1}}(A)$ in Definition 5.9, and $\mathrm{Inf}_R(A) = \mathrm{Sup}_{R^{-1}}(A)$ in Definition 5.14. As the theorems on the dual notions follow trivially by taking R^{-1} for R, we shall usually not write them down explicitly.

As a first simple result, we consider the antitony of (degrees of) upperness and lowerness.

Proposition 5.4 *The following properties are provable in FCT:*

(C1) $(X \subseteq Y)^2 \to (\mathrm{Upper}_R^Y(A) \to \mathrm{Upper}_R^X(A))$
(C2) $(X \subseteq Y)^2 \to (\mathrm{Lower}_R^Y(A) \to \mathrm{Lower}_R^X(A))$

Proof. $(X \subseteq Y)^2$ implies $x \in X \;\&\; y \in X \to x \in Y \;\&\; y \in Y$. Assuming $\mathrm{Upper}_R^Y(A)$, equivalently $(\forall x)(\forall y)(x \in X \;\&\; y \in X \;\&\; Rxy \to (Ax \to Ay))$, we can thus infer $(\forall x)(\forall y)(x \in Y \;\&\; y \in Y \;\&\; Rxy \to (Ax \to Ay)$, which proves (C1). Then (C2) follows trivially by duality. □

Note that in Proposition 5.4 we need to require an assumption twice. The following simple example demonstrates that the proof of Proposition 5.4 cannot be improved in the sense that the "doubled assumption" could only be used once.

Example 5.5 Let us consider standard Łukasiewicz logic and $U = \{x, y\}$ and define fuzzy sets A, X, Y by $Xx = Xy = Ax = 1$, $Yx = Yy = 0.9$ and $Ay = 0.8$. Using the fuzzy relation R defined as $Rxx = Ryy = Ryx = 0$ and $Rxy = 1$, we obtain that $X \subseteq Y$

is true to a degree of 0.9. Furthermore, we have $\mathrm{Upper}_R^X(A) = 0.8$ and $\mathrm{Upper}_R^Y(A) = 1$; thus the truth degree of $X \subseteq Y \to (\mathrm{Upper}_R^Y(A) \to \mathrm{Upper}_R^X(A))$ is only 0.9.

As $X \subseteq \mathrm{V}$ is always true to a degree of 1, we can infer the following simple corollary on upperness from Proposition 5.4 (by the duality of Remark 5.3, we omit the same result for lowerness).

Corollary 5.6 $\mathrm{Upper}_R(A) \to (\forall X)\,\mathrm{Upper}_R^X(A)$

Like in classical mathematics, we can define the classes of all upper (and dually, lower) bounds of a class:

Definition 5.7 The *upper cone* and the *lower cone* of a class A (with respect to R) are defined as follows:

$$\begin{aligned} R^{\triangle}A &=_{\mathrm{df}} \{x \mid (\forall a \in A)Rax\} \\ R^{\triangledown}A &=_{\mathrm{df}} \{x \mid (\forall a \in A)Rxa\} \end{aligned}$$

If we do not suppose any special conditions involving R, we write just ${}^{\triangle}A$ and ${}^{\triangledown}A$ instead of $R^{\triangle}A$ and $R^{\triangledown}A$, respectively.

Note that $R^{\triangle}A$ appears in some literature as an image operator in its own right. It is called *sub-direct image* by some authors (e.g. [29]). In [8], the systematic names of the operators ${}^{\triangle}$ and ${}^{\triangledown}$ are *subproduct image* and *superproduct preimage*, respectively.

Theorem 5.8 *The following properties of cones are provable in* FCT *for an arbitrarily fixed R:*

(C3) $\mathrm{Trans}(R) \to \mathrm{Upper}_R(R^{\triangle}A)$
(C4) $A \subseteq B \to {}^{\triangle}B \subseteq {}^{\triangle}A$
(C5) $A \subseteq {}^{\triangledown\triangle}A$
(C6) ${}^{\triangle\triangledown\triangle}A = {}^{\triangle}A$
(C7) ${}^{\triangle}(A \cup B) \subseteq {}^{\triangle}A \cap {}^{\triangle}B$
(C8) ${}^{\triangle}A \cup {}^{\triangle}B \subseteq {}^{\triangle}(A \cap B)$
(C9) $\bigcap_{A \in \mathcal{A}} {}^{\triangle}A = {}^{\triangle}\Big(\bigcup_{A \in \mathcal{A}} A\Big)$
(C10) $\bigcup_{A \in \mathcal{A}} {}^{\triangle}A \subseteq {}^{\triangle}\Big(\bigcap_{A \in \mathcal{A}} A\Big)$

(Converse inclusions and implications have crisp counter-examples.)

Proof.

(C3) $\mathrm{Trans}(R)$ implies $Rxy \to (Rax \to Ray)$, which implies $Rxy \to ((a \in A \to$

$Rax) \to (a \in A \to Ray))$, whence we get the required assertion $Rxy \to ((\forall a \in A)Rax \to (\forall a \in A)Ray)$ by generalization and quantifier shifts.

(C4) The required $(\forall x \in A)(x \in B) \to (\forall y)[(\forall x \in B)Rxy \to (\forall x \in A)Rxy]$ follows by generalization and distribution of the quantifiers from $(Ax \to Bx) \to [(Bx \to Rxy) \to (Ax \to Rxy)]$.

(C5) The required $a \in A \to (\forall x)((\forall y \in A)Ryx \to Rax)$ follows by generalization from $a \in A \to ((\forall y \in A)Ryx \to Rax)$, which is a variant of the specification axiom $(\forall y)(y \in A \to Ryx) \to (a \in A \to Rax)$.

(C6) By (C5) it is proved that ${}^{\triangle}A \subseteq {}^{\triangle\triangledown}({}^{\triangle}A)$. By (C5) and (C6), it is proved that ${}^{\triangle}({}^{\triangledown\triangle}A) \subseteq {}^{\triangle}A$. By the axiom of extensionality, we are done.

(C7) and (C8) follow directly from the antitony of cones: by (C4), ${}^{\triangle}(A \cup B) \subseteq {}^{\triangle}A$ and ${}^{\triangle}(A \cup B) \subseteq {}^{\triangle}B$, therefore ${}^{\triangle}(A \cup B) \subseteq {}^{\triangle}A \cap {}^{\triangle}B$; analogously for ${}^{\triangle}(A \cap B)$.

(C9) This assertion can be proved as follows:

$$x \in \bigcap_{A \in \mathcal{A}} {}^{\triangle}A \longleftrightarrow (\forall A \in \mathcal{A})(\forall a \in A)Rax$$
$$\longleftrightarrow (\forall a)[(\exists A \in \mathcal{A})(a \in A) \to Rax] \longleftrightarrow x \in {}^{\triangle}\Big(\bigcup_{A \in \mathcal{A}} A\Big)$$

(C10) Similarly to (C9), we can infer the following:

$$x \in \bigcup_{A \in \mathcal{A}} {}^{\triangle}A \longleftrightarrow (\exists A \in \mathcal{A})(\forall a \in A)Rax$$
$$\longrightarrow (\forall a)[(\forall A \in \mathcal{A})(a \in A) \to Rax] \longleftrightarrow x \in {}^{\triangle}\Big(\bigcap_{A \in \mathcal{A}} A\Big),$$

where the middle implication follows from Lemma B.8 (L5) by generalization and appropriate quantifier shifts. □

It is worth mentioning that the following two corollaries can be inferred directly from (C9) and (C10):

(C11) ${}^{\triangle}(A \sqcup B) = {}^{\triangle}A \sqcap {}^{\triangle}B$

(C12) ${}^{\triangle}A \sqcup {}^{\triangle}B \subseteq {}^{\triangle}(A \sqcap B)$

Theorems (C4)–(C12) as well as their duals are also corollaries of more general theorems found in [8] (submitted to this issue). Now let us move closer to the lattice-theoretical notions at which this section aims. First of all, we define maxima and minima.

Definition 5.9 The classes of all *maxima* and *minima* of a class A with respect to R are defined as follows:

$$\mathrm{Max}_R A =_{\mathrm{df}} A \cap (R^{\triangle}A)$$
$$\mathrm{Min}_R A =_{\mathrm{df}} A \cap (R^{\triangledown}A)$$

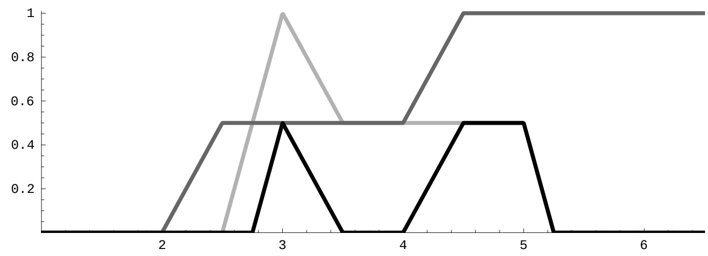

Fig. 5. The fuzzy set A from Example 4.13 (light gray), its upper cone $L_1{}^{\triangle}A$ (medium gray), and its maximum $\mathrm{Max}_{L_1} A$ (solid black).

The index R is dropped under the same conditions as noted above.

Remark 5.10 Observe that Definition 5.9 is just a more compact way of expressing the usual definition of maxima and minima as those elements of A that are larger resp. smaller than all elements in A, i.e.,

(C13) $\mathrm{Max}_R A = \{x \in A \mid (\forall y \in A) Ryx\}$
(C14) $\mathrm{Min}_R A = \{x \in A \mid (\forall y \in A) Rxy\}$

Notice further that since the property of being an upper (or lower) bound is graded in FCT, maxima (minima) have to be defined as fuzzy classes (unlike in classical mathematics, where they are determined uniquely and therefore can be defined as single elements).

Example 5.11 Let us consider the fuzzy set A from Example 4.13 and standard Łukasiewicz logic again. Further consider the fuzzy relation L_1 from Example 3.7 which is a fuzzy preorder [15]. Figure 5 shows A, $L_1{}^{\triangle}A$ and $\mathrm{Max}_{L_1} A$, while Figure 6 shows A, $L_1{}^{\triangledown}A$ and $\mathrm{Min}_{L_1} A$. The results we obtain for the lower cone and the minimum are what one may expect intuitively. Similarly intuitive results are always obtained for (unions of) fuzzy intervals. The results we obtain for the upper cone and the maximum in this case demonstrate, however, that quite peculiar results may be obtained for more unusual fuzzy sets. [5]

As the above example suggests, cones, minima and maxima may not be as intuitive and simple concepts as in classical mathematics. The following theorem demonstrates that still properties hold that one would expect intuitively.

Theorem 5.12 *The following properties of maxima are provable in* FCT*:*

(C15) $A \subseteq B \mathbin{\&} x \in \mathrm{Max}_R A \mathbin{\&} y \in \mathrm{Max}_R B \to Rxy$

[5] Although unusual, the results are nevertheless not counter-intuitive and in Figure 5 they can be explained by the shape of the membership function of A: the gradual decrease of A to the right makes the maximum subnormal (compare it with right-open crisp intervals which have no maximum at all), and the increase of the membership function in the left part induces a second peak of the maximum (as the α-cuts of A for large α have their maxima exactly there).

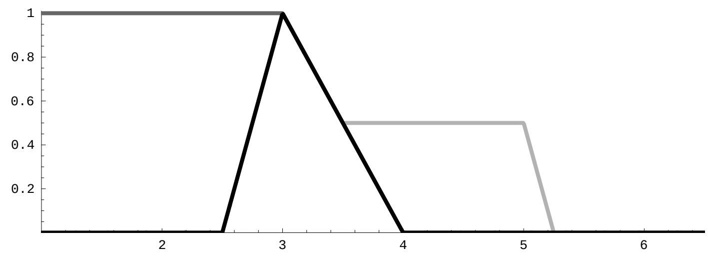

Fig. 6. The fuzzy set A from Example 4.13 (light gray), its lower cone $L_1{}^{\triangledown}A$ (medium gray), and its minimum $\mathrm{Min}_{L_1} A$ (solid black).

(C16) $x \in \mathrm{Max}_R A \,\&\, y \in \mathrm{Max}_R A \to Rxy \,\&\, Ryx$

(C17) $x \in \mathrm{Max}_R A \,\&\, y \in \mathrm{Max}_R A \,\&\, \mathrm{AntiSym}_E R \to Exy$

Proof.

(C15) We have to prove

$$(A \subseteq B) \,\&\, (x \in A \,\&\, x \in R^{\triangle}A) \,\&\, (y \in B \,\&\, y \in R^{\triangle}B) \to Rxy.$$

Now $A \subseteq B \,\&\, y \in R^{\triangle}B$ implies $y \in R^{\triangle}A$ by (C1) which, together with $x \in A$, implies Rxy.

(C16) To prove this, we simply have to combine the antecedents and consequents of the following two trivial assertions

$$x \in A \,\&\, y \in R^{\triangle}A \to Rxy$$
$$y \in A \,\&\, x \in R^{\triangle}A \to Ryx$$

and the proof is completed.

(C17) follows directly from (C16). □

A nonchalant interpretation of (C15) is that the larger (with respect to inclusion) A is, the larger (with respect to R) $\mathrm{Max}_R A$ is. The property (C16) can be interpreted as the fact that $\mathrm{Max}_R A$ is unique up to the symmetrization of R. In the case that, in a non-graded setting, R is a fuzzy preorder, it is easily possible to show that its symmetrization is a similarity [15, 67]. Then (C16) means nothing else than that $\mathrm{Max}_R A$ is a *fuzzy point* [52]. The property (C16) generalizes this to any relation R antisymmetric (to some degree) with respect to E.

The following theorem shows that maxima are upper classes inside the fuzzy class that is considered (to the degree R is transitive).

Theorem 5.13 *The following property of maxima is provable in* FCT*:*

(C18) $\mathrm{Trans}(R) \to \mathrm{Upper}_R^A(\mathrm{Max}_R A)$

Proof. By (C3), Trans(R) implies $x \in R^{\triangle}A$ & $Rxy \to y \in R^{\triangle}A$ which implies the required $x \in A$ & $y \in A$ & Rxy & $(x \in A$ & $x \in R^{\triangle}A) \to (y \in A$ & $y \in R^{\triangle}A)$. □

Now we can finally define suprema and infima. Not surprisingly, the suprema are defined as the least upper bounds, i.e., the minima of the upper cone. Again the condition of being a supremum is graded, as the notion of a bound itself is graded. Dually, the infima are defined as the greatest lower bounds.

Definition 5.14 The classes of all *suprema* and *infima* of a class A with respect to R are defined as follows:

$$\begin{aligned} \mathrm{Sup}_R A &=_{\mathrm{df}} \mathrm{Min}_R(R^{\triangle}A) \\ \mathrm{Inf}_R A &=_{\mathrm{df}} \mathrm{Max}_R(R^{\triangledown}A) \end{aligned}$$

The index R is dropped under the same conditions as noted above.

Obviously, we can rewrite the definitions in the following way:

(C19) $\mathrm{Sup}\,A = {}^{\triangle}A \cap {}^{\triangledown\triangle}A$
(C20) $\mathrm{Inf}\,A = {}^{\triangledown}A \cap {}^{\triangle\triangledown}A$

As shown by the following theorem, suprema and infima are interdefinable.

Theorem 5.15 *The following property of maxima is provable in* FCT*:*

(C21) $\mathrm{Sup}\,A = \mathrm{Inf}\,{}^{\triangle}A$

Proof. By (C20) and (C6), $\mathrm{Inf}\,{}^{\triangle}A = {}^{\triangledown\triangle}A \cap {}^{\triangle\triangledown\triangle}A = {}^{\triangledown\triangle}A \cap {}^{\triangle}A = \mathrm{Min}\,{}^{\triangle}A = \mathrm{Sup}\,A$. □

Since suprema are a special kind of minima, the general properties of the latter hold for suprema as well; further properties of suprema hold by virtue of the properties of cones. Some of such properties of suprema are summarized in the following theorem.

Theorem 5.16 *The following properties of maxima are provable in* FCT*:*

(C22) $A \subseteq B$ & $x \in \mathrm{Sup}_R A$ & $y \in \mathrm{Sup}_R B \to Rxy$
(C23) $x \in \mathrm{Sup}_R A$ & $y \in \mathrm{Sup}_R A \to Rxy$ & Ryx
(C24) $x \in \mathrm{Sup}_R A$ & $y \in \mathrm{Sup}_R A$ & $\mathrm{AntiSym}_E\, R \to Exy$
(C25) $\mathrm{Trans}(R) \to \mathrm{Upper}^{A}_{R}(\mathrm{Sup}_R A)$
(C26) $\mathrm{Trans}(R) \to \mathrm{Lower}^{R^{\triangle}A}_{R}(\mathrm{Sup}_R A)$

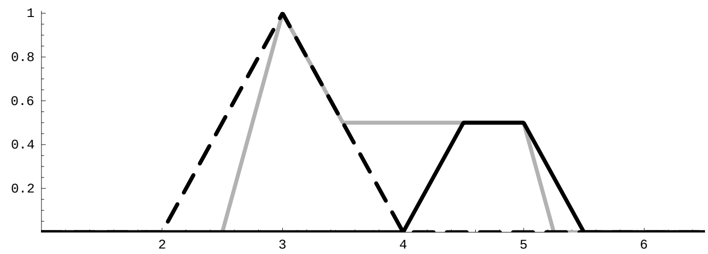

Fig. 7. The fuzzy set A from Example 4.13 (light gray), its infimum $\mathrm{Inf}_{L_1} A$ (dashed black), and its supremum $\mathrm{Sup}_{L_1} A$ (solid black).

Proof.

(C22) Follows from (C4) and the dual of (C15).
(C23) and (C24) follow respectively from the duals of (C16) and (C17).
(C25) By (C3), $\mathrm{Trans}(R)$ implies $x \in {}^{\triangle}A \mathbin{\&} Rxy \to y \in {}^{\triangle}A$. Furthermore, by (C5), $y \in A \to y \in {}^{\triangledown\triangle}A$. Combining the antecedents and consequents of these implications we get the required $x \in A \mathbin{\&} y \in A \mathbin{\&} Rxy \mathbin{\&} (x \in {}^{\triangle}A \mathbin{\&} x \in {}^{\triangledown\triangle}A) \to (y \in {}^{\triangle}A \mathbin{\&} y \in {}^{\triangledown\triangle}A)$.
(C26) Follows from the dual of (C18). □

Suprema differ from maxima already in crisp sets. The following example shows how the difference may look like in fuzzy sets.

Example 5.17 Let us revisit Example 5.11. Figure 7 shows the fuzzy set A along with $\mathrm{Inf}_{L_1} A$ and $\mathrm{Sup}_{L_1} A$. (Compare with Figures 5 and 6.)

The following theorem provides us with two results on how suprema and maxima are related to each other. For the precondition $A \subseteq A \cap A$ in (C28) see Remark 3.10.

Theorem 5.18 *The following interrelations between maxima and suprema are provable in* FCT*:*

(C27) $A \cap \mathrm{Max} A \subseteq A \cap \mathrm{Sup} A \subseteq \mathrm{Max} A$
(C28) $A \subseteq A \cap A \to \mathrm{Max} A \approx A \cap \mathrm{Sup} A$

Proof.

(C27) Using (C5), we can infer $A \cap {}^{\triangle}A \cap A \subseteq A \cap {}^{\triangle}A \cap {}^{\triangledown\triangle}A \subseteq A \cap {}^{\triangle}A$.
(C28) $A \subseteq A \cap A \to A \cap {}^{\triangle}A \subseteq A \cap {}^{\triangle}A \cap A$ which, together with the proof of (C27), yields the converse implication to $A \cap \mathrm{Sup} A \subseteq \mathrm{Max} A$ of (C27). □

By means of suprema and infima, the notion of lattice completeness can be defined [32]. A systematic study of complete lattices and fuzzy lattice completions in

FCT will be part of a subsequent paper. For some particular cases, see [2].

6 Valverde-Style Characterizations of Preorders and Similarities

This section aims at generalizing some of the most important and influential theorems in the theory of fuzzy relations to FCT—Valverde's representation theorems for fuzzy preorders and similarities [67]. In the tradition of Cantor [21], Valverde uses score functions to represent relations. Actually, he uses score functions that map into the unit interval, so these functions can also be considered as fuzzy sets. This interpretation facilitates an easy reformulation of these results in FCT.

Let us first consider the fuzzy relation R^ℓ defined as

$$R^\ell xy \equiv_{\mathrm{df}} (\forall z)(Rzx \to Rzy)$$

(for a given fuzzy relation R). This is called the *left trace* of R [35,36]. Analogously we define the *right trace* (which will be used in Section 7) as

$$R^r xy \equiv_{\mathrm{df}} (\forall z)(Ryz \to Rxz).$$

Observe the meaning of the following expressions:

$$R^\ell \subseteq R \leftrightarrow (\forall x,y)[(\forall z)(Rzx \to Rzy) \to Rxy] \tag{6.1}$$
$$R \subseteq R^\ell \leftrightarrow (\forall x,y)[Rxy \to (\forall z)(Rzx \to Rzy)] \tag{6.2}$$
$$R \approx R^\ell \leftrightarrow (\forall x,y)[Rxy \leftrightarrow (\forall z)(Rzx \to Rzy)] \tag{6.3}$$

Now we can formulate another characterization of graded reflexivity and transitivity besides those of Theorem 3.5.

Theorem 6.1 *The following properties hold in* FCT*:*

(V1) $\mathrm{Refl}(R) \leftrightarrow R^\ell \subseteq R$
(V2) $\mathrm{Trans}(R) \leftrightarrow R \subseteq R^\ell$

Proof.

(V1) To prove the first implication, we need to show that Rxy is implied by $\mathrm{Refl}(R)$ and $(\forall z)(Rzx \to Rzy)$. Specifying x for z in the latter, we get $Rxx \to Rxy$, which implies Rxy by $\mathrm{Refl}(R)$. To prove the converse implication, we can specify x for y in (6.1) and get $(\forall x)[(\forall z)(Rzx \to Rzx) \to Rxx]$, i.e. $(\forall x)(1 \to Rxx)$, i.e. $(\forall x)Rxx$.

(V2) $\mathrm{Trans}(R) \longleftrightarrow (\forall z,x,y)(Rzx \,\&\, Rxy \to Rzy) \longleftrightarrow (\forall x,y)(\forall z)[Rxy \to (Rzx \to Rzy)] \longleftrightarrow (\forall x,y)[Rxy \to (\forall z)(Rzx \to Rzy)]$ □

Corollary 6.2 *The following is provable in* FCT*:*

(V3) $\mathrm{wPreord}(R) \leftrightarrow R \approx R^{\ell}$
(V4) $\mathrm{Preord}(R) \leftrightarrow R \approxeq R^{\ell}$,
(V5) $R \approx^2 R^{\ell} \longrightarrow \mathrm{Preord}(R) \longrightarrow R \approx R^{\ell}$.

So we have obtained graded versions of Fodor's characterizations [35, Theorems 4.1, 4.3, and Corollary 4.4]. Note that, regardless of the symmetry of R, we can replace R^{ℓ} in the above characterizations by the right trace as well.

Remark 6.3 Observe that the following holds obviously (cf. Definitions B.7 and 5.7):

$$R^{\leftarrow}\{x\} = \{z \mid (\exists y \in \{x\})Rzy\} = \{z \mid Rzx\}$$
$$R^{\triangledown}\{x\} = \{z \mid (\forall a \in \{x\})Rza\} = \{z \mid Rzx\}$$

So we can rewrite (V3) as follows:

$$\mathrm{wPreord}(R) \leftrightarrow (\forall x,y)\big(Rxy \leftrightarrow R^{\leftarrow}\{x\} \subseteq R^{\leftarrow}\{y\}\big)$$
$$\mathrm{wPreord}(R) \leftrightarrow (\forall x,y)\big(Rxy \leftrightarrow R^{\triangledown}\{x\} \subseteq R^{\triangledown}\{y\}\big)$$

In words, a relation R is a weak preorder to the degree it coincides with graded inclusion between the cones (or preimages) of crisp singletons.

Now we have all prerequisites for formulating and proving a graded version of Valverde's representation theorem for preorders. In order to make notations more compact, let us define two properties of *Valverde preorder representability* (a strong one and a weak one) for a given fuzzy relation R as

$$\mathrm{ValP}(R) \equiv_{\mathrm{df}} (\exists \mathcal{A})(R \approxeq \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \to Ay)\})$$
$$\mathrm{wValP}(R) \equiv_{\mathrm{df}} (\exists \mathcal{A})(R \approx \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \to Ay)\})$$

Then we can prove the following essential result for preorders and weak preorders.

Theorem 6.4 FCT *proves the following:*

(V6) $\mathrm{ValP}^2(R) \longrightarrow \mathrm{Preord}(R) \longrightarrow \mathrm{ValP}(R)$
(V7) $\mathrm{wValP}^3(R) \longrightarrow \mathrm{wPreord}(R) \longrightarrow \mathrm{wValP}(R)$

Proof. We prove just (V6), the proof of (V7) is analogous. To show the first implication we define $S_{\mathcal{A}} = \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \to Ay)\}$. If we show $\mathrm{Preord}(S_{\mathcal{A}})$, then the application of (R24) and some quantifier shifts complete the proof.

Obviously, $\mathrm{Refl}(S_{\mathcal{A}})$ is a theorem, now we show $\mathrm{Trans}(S_{\mathcal{A}})$: $S_{\mathcal{A}}xy \mathbin{\&} S_{\mathcal{A}}yz \longleftrightarrow (\forall A \in \mathcal{A})(Ax \to Ay) \mathbin{\&} (\forall A \in \mathcal{A})(Ay \to Az) \longrightarrow [(Ax \to Ay) \mathbin{\&} (Ay \to Az)] \longrightarrow (Ax \to Az)$. By generalization we get: $S_{\mathcal{A}}xy \mathbin{\&} S_{\mathcal{A}}yz \longrightarrow (\forall A \in \mathcal{A})(Ax \to Az) \longleftrightarrow S_{\mathcal{A}}xz$.

To prove the second implication just take $\mathcal{A} = \{A \mid (\exists z)(A = \{x \mid Rzx\})\}$ and use (V4). □

Obviously, (V6) is more complicated than Valverde's original result; it is an example where the graded framework does not provide us with just a plain copy of the non-graded (or crisp) result. The following corollary gives us a result that is comparable with Valverde's original theorem.

Corollary 6.5 FCT *proves the following:*

(V8) $$\begin{aligned} &\triangle\,\mathrm{Preord}(R) \longleftrightarrow \triangle\,\mathrm{wPreord}(R) \longleftrightarrow R = R^{\ell} \\ &\longleftrightarrow \triangle\,\mathrm{ValP}(R) \longleftrightarrow \triangle\,\mathrm{wValP}(R) \\ &\longleftrightarrow (\exists \mathcal{A})\ (R = \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \to Ay)\}) \\ &\longleftrightarrow (\exists \mathcal{A})(\mathrm{Crisp}(\mathcal{A}) \mathbin{\&} (R = \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \to Ay)\})) \end{aligned}$$

Proof. The first four equivalences are trivial consequences of results above, to prove the last two we prove three implications: clearly the seventh formula implies the sixth one and that the sixth one implies the fifth one. To complete the proof we show that

$$\triangle\,\mathrm{Preord}(R) \to (\exists \mathcal{A})(R = \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \to Ay)\}).$$

Take $\mathcal{A} = \{A \mid (\exists z)(A = \{x \mid Rzx\})\}$ and apply $\triangle$-necessitation to (V4). □

Although the last formula in (V8) is a perfect copy of Valverde's non-graded representation, the corollary still has graded elements—note that in the sixth equivalent formula, the class $\mathcal{A}$ may be a fuzzy class of fuzzy classes (unlike Valverde's theorem, in which a crisp family of functions is used). The degree of $A \in \mathcal{A}$ may be considered as a weighting factor that controls the influence of a specific A on the final result.

Example 6.6 Let us shortly revisit Example 3.2 (in which we use standard Łukasiewicz logic). The fuzzy relation P_1 was actually constructed from the following crisp family of three fuzzy sets $\mathcal{A} = \{A_1, A_2, A_3\}$ that are defined as follows (for convenience, in vector notation):

$$\begin{aligned} A_1 &= (0.7, 0.8, 0.2, 0.5, 0.4, 0.6) \\ A_2 &= (0.3, 0.5, 0.6, 0.4, 0.7, 1.0) \\ A_3 &= (1.0, 1.0, 0.6, 0.4, 0.3, 0.0) \end{aligned}$$

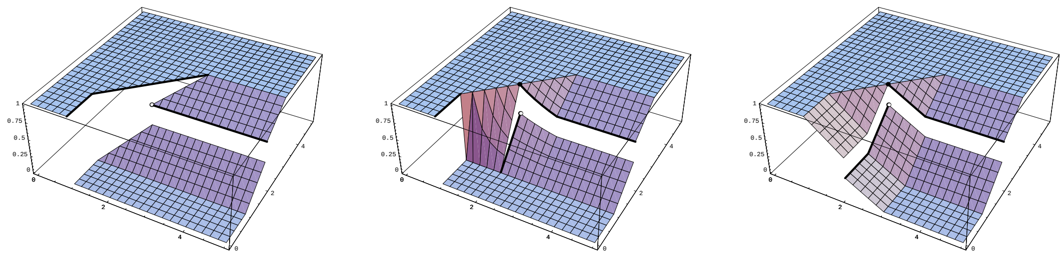

Fig. 8. Plots of fuzzy preorders that are obtained if we interpret (6.4) in standard Gödel logic (left), in standard product logic (middle), and in standard Łukasiewicz logic (right).

Example 6.7 In [17], some Valverde-style constructions are investigated. For illustrative purpose, let us quote the following example with $U = \mathbb{R}$. We consider a one-element crisp family $\mathcal{B} = \{B\}$, where the fuzzy set B is defined as follows:

$$Bx = \begin{cases} 0 & \text{if } x \in [0,1[\\ 0.4 \cdot (x-1) & \text{if } x \in [1,2[\\ 0.7 + 0.3 \cdot (x-2) & \text{if } x \in [2,3[\\ 1 & \text{if } x \in [3,5] \end{cases}$$

Figure 8 shows the three fuzzy preorders that are obtained if we interpret the definition

$$R = \{\langle x,y\rangle \mid (\forall C \in \mathcal{B})(Cx \to Cy)\} = \{\langle x,y\rangle \mid Bx \to By\} \qquad (6.4)$$

respectively in standard Gödel logic, standard product logic, and standard Łukasiewicz logic.

In his landmark paper [67], Valverde not only considers fuzzy preorders, but also similarities (as obvious from the title of this paper). So the question naturally arises how we can modify the above results in the presence of symmetry. As will be seen next, the modifications are not as straightforward as in the non-graded case. Let us first define the fuzzy relation $R^{\ell s}$ as

$$R^{\ell s}xy =_{\mathrm{df}} (\forall z)(Rzx \leftrightarrow Rzy)$$

(for a given fuzzy relation R). This is called the *left symmetric trace* of R.

The following lemma demonstrates how this notion is related to the defining properties of similarity. More or less unexpectedly, the result is not that straightforward for symmetry.

Theorem 6.8 *The following are theorems of* FCT*:*

(V9) $R^{\ell s} \subseteq R \leftrightarrow \mathrm{Refl}(R)$
(V10) $R \subseteq R^{\ell s} \to \mathrm{Trans}(R)$
(V11) $R \approxeq R^{\ell s} \to \mathrm{Sym}(R)$
(V12) $\mathrm{Sym}(R) \,\&\, \mathrm{Trans}(R) \to R \subseteq R^{\ell s}$

Proof.

(V9) Analogous to the proof of (V1).

(V10) Follows from (V2) by observation that $R^{\ell s} \subseteq R^{\ell}$.

(V11) Obviously we can get $R \subseteq R^{\ell s} \longrightarrow (Rxy \to (Ryx \leftrightarrow Ryy)) \longrightarrow (Rxy \to (Ryy \to Ryx))$. So $R \subseteq R^{\ell s} \,\&\, \mathrm{Refl}(R) \to (\forall y)(Rxy \to Ryx)$. Finally (V9) completes the proof.

(V12) We need to show that $Rzx \leftrightarrow Rzy$ is implied by $\mathrm{Sym}(R)$, $\mathrm{Trans}(R)$, and Rxy. First by $\mathrm{Trans}(R)$ and Rxy we get $Rzx \to Rzy$; secondly, by $\mathrm{Sym}(R)$ and Rxy we get Ryx, whence by $\mathrm{Trans}(R)$ we get $Rzy \to Rzx$. □

The following theorem provides us with an analogue of Corollary 6.2, unfortunately, with looser bounds on the left-hand side.

Corollary 6.9 FCT *proves:*

(V13) $R \approx^4 R^{\ell s} \longrightarrow R \approxeq^2 R^{\ell s} \longrightarrow \mathrm{Sim}(R) \longrightarrow R \approxeq R^{\ell s} \longrightarrow R \approx R^{\ell s}$

(V14) $R \approx^2 R^{\ell s} \longrightarrow R \approxeq R^{\ell s} \longrightarrow \mathrm{wSim}(R)$

(V15) $\mathrm{wSim}^2(R) \longrightarrow R \approx R^{\ell s}$

The question arises whether it is really necessary to require $\approxeq$ rather than $\approx$ in (V11). The following example tells us that this is indeed the case. It also implies that $R \approx R^{\ell s} \to \mathrm{wSim}(R)$ does *not* hold in general.

Example 6.10 Consider $U = \{1,2\}$, standard Łukasiewicz logic, and the following fuzzy relation:

$$R = \begin{pmatrix} 0.5 & 1.0 \\ 0.0 & 0.5 \end{pmatrix}$$

It is obvious that $\mathrm{Refl}(R) = 0.5$ and $\mathrm{Sym}(R) = 0$. Moreover, routine calculations show that $\mathrm{Trans}(R) = 1$. To compute $R \approxeq R^{\ell s}$, we have to consider the truth values of $Rxy \leftrightarrow (\forall z)(Rzx \leftrightarrow Rzy)$ for all $x, y \in U$:

$$\begin{array}{ll}
x=1, y=1: & \min(\overbrace{0.5 \leftrightarrow (0.5 \leftrightarrow 0.5)}^{z=1}, \overbrace{0.5 \leftrightarrow (0.0 \leftrightarrow 0.0)}^{z=2}) = 0.5 \\
x=1, y=2: & \min(1.0 \leftrightarrow (0.5 \leftrightarrow 1.0), 1.0 \leftrightarrow (1.0 \leftrightarrow 0.5)) = 0.5 \\
x=2, y=1: & \min(0.0 \leftrightarrow (1.0 \leftrightarrow 0.5), 0.0 \leftrightarrow (0.5 \leftrightarrow 0.0)) = 0.5 \\
x=2, y=2: & \min(0.5 \leftrightarrow (1.0 \leftrightarrow 1.0), 0.5 \leftrightarrow (0.5 \leftrightarrow 0.5)) = 0.5
\end{array}$$

So, we finally obtain $R \approx R^{\ell s} = 0.5$ and $R \approxeq R^{\ell s} = 0$.

Now we can formulate a graded version of Valverde's representation theorem for similarities. Analogously to the above considerations, let us define the property of *Valverde similarity representability* (strong one and weak one) for a given fuzzy relation R as

$$\mathrm{ValS}(R) =_{\mathrm{df}} (\exists\mathcal{A})(R \approxeq \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \leftrightarrow Ay)\}),$$
$$\mathrm{wValS}(R) =_{\mathrm{df}} (\exists\mathcal{A})(R \approx \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \leftrightarrow Ay)\}).$$

In the same way as for preorders, we can prove Valverde's representation theorem of similarities and weak similarities.

Theorem 6.11 FCT *proves the following:*

(V16) $\mathrm{ValS}^3(R) \longrightarrow \mathrm{Sim}(R) \longrightarrow \mathrm{ValS}(R)$
(V17) $\mathrm{wValS}^3(R) \rightarrow \mathrm{wSim}(R)$
(V18) $\mathrm{wSim}^2(R) \rightarrow \mathrm{wValS}(R)$

Again, (V16) is more complicated than Valverde's original representation of similarities. In the following corollary, analogously to preorders, we can infer a result very similar to Valverde's original theorem in case that the corresponding properties are fulfilled to degree 1.

Corollary 6.12 FCT *proves the following:*

(V19) $\triangle\,\mathrm{Sim}(R) \longleftrightarrow \triangle\,\mathrm{wSim}(R) \longleftrightarrow R = R^{\ell s}$
$\longleftrightarrow \triangle\,\mathrm{ValS}(R) \longleftrightarrow \triangle\,\mathrm{wValS}(R)$
$\longleftrightarrow (\exists\mathcal{A})\ (R = \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \leftrightarrow Ay)\})$
$\longleftrightarrow (\exists\mathcal{A})(\mathrm{Crisp}(\mathcal{A})\ \&\ (R = \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \leftrightarrow Ay)\}))$

Again, like in the case of preorders, (V19) has a graded ingredient—the class $\mathcal{A}$ may be a fuzzy class of fuzzy classes.

Example 6.13 Consider $U = [0,3]$, standard Łukasiewicz logic, and the following four fuzzy sets:

$$A_1x = \max(0,\min(1,x))$$
$$A_2x = \max(0,\min(1,x-1))$$
$$A_3x = \max(0,\min(1,x-2))$$
$$A_4x = \max(0,\min(1,x-3))$$

Figure 9 shows plots of two fuzzy similarities that we obtain by the construction that is provided by (V19):

$$E_1xy = (\forall A \in \mathcal{A}_1)(Ax \leftrightarrow Ay)$$
$$E_2xy = (\forall A \in \mathcal{A}_2)(Ax \leftrightarrow Ay)$$

where $\mathcal{A}_1 = \{A_1,A_2,A_3,A_4\}$, i.e. a crisp finite family of fuzzy sets. Hence, E_1 is the fuzzy relation obtained from Valverde's original construction. The fuzzy class $\mathcal{A}_2$, however, is defined such that $\mathcal{A}_2A_1 = \mathcal{A}_2A_3 = \mathcal{A}_2A_4 = 1$ and $\mathcal{A}_2A_2 = 0.6$, i.e. we assign a lower weight of 0.6 to the second fuzzy set.

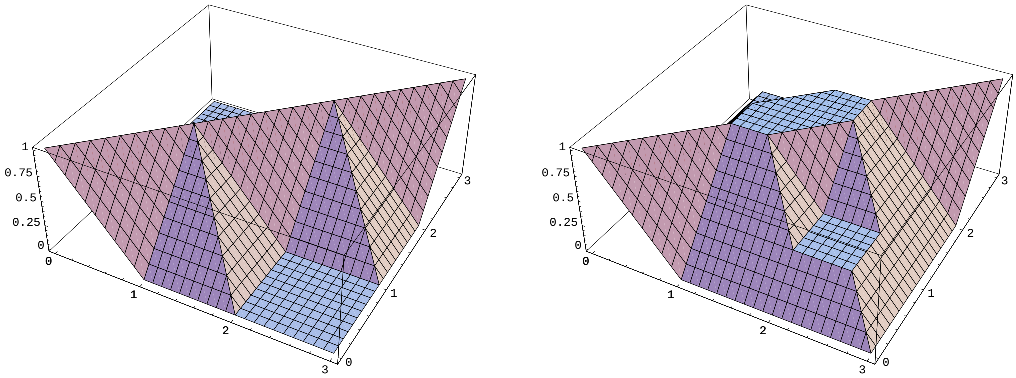

Fig. 9. Plots of the two fuzzy relations E_1 (left) and E_2 (right) from Example 6.13.

7 Similarities and Partitions

The one-to-one correspondence between equivalence relations and partitions is one of the most fundamental correspondences in classical mathematics. It is clear, therefore, that fuzzy partitions have been studied intensively in connection with similarity relations. The first approach to fuzzy partitions by Ruspini [63] does not facilitate a direct correspondence with similarity relations. Only more logically oriented approaches to fuzzy partitions that were introduced more recently are able to provide a smooth interplay with similarity relations. In this section, we demonstrate how the well-accepted (non-graded) approach by De Baets and Mesiar [26] (for similar or complementary studies, see also [12, 30, 41, 42, 45, 51, 52]) can be transferred to our graded framework.

Definition 7.1 Consider a fuzzy relation R. For a given element x, we define the *afterset* of x (with respect to R) as

$$[x]_R =_{\mathrm{df}} \{y \mid Rxy\}.$$

It is clear that, if R is a similarity, $[x]_R$ can be understood as the *equivalence class* of x. Note that Gottwald, in his studies [40, 41], defines the equivalence class of x inversely as the *foreset* $\{y \mid Ryx\}$. We stick to the afterset-based definition in this section. The choice is immaterial, since the aftersets of R are the foresets of R^{-1} and vice versa, and R and R^{-1} satisfy Refl, Sym, and Trans both to the same degrees (see Section 3).

The following lemma provides us with some easy-to-see links to concepts we have introduced earlier in this paper.

Lemma 7.2 *The following properties of aftersets are provable in* FCT*:*

(P1) $[x]_R = R^{\triangle}\{x\} = R^{\uparrow}\{x\}$

(P2) $[x]_R \subseteq [y]_R \longleftrightarrow (\forall z)(Rxz \to Ryz) \longleftrightarrow R^r xy$

Now we can prove some basic properties of aftersets (note that semantically equivalent results for left-continuous t-norms can be found in [41, Section 18.6]).

Theorem 7.3 *The following properties are provable in* FCT*:*

(P3) $\mathrm{Refl}(R) \leftrightarrow (\forall x)(x \in [x]_R)$
(P4) $\mathrm{Refl}(R) \leftrightarrow (\forall x,y)([x]_R \subseteq [y]_R \to Rxy)$
(P5) $\mathrm{Refl}(R)\ \&\ \mathrm{Sym}(R) \to (\forall x,y)([y]_R \subseteq [x]_R \to Rxy)$
(P6) $\mathrm{Refl}(R) \to (\forall x,y)([x]_R \approx [y]_R \to Rxy)$
(P7) $\mathrm{Refl}^2(R)\ \&\ \mathrm{Sym}(R) \to (\forall x,y)([y]_R \approxeq [x]_R \to R^2xy)$
(P8) $\mathrm{Trans}(R) \leftrightarrow (\forall x,y)(Rxy \to [y]_R \subseteq [x]_R)$
(P9) $\mathrm{Trans}(R)\ \&\ \mathrm{Sym}(R) \to (\forall x,y)(Rxy \to [x]_R \subseteq [y]_R)$
(P10) $\mathrm{Trans}(R)\ \&\ \mathrm{Sym}(R) \to (\forall x,y)(Rxy \to [x]_R \approx [y]_R)$
(P11) $\mathrm{Trans}^2(R)\ \&\ \mathrm{Sym}(R) \to (\forall x,y)(R^2xy \to [x]_R \approxeq [y]_R)$

Proof.

(P3) Follows directly from the definition of $\mathrm{Refl}(R)$.
(P4) Follows from $\mathrm{Refl}(R) \leftrightarrow R^r \subseteq R$ (compare with (V1)) and (P2).
(P5) Take (P4) and apply symmetry.
(P6) Trivial consequence of (P4).
(P7) Use (P4) and (P5).
(P8) Follows from $\mathrm{Trans}(R) \leftrightarrow R \subseteq R^r$ (compare with (V2)) and (P2).
(P9) Use (P8) and symmetry.
(P10) and (P11) both follow from (P8) and (P9). □

From Theorem 7.3, we can now infer a first important result—that similarities can be represented by their aftersets (i.e., equivalence classes).

Corollary 7.4 *The following can be proved in* FCT*:*

(P12) $\mathrm{Sim}(R) \to (\forall x,y)(Rxy \leftrightarrow [x]_R \approx [y]_R)$
(P13) $\mathrm{Sim}^2(R) \to (\forall x,y)(R^2xy \leftrightarrow [x]_R \approxeq [y]_R)$

In classical mathematics, the notion of quotient set is essential for the study of the correspondence between equivalence relations and partitions. As also in previous literature, we define quotient classes in perfect analogy to the crisp case.

Definition 7.5 For a given fuzzy relation R, we define the *quotient class* V/R as the class of all aftersets (equivalence classes):

$$\mathrm{V}/R =_{\mathrm{df}} \{A \mid (\exists x)(A = [x]_R)\}$$

It is clear that the name *quotient class* is best justified if R is a similarity. Let $\mathcal{A}$ be a class of (fuzzy) classes resulting from some similarity in this way. By investigating properties of $\mathcal{A}$, we found four constituting properties: crispness, normality of its elements, covering, and disjointness (in a wider sense). They are defined as follows.

Definition 7.6 Let $\mathcal{A}$ be a fuzzy class of fuzzy classes. We define the following properties of $\mathcal{A}$:

$$\begin{aligned}
\mathrm{NormM}(\mathcal{A}) &\equiv_{\mathrm{df}} (\forall A \in \mathcal{A})(\exists x)\triangle Ax \\
\mathrm{Cover}(\mathcal{A}) &\equiv_{\mathrm{df}} (\forall x)(\exists A \in \mathcal{A})\triangle Ax \\
\mathrm{Disj}(\mathcal{A}) &\equiv_{\mathrm{df}} (\forall A, B \in \mathcal{A})(A \parallel B \to A \approx B)
\end{aligned}$$

Correspondingly, we can define the degree to which $\mathcal{A}$ is a partition as

$$\mathrm{Part}(\mathcal{A}) \equiv_{\mathrm{df}} \mathrm{Crisp}(\mathcal{A}) \,\&\, \mathrm{NormM}(\mathcal{A}) \,\&\, \mathrm{Cover}(\mathcal{A}) \,\&\, \mathrm{Disj}(\mathcal{A})$$

The first three properties are self-explanatory, $\mathrm{Disj}(\mathcal{A})$ is a straightforward (graded) generalization of the disjointness criterion that is well-known from the literature [26, 45, 51, 52]. Without explicitly referring to this as a notion of fuzzy partition, some authors [45, 51, 52] study the disjointness property in conjunction with normality (and crispness, as they are working in a non-graded framework). The covering property was later introduced by De Baets and Mesiar [26] and similarly studied by Demirci [30] and Bělohlávek [12]. The degree $\mathrm{Part}(\mathcal{A})$ to which a class of classes $\mathcal{A}$ is a partition is thus a straightforward (graded) generalization of the concept of *T-partition* introduced by De Baets and Mesiar [26]. [6]

Observe that the properties $\mathrm{Crisp}(\mathcal{A}), \mathrm{NormM}(\mathcal{A})$, and $\mathrm{Cover}(\mathcal{A})$ are crisp. Thus, we have

$$\mathrm{Part}(\mathcal{A}) \leftrightarrow \mathrm{Crisp}(\mathcal{A}) \wedge \mathrm{NormM}(\mathcal{A}) \wedge \mathrm{Cover}(\mathcal{A}) \wedge \mathrm{Disj}(\mathcal{A}),$$

i.e. there is no need to define a separate concept of a "weak fuzzy partition". Moreover, it follows that

$$(\mathrm{Part}(\mathcal{A}) \leftrightarrow 0) \vee (\mathrm{Part}(\mathcal{A}) \leftrightarrow \mathrm{Disj}(\mathcal{A})).$$

In other words, the truth value of $\mathrm{Part}(\mathcal{A})$ for a given $\mathcal{A}$ is either 0 or equal to the truth value of $\mathrm{Disj}(\mathcal{A})$.

---

[6] An alternative option in Definition 7.5 is taking $\{A \mid (\exists x)(A \approx [x]_R)\}$ for the quotient class. This would yield a meaningful, fully fuzzified notion of quotient class and the results of this section would only need a slight adaptation (the $\triangle$'s in Definition 7.6 could be dropped in exchange for some more exponents in definitions and proofs). The usage of $=$ in Definition 7.5 is motivated mainly by keeping the direct correspondence with De Baets and Mesiar's notion.

Theorem 7.7 FCT *proves the following properties of the quotient* V/R*:*

(P14) $\mathrm{Crisp}(\mathrm{V}/R)$
(P15) $\triangle\,\mathrm{Refl}(R) \rightarrow \mathrm{Cover}(\mathrm{V}/R)$
(P16) $\triangle\,\mathrm{Refl}(R) \rightarrow \mathrm{NormM}(\mathrm{V}/R)$
(P17) $\mathrm{Trans}^2(R)\ \&\ \mathrm{Sym}(R) \rightarrow \mathrm{Disj}(\mathrm{V}/R)$
(P18) $\mathrm{Trans}^2(R)\ \&\ \mathrm{Sym}(R)\ \&\ \triangle\,\mathrm{Refl}(R) \rightarrow \mathrm{Part}(\mathrm{V}/R)$

Proof.

(P14)–(P16) are straightforward to prove.

(P17) From $\mathrm{Trans}(R)$ and $\mathrm{Sym}(R)$, we get $Ryx\ \&\ Rzx \rightarrow Ryz$, which, using the definition, can be written as $x \in [y]_R\ \&\ x \in [z]_R \rightarrow Ryz$. Using (P8) and $\mathrm{Trans}(R)$ again, we get $x \in [y]_R\ \&\ x \in [z]_R \rightarrow [y]_R \subseteq [z]_R$. In the same way, we get $x \in [z]_R\ \&\ x \in [y]_R \rightarrow [z]_R \subseteq [y]_R$. Combining these two formulae, we get $\mathrm{Trans}^2(R)\ \&\ \mathrm{Sym}(R) \rightarrow (x \in [z]_R\ \&\ x \in [y]_R \rightarrow [z]_R \approx [y]_R)$. Then applying generalization (for z), quantifier shifts, and the definition of $\|$ completes the proof.

(P18) Immediate consequence of (P14)–(P17). □

Now, after we have studied the properties of the quotient of a given fuzzy relation, the question arises how we can extract a fuzzy relation (a similarity in the ideal case) from a given fuzzy partition.

Definition 7.8 For a given fuzzy class of fuzzy classes $\mathcal{A}$ we define a fuzzy relation $R^{\mathcal{A}}$ in the following way:

$$R^{\mathcal{A}} =_{\mathrm{df}} \{\langle x,y\rangle \mid (\exists A \in \mathcal{A})(Ax\ \&\ Ay)\}$$

Note that the definition $R^{\mathcal{A}}$ is not the only possible definition of how to "extract" a fuzzy relation from a family of subsets. Another often-used way to do that is

$$R = \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \leftrightarrow Ay)\}$$

(see [26, 51, 67] and many other papers). The latter actually means that we define a fuzzy relation R such that $\mathrm{ValS}(R)$ is fulfilled (to a degree of 1). Note, however, that $\triangle\,\mathrm{ValS}(R) \longleftrightarrow \triangle\,\mathrm{wValS}(R) \longleftrightarrow \triangle\,\mathrm{Sim}(R)$ holds by (V19). Thus we obtain a similarity regardless of the properties of the class $\mathcal{A}$. So this definition would not allow us to relate properties of partitions with properties of the induced relations in a meaningful graded manner. That is why we use the construction $R^{\mathcal{A}}$. The following theorem provides us with these relationships.

Theorem 7.9 *The following properties of* $R^{\mathcal{A}}$ *are provable in* FCT*:*

(P19) $\mathrm{Sym}(R^{\mathcal{A}})$
(P20) $\mathrm{Cover}(\mathcal{A}) \to \triangle\,\mathrm{Refl}(R^{\mathcal{A}})$
(P21) $\mathrm{Disj}(\mathcal{A}) \to \mathrm{Trans}(R^{\mathcal{A}})$
(P22) $\mathrm{Part}(\mathcal{A}) \longrightarrow \triangle\,\mathrm{Sym}(R^{\mathcal{A}})\ \&\ \triangle\,\mathrm{Refl}(R^{\mathcal{A}})\ \&\ \mathrm{Trans}(R^{\mathcal{A}}) \longrightarrow \mathrm{Sim}(R^{\mathcal{A}})$

Proof.

(P19) Trivial.
(P20) $\mathrm{Cover}(\mathcal{A}) \longrightarrow (\exists X \in \mathcal{A})\triangle(x \in X) \longrightarrow \triangle(\exists X \in \mathcal{A})(x \in X\ \&\ x \in X) \longleftrightarrow \triangle R^{\mathcal{A}}xx$
(P21) From $R^{\mathcal{A}}xy\ \&\ R^{\mathcal{A}}yz$, we get $(\exists A, B \in \mathcal{A})(Ax\ \&\ Ay\ \&\ By\ \&\ Bz)$. Since from $\mathrm{Disj}(\mathcal{A})$ we get $(Ay\ \&\ By) \to A \approx B$, we have $(\exists A, B \in \mathcal{A})(Ax\ \&\ A \approx B\ \&\ Bz)$. As $A \approx B\ \&\ Bz \to Az$, we obtain $(\exists A \in \mathcal{A})(Ax\ \&\ Az) \leftrightarrow R^{\mathcal{A}}xz$, and the proof is done.
(P22) Immediate consequence of (P19)–(P21). □

The property (P18) has told us that the quotient of a similarity is a partition. Now (P22) entails that partitions induce similarities. Note, however, that this is not yet a proof of one-to-one correspondence. We do not know yet whether these correspondences are invertible, i.e., (i) whether the quotient of a similarity induced by a partition is the same as the original partition, and (ii) whether the quotient of a given similarity induces the same similarity. The following final theorem gives answers to these questions—fortunately in a fully graded manner.

Theorem 7.10 FCT *proves the following:*

(P23) $\mathrm{Sim}(R) \to (R^{\mathrm{V}/R} \approxeq R)$
(P24) $\mathrm{Part}(\mathcal{A}) \longrightarrow \mathrm{Crisp}(\mathcal{A})\ \&\ \mathrm{NormM}(\mathcal{A})\ \&\ \mathrm{Disj}(\mathcal{A}) \longrightarrow (\forall A \in \mathcal{A})(\exists B \in \mathrm{V}/R^{\mathcal{A}})(A \approxeq B)$
(P25) $\mathrm{Part}(\mathcal{A}) \longrightarrow \mathrm{Crisp}(\mathcal{A})\ \&\ \mathrm{Cover}(\mathcal{A})\ \&\ \mathrm{Disj}(\mathcal{A}) \longrightarrow (\forall B \in \mathrm{V}/R^{\mathcal{A}})(\exists A \in \mathcal{A})(A \approxeq B)$

Proof.

(P23) We shall show that $\mathrm{Sym}(R)\ \&\ \mathrm{Trans}(R) \to R^{\mathrm{V}/R} \subseteq R$ and that $\mathrm{Refl}(R) \to R \subseteq R^{\mathrm{V}/R}$. The first part is proved by the following steps:

$$
\begin{aligned}
R^{\mathrm{V}/R}xy &\longrightarrow (\exists A \in \mathrm{V}/R)(Ax\ \&\ Ay)\\
&\longrightarrow (\exists A)((\exists z)([z]_R = A)\ \&\ Ax\ \&\ Ay)\\
&\longrightarrow (\exists A)(\exists z)([z]_R = A\ \&\ Ax\ \&\ Ay)\\
&\longrightarrow (\exists A)(\exists z)([z]_R = A\ \&\ Ax\ \&\ [z]_R = A\ \&\ Ay)\\
&\longrightarrow (\exists z)(x \in [z]_R\ \&\ y \in [z]_R)\\
&\longrightarrow (\exists z)(Rzx\ \&\ Rzy)\\
&\longrightarrow (\exists z)(Rxz\ \&\ Rzy), \text{ by } \mathrm{Sym}(R),
\end{aligned}
$$

$\longrightarrow Rxy$, by Trans(R).

The second part is proved by the following steps:

$Rxy \longrightarrow [x]_R = [x]_R \,\&\, x \in [x]_R \,\&\, y \in [x]_R$, by Refl$(R)$,

$\longrightarrow (\exists z)([z]_R = [z]_R \,\&\, x \in [z]_R \,\&\, y \in [z]_R)$

$\longrightarrow (\exists A)(\exists z)([z]_R = A \,\&\, Ax \,\&\, Ay)$

$\longrightarrow (\exists A)((\exists z)([z]_R = A) \,\&\, Ax \,\&\, Ay)$

$\longrightarrow (\exists A \in \mathrm{V}/R)(Ax \,\&\, Ay)$

$\longrightarrow R^{\mathrm{V}/R}xy$.

(P24) Let us choose a fuzzy set $A \in \mathcal{A}$. Since Crisp$(\mathcal{A})$ is fulfilled, $\triangle A \in \mathcal{A}$ holds. Since NormM$(\mathcal{A})$ holds, we know that there exists an x such that $\triangle Ax$. Now we choose $B = [x]_{R^{\mathcal{A}}}$, i.e. $By \longleftrightarrow R^{\mathcal{A}}xy \longleftrightarrow (\exists C \in \mathcal{A})(Cx \,\&\, Cy)$. Since $\triangle Ax$ and $\triangle A \in \mathcal{A}$ we get:

$$Ay \longrightarrow A \in \mathcal{A} \,\&\, Ax \,\&\, Ay \longrightarrow (\exists C \in \mathcal{A})(Cx \,\&\, Cy),$$

i.e. we have proved

$$\text{Crisp}(\mathcal{A}) \,\&\, \text{NormM}(\mathcal{A}) \rightarrow A \subseteq B \tag{7.1}$$

Conversely, we can prove the following:

$By \longleftrightarrow (\exists C \in \mathcal{A})(Cx \,\&\, Cy)$

$\longrightarrow (\exists C \in \mathcal{A})(Ax \,\&\, Cx \,\&\, Cy)$, by $\triangle Ax$,

$\longrightarrow (\exists C \in \mathcal{A})(A \approx C \,\&\, Cy)$, by Disj$(\mathcal{A})$,

$\longrightarrow (\exists C \in \mathcal{A})\ Ay$

$\longrightarrow Ay$

So we have proved

$$\text{Crisp}(\mathcal{A}) \,\&\, \text{NormM}(\mathcal{A}) \,\&\, \text{Disj}(\mathcal{A}) \rightarrow B \subseteq A. \tag{7.2}$$

Finally, we can join (7.1) and (7.2) to complete the proof (as the properties Crisp and NormM are crisp).

(P25) Let us consider an arbitrary $B \in \mathrm{V}/R^{\mathcal{A}}$. Since Crisp$(\mathrm{V}/R^{\mathcal{A}})$ holds by (P14), we have $\triangle(B \in \mathrm{V}/R^{\mathcal{A}})$, which means that there exists an x such that $B = [x]_{R^{\mathcal{A}}} = \{y \mid R^{\mathcal{A}}xy\}$. By (P20), we have Cover$(\mathcal{A}) \rightarrow \triangle$Refl$(R^{\mathcal{A}})$. Hence, we have $\triangle Bx$. From Cover$(\mathcal{A})$ and Crisp$(\mathcal{A})$ we can deduce that we can choose an $A \in \mathcal{A}$ such that $\triangle Ax$. Hence, we can deduce the following:

$$Ay \longrightarrow A \in \mathcal{A} \,\&\, Ax \,\&\, Ay \longrightarrow (\exists C \in \mathcal{A})(Cx \,\&\, Cy) \longrightarrow By$$

So we have proved the following:

$$\text{Crisp}(\mathcal{A}) \,\&\, \text{NormM}(\mathcal{A}) \,\&\, \text{Cover}(\mathcal{A}) \rightarrow A \subseteq B \tag{7.3}$$

Conversely, we can prove

$$\text{Crisp}(\mathcal{A}) \,\&\, \text{Cover}(\mathcal{A}) \,\&\, \text{Disj}(\mathcal{A}) \rightarrow B \subseteq A. \tag{7.4}$$

completely analogously to the proof of (7.2) (just to get $\triangle Ax$ we use Cover instead of NormM). Finally, we can join (7.3) and (7.4) to complete the proof. □

Nonchalantly speaking, we can say that (P24) and (P25) together mean that the more $\mathcal{A}$ is a partition, the more similar $\mathcal{A}$ and $\mathrm{V}/R^{\mathcal{A}}$ are. The question arises, whether they are equal if $\mathcal{A}$ is a partition to a degree of 1. The next corollary gives a positive answer and lists some other well-known non-graded results [12, 26, 30] that are consequences of graded results from above.

Corollary 7.11 FCT *proves the following:*

(P26) $\triangle\,\mathrm{Sim}(R) \to \triangle\,\mathrm{Part}(\mathrm{V}/R)$
(P27) $\triangle\,\mathrm{Sim}(R) \to R^{\mathrm{V}/R} = R$
(P28) $\triangle\,\mathrm{Part}(\mathcal{A}) \to \triangle\,\mathrm{Sim}(R^{\mathcal{A}})$
(P29) $\triangle\,\mathrm{Part}(\mathcal{A}) \to \mathrm{V}/R^{\mathcal{A}} = \mathcal{A}$

Proof. The assertions (P26), (P27) and (P28) are immediate consequences of (P18), (P23) and (P22), respectively. The assertion (P29) can be proved as follows: from $\mathrm{Part}(\mathcal{A})$, we know that $\mathcal{A}$ is a crisp set and, by (P14), we know that $\mathrm{V}/R^{\mathcal{A}}$ is crisp too. Then, using $\triangle\,\mathrm{Part}(\mathcal{A})$, (P24) implies $\mathcal{A} \subseteq \mathrm{V}/R^{\mathcal{A}}$ and (P25) implies $\mathrm{V}/R^{\mathcal{A}} \subseteq \mathcal{A}$, which completes the proof. □

Let us close this section with a simple example that illustrates the above results.

Example 7.12 Let us consider $U = \{1,2,3,4\}$, standard Łukasiewicz logic, and the crisp class $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$, where A_1, A_2, A_3, A_4 are fuzzy sets defined in the following way:

$$A_1 = (1.0, 0.4, 0.3, 0.0)$$
$$A_2 = (0.0, 1.0, 0.7, 0.0)$$
$$A_3 = (0.1, 0.2, 1.0, 0.5)$$
$$A_4 = (0.0, 0.1, 0.5, 1.0)$$

Obviously, $\mathrm{Crisp}(\mathcal{A}) = \mathrm{Cover}(\mathcal{A}) = \mathrm{NormM}(\mathcal{A}) = 1$. To compute $\mathrm{Disj}(\mathcal{A})$, we first compute the degrees of compatibility (overlapping) and equality:

$\Vert$	A_1	A_2	A_3	A_4
A_1	1.0	0.4	0.3	0.0
A_2	0.4	1.0	0.7	0.2
A_3	0.3	0.7	1.0	0.5
A_4	0.0	0.2	0.5	1.0

$\approx$	A_1	A_2	A_3	A_4
A_1	1.0	0.0	0.1	0.0
A_2	0.0	1.0	0.2	0.0
A_3	0.1	0.2	1.0	0.5
A_4	0.0	0.0	0.5	1.0

From these values, we see that the pair (A_2, A_3) is the one for which compatibility exceeds equality to the largest extent. So, we obtain

$$\mathrm{Disj}(\mathcal{A}) = A_2 \parallel A_3 \to A_2 \approx A_3 = 0.7 \to 0.2 = 0.5$$

which implies $\mathrm{Part}(\mathcal{A}) = 0.5$. We can derive $R^{\mathcal{A}}$ as follows:

$$R^{\mathcal{A}} = \begin{pmatrix} 1.0 & 0.4 & 0.3 & 0.0 \\ 0.4 & 1.0 & 0.7 & 0.1 \\ 0.3 & 0.7 & 1.0 & 0.5 \\ 0.0 & 0.1 & 0.5 & 1.0 \end{pmatrix}$$

Obviously $\mathrm{Refl}(R^{\mathcal{A}}) = \mathrm{Sym}(R^{\mathcal{A}}) = 1$ (any other result would contradict our findings above). Straightforward calculations show that

$$\mathrm{Sim}(R^{\mathcal{A}}) = \mathrm{wSim}(R^{\mathcal{A}}) = \mathrm{Trans}(R^{\mathcal{A}}) = 0.9.$$

Hence, we can conclude that the bounds in (P21) are not necessarily tight (which proves that the converse implication cannot generally hold).

Now let us consider the quotient $U/R^{\mathcal{A}}$. Obviously, $U/R^{\mathcal{A}} = \{B_1, B_2, B_3, B_4\}$ with

$$\begin{aligned} B_1 &= (1.0, 0.4, 0.3, 0.0) \\ B_2 &= (0.4, 1.0, 0.7, 0.1) \\ B_3 &= (0.3, 0.7, 1.0, 0.5) \\ B_4 &= (0.0, 0.1, 0.5, 1.0) \end{aligned}$$

and we immediately see the discrepancy between $\mathcal{A}$ and $U/R^{\mathcal{A}}$. Interestingly, we have $A_1 \subseteq B_1$, $A_2 \subseteq B_2$, $A_3 \subseteq B_3$, and $A_4 \subseteq B_4$. This is not surprising, however, if one looks at the proofs of (P24) and (P25), where we show that, for an $A \in \mathcal{A}$, we can find a $B \in \mathrm{V}/R^{\mathcal{A}}$ such that $A \subseteq B$. Not surprisingly either, A_1 is most similar to B_1, just as A_2 is most similar to B_2, and so on. Simple calculations show that the truth values of the formulae on the right-hand sides of (P24) and (P25) are both 0.5.

If we compute $R^{U/R^{\mathcal{A}}}$, we obtain the following fuzzy relation:

$$R^{U/R^{\mathcal{A}}} = \begin{pmatrix} 1.0 & 0.4 & 0.3 & 0.0 \\ 0.4 & 1.0 & 0.7 & 0.1 \\ 0.3 & 0.7 & 1.0 & 0.5 \\ 0.0 & 0.1 & 0.5 & 1.0 \end{pmatrix}$$

Then routine computations show that this fuzzy relation is a similarity. So, at least in the setting of this example, successive application of computing quotients and induced similarities yields increasing degrees to which the relations are similarities and the classes of fuzzy sets are partitions.

8 Concluding Remarks

In this paper, we have rephrased and generalized results on binary fuzzy relations to the graded framework of Fuzzy Class Theory (FCT). While Section 3 was more or less concerned with rewriting Gottwald's previously published results, Sections 4–7 have generalized results that were known in the non-graded framework of traditional theory of fuzzy relations to the fully fledged graded framework of FCT. These new results hereby demonstrate that Fuzzy Class Theory is indeed a very powerful and easy-to-use framework for handling fuzzified properties of fuzzy relations.

This paper has never been intended as a comprehensive treatise that covers the whole theory of crisp or fuzzy relations. We only tried to communicate the idea of how to apply Fuzzy Class Theory to generalizing existing (and possibly discovering new) results on fuzzy relations in the fully graded framework of FCT. Obviously, much is left for future studies, and we would like to encourage everybody interested in this topic to adopt the framework and advance the results.

A First-Order MTL$_\triangle$: Basic Definitions

Monoidal t-norm based logic (MTL for short) was introduced by Esteva and Godo in [33] as an extension of Höhle's monoidal logic [46] by the axiom of prelinearity (i.e., the axiom (A6) below). In this appendix we recall the definitions and some of the basic properties of MTL and its expansion by the connective $\triangle$. We start with the propositional variant and then expand it to the first-order predicate variant.

The formulae of propositional logic MTL are composed from a countable set of propositional atoms by using three basic binary connectives $\rightarrow$, $\wedge$, and $\&$, and a nullary connective 0. Further connectives can be defined as:

$$
\begin{aligned}
\varphi \vee \psi &\equiv_{\mathrm{df}} ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi),\\
\neg\varphi &\equiv_{\mathrm{df}} \varphi \rightarrow 0,\\
\varphi \leftrightarrow \psi &\equiv_{\mathrm{df}} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi),\\
1 &\equiv_{\mathrm{df}} \neg 0.
\end{aligned}
$$

Convention A.1 In order to avoid unnecessary parentheses, we stipulate that unary connectives take precedence over $\wedge$, $\vee$, and $\&$, which in turn bind more closely than $\rightarrow$ and $\leftrightarrow$.

The deduction rule of MTL is Modus Ponens (from φ and $\varphi \rightarrow \psi$ infer ψ) and the following formulae are the axioms of MTL:

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2) $\varphi \,\&\, \psi \rightarrow \varphi$

(A3) $\varphi \& \psi \to \psi \& \varphi$
(A4a) $\varphi \& (\varphi \to \psi) \to \varphi \wedge \psi$
(A4b) $\varphi \wedge \psi \to \varphi$
(A4c) $\varphi \wedge \psi \to \psi \wedge \varphi$
(A5a) $(\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi)$
(A5b) $(\varphi \& \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A7) $0 \to \varphi$

The logic $\mathrm{MTL}_\triangle$ was introduced in [33] as an expansion of the logic MTL by a new unary connective $\triangle$, the deduction rule of necessitation (from φ infer $\triangle\varphi$), and the following axioms:

(A$\triangle$1) $\triangle\varphi \vee \neg\triangle\varphi$
(A$\triangle$2) $\triangle(\varphi \vee \psi) \to (\triangle\varphi \vee \triangle\psi)$
(A$\triangle$3) $\triangle\varphi \to \varphi$
(A$\triangle$4) $\triangle\varphi \to \triangle\triangle\varphi$
(A$\triangle$5) $\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$

Formulae derived from these axioms by means of the mentioned deduction rules are called *theorems* of $\mathrm{MTL}_\triangle$.

Definition A.2 An MTL-*algebra* is a structure $\mathbf{L} = (L, *, \Rightarrow, \wedge, \vee, 0, 1)$, where

(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice
(2) $(L, *, 1)$ is a commutative monoid
(3) $x \leq (y \Rightarrow z)$ if and only if $x * y \leq z$ for all $x, y, z \in L$ (residuation)
(4) $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$ for all $x, y \in L$ (prelinearity)

Definition A.3 A structure $\mathbf{L} = (L, *, \Rightarrow, \wedge, \vee, 0, 1, \triangle)$ is called an $\mathrm{MTL}_\triangle$-*algebra* if $(L, *, \Rightarrow, \wedge, \vee, 0, 1)$ is an MTL-algebra and if the additional connective $\triangle$ has the following properties (for all $x, y \in L$):

(1) $\triangle x \vee (\triangle x \Rightarrow 0) = 1$
(2) $\triangle(x \vee y) \leq (\triangle x \vee \triangle y)$
(3) $\triangle x \leq x$
(4) $\triangle x \leq \triangle\triangle x$
(5) $\triangle(x \Rightarrow y) \leq \triangle x \Rightarrow \triangle y$
(6) $\triangle 1 = 1$

If the lattice order of $\mathbf{L}$ is linear, we say that $\mathbf{L}$ is an $\mathrm{MTL}_\triangle$-*chain*. If the lattice reduct of $\mathbf{L}$ is the real unit interval with the usual order, we say that $\mathbf{L}$ is a *standard* $\mathrm{MTL}_\triangle$-*chain*. It can be easily shown that in each $\mathrm{MTL}_\triangle$-chain the following holds:

$$\triangle x = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

The structure $([0,1], *, \Rightarrow, \min, \max, 0, 1, \triangle)$ is a standard MTL$_\triangle$-chain if and only if $*$ is a left-continuous t-norm and $\Rightarrow$ its residuum.

Given an MTL$_\triangle$-algebra, we can evaluate formulae of MTL$_\triangle$ by assigning elements of L to propositional atoms and computing values of compound formulae using operations of $\mathbf{L}$. A formula is a *tautology* of a given MTL$_\triangle$-algebra if it always evaluates to 1.

The completeness theorem for MTL and MTL$_\triangle$ with respect to standard algebras was proved in [49]: a formula is a theorem in MTL$_\triangle$ if and only if it is a tautology of each standard MTL$_\triangle$-algebra.

Now we introduce the language of first-order MTL$_\triangle$ logic (we give a slightly simplified account, omitting the subsumption of sorts; for full details see [4]).

Definition A.4 A *predicate language* $\mathbf{\Gamma}$ is a tuple $(\mathbf{S}, \mathbf{P}, \mathbf{F}, \mathbf{a})$, where $\mathbf{S}$ is a non-empty set of sorts of variables, $\mathbf{P}$ is a non-empty set of predicate symbols, $\mathbf{F}$ is a set of function symbols, and $\mathbf{a}$ is an *arity function* which assigns a sequence of sorts $(s_1, \ldots, s_k)$ to each predicate symbol and a sequence of sorts $(s_1, \ldots, s_k, s_{k+1})$ to each function symbol ($k \geq 0$ in both cases). Functions with arity (s_1) are called *object constants* of sort s_1. The set $\mathbf{P}$ is supposed to contain a symbol $=$ of arity (s, s) for each sort s. For each sort s, there are countably many variables $x_1^s, x_2^s, \ldots$.

For the rest of this appendix, fix a predicate language $\mathbf{\Gamma}$ and an MTL$_\triangle$-chain $\mathbf{L}$.

Definition A.5 Any variable x^s of sort s is a *term* of sort s. If $F \in \mathbf{F}$ is a function symbol of arity $(s_1, \ldots, s_k, s_{k+1})$, then for any terms $t_1, \ldots, t_k$ of respective sorts $s_1, \ldots, s_k$, the expression $F(t_1, \ldots, t_k)$ is a term of sort s_{k+1}.

Atomic formulae have the form $P(t_1, \ldots, t_k)$, where $t_1, \ldots, t_k$ are terms of respective sorts $s_1, \ldots, s_k$ and $P \in \mathbf{P}$ is a predicate symbol of arity $(s_1, \ldots, s_k)$. Where convenient, we switch to infix notation for binary predicate symbols.

Formulae are built from atomic formulae by using the connectives of MTL$_\triangle$ and the quantifiers $\forall, \exists$ (for a formula φ and a variable x, both $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulae).

Definition A.6 An occurrence of a variable x in a formula φ is *bound* if it is in the scope of a quantifier over x; otherwise it is called *free.* A formula φ is called a *sentence* if all occurrences of variables in φ are bound.

A term t is *substitutable* for the object variable x^s of sort s in a formula $\varphi(x^s)$ if and only if t is also of sort s and no variable occurring in t becomes bound in $\varphi(t)$.

Definition A.7 First-order MTL$_\triangle$ logic (with crisp identity) has the following axioms:

(P) The axioms resulting from the axioms of $\text{MTL}_\triangle$ by substituting first-order formulae for propositional formulae

($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$, where t is substitutable for x in φ

($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$, where t is substitutable for x in φ

($\forall$2) $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, where x is not free in χ

($\exists$2) $(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, where x is not free in χ

($\forall$3) $(\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$, where x is not free in χ

(=1) $x = x$

(=2) $x = y \to (\varphi(x) \leftrightarrow \varphi(y))$, where y is substitutable for x in φ

The deduction rules are those of $\text{MTL}_\triangle$ and *generalization*: from φ infer $(\forall x)\varphi$.

We define the notion of a *theorem* in the same way as in the propositional case. We can also define a more general notion of a theory.

Definition A.8 A *theory* is a set of sentences. A formula is *provable in a theory T* if it is derivable from the axioms of first-order $\text{MTL}_\triangle$ and sentences belonging to T by the deduction rules. We denote this fact by $T \vdash \varphi$.

Definition A.9 An **L**-*structure* **M** has the form: $\mathbf{M} = ((M_s)_{s\in\mathbf{S}}, (P_\mathbf{M})_{P\in\mathbf{P}}, (F_\mathbf{M})_{F\in\mathbf{F}})$, where each M_s is a non-empty set; each $P_\mathbf{M}$ is a k-ary fuzzy relation $P_\mathbf{M}\colon \prod_{i=1}^{k} M_{s_i} \to \mathbf{L}$ for each predicate symbol $P \in \mathbf{P}$ of arity $(s_1,\ldots,s_k)$; and $F_\mathbf{M}$ is a k-ary function $F_\mathbf{M}\colon \prod_{i=1}^{k} M_{s_i} \to M_{s_{k+1}}$ for each function symbol $F \in \mathbf{F}$ of arity $(s_1,\ldots,s_k,s_{k+1})$. Furthermore, $=_\mathbf{M}$ is the crisp identity of the elements of M_s for each $s \in \mathbf{S}$.

In words: an **L**-structure consists of (i) domains for all sorts of variables, (ii) an interpretation of all predicate symbols by **L**-fuzzy relations defined on appropriate domains, and (iii) an interpretation of all function symbols by crisp functions between appropriate domains.

Definition A.10 Let **M** be an **L**-structure. An **M**-*evaluation* is a mapping v which assigns an element from M_s to each object variable x of sort s. For an **M**-evaluation v, a variable x of sort s, and $a \in M_s$ we define the **M**-evaluation $v[x \mapsto a]$ as

$$v[x \mapsto a](y) = \begin{cases} a \text{ if } y = x \\ v(y) \text{ otherwise} \end{cases}$$

Definition A.11 Let **M** be an **L**-structure and v an **M**-evaluation. We define the *values* of terms and the *truth values* of formulae in **M** for an **M**-evaluation v as:

$$\begin{array}{rcll}
\|x\|^{\mathbf{L}}_{\mathbf{M},v} & = & v(x) & \\
\|F(t_1,\dots,t_n)\|^{\mathbf{L}}_{\mathbf{M},v} & = & F_{\mathbf{M}}(\|t_1\|^{\mathbf{L}}_{\mathbf{M},v},\dots,\|t_n\|^{\mathbf{L}}_{\mathbf{M},v}) & \text{for each } F\in\mathbf{F} \\
\|P(t_1,\dots,t_n)\|^{\mathbf{L}}_{\mathbf{M},v} & = & P_{\mathbf{M}}(\|t_1\|^{\mathbf{L}}_{\mathbf{M},v},\dots,\|t_n\|^{\mathbf{L}}_{\mathbf{M},v}) & \text{for each } P\in\mathbf{P} \\
\|c(\varphi_1,\dots,\varphi_n)\|^{\mathbf{L}}_{\mathbf{M},v} & = & c_{\mathbf{L}}(\|\varphi_1\|^{\mathbf{L}}_{\mathbf{M},v},\dots,\|\varphi_n\|^{\mathbf{L}}_{\mathbf{M},v}) & \text{for each connective } c \\
\|(\forall x)\varphi\|^{\mathbf{L}}_{\mathbf{M},v} & = & \inf_{a\in M}\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v[x\to a]} & \\
\|(\exists x)\varphi\|^{\mathbf{L}}_{\mathbf{M},v} & = & \sup_{a\in M}\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v[x\to a]} &
\end{array}$$

If an infimum or supremum does not exist, we consider its value as undefined. We say that a structure **M** is *safe* if and only if $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ is defined for each formula φ and each **M**-evaluation v. Note that, in a standard MTL$_\triangle$-algebra (or more generally in any MTL$_\triangle$-algebra whose lattice reduct is a complete lattice), the *safeness* of a structure is a superfluous condition, as the suprema and infima of *all* sets exist.

Definition A.12 A formula φ is *valid* in a structure **M** (denoted as $\mathbf{M}\models\varphi$) if $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}=1$ for each **M**-evaluation v. A structure **M** is a *model* of a theory T if $\mathbf{M}\models\varphi$ for each φ in T.

Finally we present the (strong) completeness theorem which relates syntactical and semantical aspects of the first-order MTL$_\triangle$ logic (see [33, 53] for a proof). Recall that the direction from provability to validity is usually called *soundness*, whereas the converse direction one is called *completeness*.

Theorem A.13 *Let* Γ *be a predicate language,* T *a theory, and* φ *a formula. Then the following are equivalent:*

(1) $T\vdash\varphi$.
(2) $\mathbf{M}\models\varphi$ *for each* MTL-*chain* **L** *and each safe* **L**-*model* **M** *of* T.
(3) $\mathbf{M}\models\varphi$ *for each standard* MTL-*chain* **L** *and each* **L**-*model* **M** *of* T.

Thus by (1)⇒(2) we get that if a formula is provable in a given theory T, then it is valid in all models of T over *all* MTL$_\triangle$-chains. Conversely, by (3)⇒(1) we get that if a formula is valid in all models of T over all all *standard* MTL$_\triangle$-chains, then it is provable in T.

B Fuzzy Class Theory: Basic Definitions

Fuzzy Class Theory has the aim to axiomatize the notion of fuzzy set. In the first paper [4], it was based on the logic ŁΠ [34]. In this paper, we use the logic MTL$_\triangle$; obviously all definitions and basic results of [4] can be transferred from ŁΠ to MTL$_\triangle$. For an introduction to MTL$_\triangle$, see Appendix A (for a more extensive overview of propositional MTL, see [33]; a more detailed treatment on first-order MTL$_\triangle$ with crisp equality can be found in [43]).

In this section, we present an overview of Fuzzy Class Theory (FCT) in order to provide the reader with the necessary background. Note that this is only a brief introduction to the most basic concepts of FCT with the aim to keep the paper self-contained. Readers who want to understand all proof details or even to make proofs in FCT themselves should not expect to find all necessary material in this paper. Instead, they are referred to the freely available primer [6].

Definition B.1 *Fuzzy Class Theory* (over $\mathrm{MTL}_\triangle$) is a theory over multi-sorted first-order logic $\mathrm{MTL}_\triangle$ with crisp equality. We have sorts for individuals of the zeroth order (i.e., atomic objects) denoted by lowercase variables $a,b,c,x,y,z,\ldots$; individuals of the first order (i.e., fuzzy classes) denoted by uppercase variables $A,B,X,Y,\ldots$; individuals of the second order (i.e., fuzzy classes of fuzzy classes) denoted by calligraphic variables $\mathcal{A},\mathcal{B},\mathcal{X},\mathcal{Y},\ldots$; etc. Individuals $\xi_1,\ldots,\xi_k$ of each order can form k-tuples (for any $k \geq 0$), denoted by $\langle \xi_1,\ldots,\xi_k \rangle$; tuples are governed by the usual axioms known from classical mathematics (e.g., that tuples equal if and only if their respective constituents equal). Furthermore, for each variable x of any order n and for each formula φ there is a class term $\{x \mid \varphi\}$ of order $n+1$.

Besides the logical predicate of identity, the only primitive predicate is the membership predicate $\in$ between successive sorts (i.e., between individuals of the n-th order and individuals of the $(n+1)$-st order, for any n).[7] The axioms for $\in$ are the following (for variables of all orders):

($\in$1) $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$, for each formula φ (comprehension axioms)
($\in$2) $(\forall x)\triangle(x \in A \leftrightarrow x \in B) \rightarrow A = B$ (extensionality)

Moreover, we use all axioms and deduction rules of first-order $\mathrm{MTL}_\triangle$. Theorems, theories, proofs, etc., can be defined completely analogously.

Observation B.2 Since the language of FCT is the same at each order, defined symbols of any order can be shifted to all higher orders as well. Since furthermore the axioms of FCT have the same form at each order, all theorems on FCT-definable notions are preserved by uniform upward order-shifts.

Convention B.3 For better readability, let us make the following conventions:

- We use the notations $(\forall x \in A)\varphi$, $(\exists x \in A)\varphi$ as abbreviations for $(\forall x)(x \in A \rightarrow \varphi)$ and $(\exists x)(x \in A \mathbin{\&} \varphi)$, respectively.
- The notation $\{x \in A \mid \varphi\}$ is short for $\{x \mid x \in A \mathbin{\&} \varphi\}$.
- We use $\{\langle x_1,\ldots,x_k \rangle \mid \varphi\}$ as abbreviation for $\{x \mid (\exists x_1)\ldots(\exists x_k)(x = \langle x_1,\ldots,x_k \rangle \mathbin{\&} \varphi)\}$.
- The formulae $\varphi \mathbin{\&} \ldots \mathbin{\&} \varphi$ (n times) are abbreviated φ^n; instead of $(x \in A)^n$, we can write $x \in^n A$ (analogously for other predicates).

7 By this requirement, Russell's paradox is avoided in a similar fashion as in type theory [64].

- Furthermore, $x \notin A$ is shorthand for $\neg(x \in A)$; analogously for other binary predicates.
- We use Ax and $Rx_1 \dots x_n$ synonymously for $x \in A$ and $\langle x_1, \dots, x_n \rangle \in R$, respectively.
- A chain of implications $\varphi_1 \to \varphi_2, \varphi_2 \to \varphi_3, \dots, \varphi_{n-1} \to \varphi_n$ will for short be written as $\varphi_1 \longrightarrow \varphi_2 \longrightarrow \cdots \longrightarrow \varphi_n$; analogously for the equivalence connective.

Definition B.4 In FCT, we define the following elementary fuzzy set operations:

$\emptyset$	$=_{\text{df}}$	$\{x \mid 0\}$	empty class
V	$=_{\text{df}}$	$\{x \mid 1\}$	universal class
$\mathrm{Ker}(A)$	$=_{\text{df}}$	$\{x \mid \triangle(x \in A)\}$	kernel
$\mathrm{Supp}(A)$	$=_{\text{df}}$	$\{x \mid \neg\triangle\neg(x \in A)\}$	support
$\backslash A$	$=_{\text{df}}$	$\{x \mid x \notin A\}$	complement
$A \cap B$	$=_{\text{df}}$	$\{x \mid x \in A \mathbin{\&} x \in B\}$	intersection
$A \sqcap B$	$=_{\text{df}}$	$\{x \mid x \in A \wedge x \in B\}$	min-intersection
$A \sqcup B$	$=_{\text{df}}$	$\{x \mid x \in A \vee x \in B\}$	max-union
$A \setminus B$	$=_{\text{df}}$	$\{x \mid x \in A \mathbin{\&} x \notin B\}$	difference

Definition B.5 Further we define in FCT the following elementary relations between fuzzy sets:

$\mathrm{Hgt}(A)$	$\equiv_{\text{df}}$	$(\exists x)(x \in A)$	height
$\mathrm{Norm}(A)$	$\equiv_{\text{df}}$	$(\exists x)\triangle(x \in A)$	normality
$\mathrm{Crisp}(A)$	$\equiv_{\text{df}}$	$(\forall x)\triangle(x \in A \vee x \notin A)$	crispness
$\mathrm{Fuzzy}(A)$	$\equiv_{\text{df}}$	$\neg\mathrm{Crisp}(A)$	fuzziness
$A \subseteq B$	$\equiv_{\text{df}}$	$(\forall x)(x \in A \to x \in B)$	inclusion
$A \approxeq B$	$\equiv_{\text{df}}$	$(A \subseteq B) \mathbin{\&} (B \subseteq A)$	(strong) bi-inclusion
$A \approx B$	$\equiv_{\text{df}}$	$(\forall x)(x \in A \leftrightarrow x \in B)$	weak bi-inclusion
$A \parallel B$	$\equiv_{\text{df}}$	$(\exists x)(x \in A \mathbin{\&} x \in B)$	compatibility

Definition B.6 The union and intersection of a class of classes are functions defined as

$$\bigcup \mathcal{A} =_{\text{df}} \{x \mid (\exists A \in \mathcal{A})(x \in A)\}$$
$$\bigcap \mathcal{A} =_{\text{df}} \{x \mid (\forall A \in \mathcal{A})(x \in A)\}$$

Definition B.7 In FCT, we define the following operations:

$A \times B$	$=_{\mathrm{df}}$	$\{\langle x,y\rangle \mid x \in A \mathbin{\&} y \in B\}$	Cartesian product
$\mathrm{Dom}(R)$	$=_{\mathrm{df}}$	$\{x \mid Rxy\}$	domain
$\mathrm{Rng}(R)$	$=_{\mathrm{df}}$	$\{y \mid Rxy\}$	range
$R^{\leftarrow}A$	$=_{\mathrm{df}}$	$\{x \mid (\exists y)(y \in A \mathbin{\&} Rxy)\}$	pre-image
$R \circ S$	$=_{\mathrm{df}}$	$\{\langle x,y\rangle \mid (\exists z)(Rxz \mathbin{\&} Szy)\}$	composition
R^{-1}	$=_{\mathrm{df}}$	$\{\langle x,y\rangle \mid Ryx\}$	converse relation
Id	$=_{\mathrm{df}}$	$\{\langle x,y\rangle \mid x = y\}$	identity relation

The following lemma lists a collection of results that are employed helpful later in this paper.

Lemma B.8 *The following results are provable in* FCT*:*

(L4) $\triangle(\varphi \vee \neg\varphi) \to [(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))]$
(L5) $\varphi \mathbin{\&} (\psi \to \chi) \to ((\varphi \to \psi) \to \chi)$
(L6) $\bigcup\{B \mid \varphi(B)\} \subseteq A \leftrightarrow (\forall B)(\varphi(B) \to B \subseteq A)$
(L7) $A \subseteq \bigcap\{B \mid \varphi(B)\} \leftrightarrow (\forall B)(\varphi(B) \to A \subseteq B)$
(L8) $\varphi(C) \to \bigcap\{B \mid \varphi(B)\} \subseteq C$
(L9) $\varphi(C) \to C \subseteq \bigcup\{B \mid \varphi(B)\}$
(L10) $(\exists x)(\varphi \vee \psi) \leftrightarrow ((\exists x)\varphi \vee (\exists x)\varphi)$
(L11) $(\forall x)(\varphi \wedge \psi) \leftrightarrow ((\forall x)\varphi \wedge (\forall x)\psi)$
(L12) $(\exists x)(\varphi \wedge \psi) \to ((\exists x)\varphi \wedge (\exists x)\psi)$
(L13) $((\forall x)\varphi \vee (\forall x)\psi) \to (\forall x)(\varphi \vee \psi)$
(L14) $(\forall x \in A)(\chi \to \psi) \to (\chi \to (\forall x \in A)\psi)$, *where x is free in χ*
(L15) $(\forall x \in A)(\varphi \to \psi) \to ((\forall x \in A)\varphi \to (\forall x \in A \cap A)\psi)$
(L16) $(\forall x \in A)(\varphi \to \psi) \to ((\exists x \in A \cap A)\varphi \to (\exists x \in A)\psi)$

The models of FCT are systems (closed under definable operations) of fuzzy sets (and fuzzy relations) of all orders over some crisp universe U, where the membership functions of fuzzy subsets take values in some $\mathrm{MTL}_{\triangle}$-chain (see [33] and Appendix A). Intended models are those which contain *all* fuzzy subsets and fuzzy relations over U (of all orders); we call such models *full*. Models in which moreover the $\mathrm{MTL}_{\triangle}$-chain is standard (i.e., given by a left-continuous t-norm on the unit interval $[0,1]$) correspond to Zadeh's [68] original notion of fuzzy set; therefore we call them *Zadeh models*.

FCT is sound with respect to Zadeh (or full) models; thus, whatever we prove in FCT is true about real-valued (or $\mathbf{L}$-valued for any $\mathrm{MTL}_{\triangle}$-chain $\mathbf{L}$) fuzzy sets and relations. Although the theory of Zadeh models is not *completely* axiomatizable, [8] the axiomatic system of FCT approximates it very well: the comprehension axioms ensure the existence of (at least) all fuzzy sets which are *definable* (by a formula of FCT), and the axioms of extensionality ensure that fuzzy sets are determined by their membership functions. This axiomatization is sufficient for almost all practi-

[8] Due to Gödel's Incompleteness Theorem [37], since natural numbers are definable in Zadeh models over $\mathrm{MTL}_{\triangle}$.

cal purposes; it can be characterized as *simple type theory over fuzzy logic* (cf. [58]) or *Henkin-style higher-order fuzzy logic*.

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