Problems and Gaps in Selected Papers

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Cuiusvis hominis est errare, nullius nisi insipientis in errore perseverare.
Any human can make a mistake, but none but a fool will persist in error.
—M. Tullius Cicero

This report documents errors identified during an error-checking effort covering a number of selected papers. The report was prepared using a centaur-style workflow: prompted by the human author, the AI model reviewed the papers and drafted the errata. The human author then checked the findings and made the final edits, in some cases clarifying furhter details with the model. GPT-6 Astra assisted in refining the main prompt used for the reviews.

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†OpenAI, models GPT-5.6 Sol, GPT-6 Astra, and GPT-6.1 Sol. (Not being a published work, this note intentionally departs from the academic requirements of co-authorship.)

1 *A predicate variant of two-layered many-valued probability logic* (Běhounek, 2025)

The corrections below address errors found in [18], a conference treatment with some proofs omitted. GPT-6.1 Sol carried out the review.

1.1 Summary

One substantial finding concerns the probability-measure conditions in Definition III.2. Four moderate findings concern the description of earlier probability logics, the comparison with the logic of Fagin, Halpern, and Megiddo, the transfer of first-order inference rules, and the characterization of rigidity using actuality. Five minor issues concern the lower-layer axioms, two semantic transcription errors, a variable convention, and a bibliographic place name.

1.2 Findings in more detail

1. Substantial: *Truncated additivity does not define the intended probability measures.*

   Definition III.2 requires only $\pi(W) = 1^{\boldsymbol{L}}$ and $\pi(A \cup B) = \pi(A) \oplus^{\boldsymbol{L}} \pi(B)$ for disjoint $A, B \in \mathcal{U}$. These conditions do not make a standard-valued π a finitely additive probability measure, contrary to the description following the definition. For example, let $W = \{w_0, w_1\}$, $\mathcal{U} = 2^W$, and

   $$\pi(\emptyset) = 0, \qquad \pi(\{w_0\}) = \pi(\{w_1\}) = \tfrac{3}{4}, \qquad \pi(W) = 1.$$

   This map satisfies both printed conditions, since $\frac{3}{4} \oplus \frac{3}{4} = 1$. On a singleton individual domain, let a unary predicate Q be true precisely at w_0, and put $\varphi \coloneqq Q(x)$. The resulting model is safe, but $\|\mathsf{P}\varphi\| = \|\mathsf{P}{\sim}\varphi\| = 3/4$. Consequently, the formulas in (4), (17), and (18) all have value $1/2$. Formula (13) also has value $1/2$ when ψ is a contradiction. These failures persist despite $\pi(\emptyset) = 0$. Moreover, the printed conditions even admit $\pi(A) = 1$ for every $A \in \mathcal{U}$. In such a model a contradiction φ refutes (8) and (24). These are consequences of the same defect in the measure definition.

   The intended probability semantics can be restored by adding complement preservation,

   $$\pi(W \setminus A) = {\sim}^{\boldsymbol{L}}\pi(A) \qquad (A \in \mathcal{U}).$$

   Indeed, the printed additivity condition implies monotonicity. For disjoint A, B, it follows that $\pi(B) \leq \pi(W \setminus A) = {\sim}^{\boldsymbol{L}}\pi(A)$, so $\pi(A) \otimes^{\boldsymbol{L}} \pi(B) = 0^{\boldsymbol{L}}$. Thus the displayed $\oplus$-sum is an untruncated sum in the ordered-group representation of the MV-chain. In the standard algebra this gives ordinary finite additivity; also $\pi(\emptyset) = 0$. Equivalently, the disjoint-additivity clause can require both its printed equality and $\pi(A) \otimes^{\boldsymbol{L}} \pi(B) = 0^{\boldsymbol{L}}$.

2. Moderate: *Earlier fuzzy probability logics were not all propositional.*

   The unrestricted assertion on p. 32 that all known $[0, 1]$-valued probability logics are propositional is false. Wild, Schröder, Pattinson, and König [35, Section 2] already introduced a probabilistic fuzzy first-order logic $\mathsf{FO}(\mathsf{P})$ in 2019. It has individual quantification and a probability binder interpreted by expected truth values; for crisp arguments, that binder gives probabilities of events.

   The claim can instead be restricted to the propositional two-layered systems recalled in Section II-B. The earlier $\mathsf{FO}(\mathsf{P})$ uses a different probabilistic structure and does not identify the particular two-layered construction proposed here with an existing system.

3. Moderate: *The expressivity comparison conflates formula translations with translations into hypersequents.*

   Section II-B (p. 34) attributes to [1], the paper's reference 16, a mutual faithful interpretation between the formula-based logic FP(Ł) and the probability logic of Fagin, Halpern, and Megiddo. The related claim on p. 32 is likewise unqualified. Theorems 1–2 of [1] give mutual formula translations for the expansion by $\triangle$, denoted there by $\mathrm{Pr}_{\mathrm{Ł}_\triangle}$. The translation without $\triangle$ in [1, Theorem 10] uses hypersequents of relations, including strict inequalities, rather than single formulas of FP(Ł).

   This distinction is substantive. Over a language with one event atom p, every FP(Ł) formula has a continuous truth-value function of $r = \pi(p)$, so its truth-at-1 set is closed in $[0, 1]$. The classical probability formula $\neg(\pi(p) \geq 1/2)$ has truth set $[0, 1/2)$ and therefore has no such formula representation on the same probability models. The corrected comparison uses either the $\triangle$-expansion for formula translations or the hypersequent presentation for the $\triangle$-free result.

4. Moderate: *First-order rules do not transfer with unrestricted term substitution.*

   The opening paragraph of p. 36 asserts that all rule schemata valid in Ł$\forall$ are valid for modal formulas. This fails for universal instantiation with nonrigid terms, even when the instantiated formula is a modal atom and the measure is an ordinary finitely additive probability measure. Let $D = W = \mathbb{N}$, interpret a constant C by w at world w, and use the finite–cofinite set algebra with probability 0 on finite sets and 1 on cofinite sets. Put

   $$\Psi(x) \coloneqq \mathsf{P}{\sim}(x = C).$$

   For every fixed value of x, its inequality to C holds at all but one world. Hence $\mathbf{M} \models (\forall x)\Psi(x)$, whereas $\Psi(C)$ has value 0. In the pure equality language with C, all definable events are finite or cofinite, and all upper quantifier values exist, so this model is safe. Thus the ordinary first-order rule $(\forall x)\Psi(x) \models \Psi(C)$ fails here.

   The propositional rules listed in that paragraph and the rule of generalization remain sound. For rules involving instantiation, substituting an individual variable or a term rigid in the model gives the usual substitution lemma because its denotation is fixed across every world inspected by a modal atom. Induction through the upper connectives and quantifiers then gives the corresponding instantiation rule, with the usual freedom-for-substitution condition. An unrestricted transfer assertion must therefore be replaced by one respecting these term restrictions, consistently with the rigidity premise in Proposition III.5.

5. Moderate: *The actuality implication is not a characterization of rigidity.*

   In the discussion following Definition III.4 (p. 36), the condition $\mathbf{M} \models @\varphi \to \Box\varphi$ is said to be equivalent to rigidity. Let a closed formula φ be false at the actual world and true at another world. Then $@\varphi \to \Box\varphi$ has value 1, whereas $\Diamond\varphi \to \Box\varphi$ has value 0.

   A correct actuality-based characterization is

   $$\mathbf{M} \models (@\varphi \to \Box\varphi) \wedge (@{\sim}\varphi \to \Box{\sim}\varphi).$$

   The other characterization given in the same paragraph, $\mathbf{M} \models \Box\varphi \vee \Box{\sim}\varphi$, is valid as stated.

6. Minor issues:

   (a) The axiomatization recalled in Section II-B (p. 34) omits the classical propositional axioms for the nonmodal layer. These are needed alongside the Łukasiewicz axioms for modal formulas and (1)–(3), so that necessitation yields, for example, $\mathsf{P}(p \vee \sim p)$ from its classical event tautology.

   (b) In Definition III.2, the interpretation type of a function symbol is printed as $\mathscr{I}(F^{(n)}) \colon W \times D^n \to D^n$. Its codomain should be D, including for $n = 0$.

   (c) The predicate-application clause in Definition III.2 repeats $[\![t_1]\!]_w^{\mathbf{M},v}$ in its last argument. That argument should be $[\![t_n]\!]_w^{\mathbf{M},v}$.

   (d) In the term clause of Definition III.4, y must be chosen not to occur in t. Without this convention, on $D = \mathbb{Z}/3\mathbb{Z}$ the two interpretations $F_{w_0}(d) = d+1$ and $F_{w_1}(d) = d+2$ make $F(y) = y$ constantly false, although $F(y)$ is not rigid.

   (e) In reference 18, the place of publication is misspelled "Dordercht". The correct spelling is Dordrecht.

1.3 Status of the results

Restoring the probability condition in Finding 1 gives the intended standard probability measures and their MV-chain-valued counterparts. With that correction, the affected laws (4), (8), (13), (17), (18), and (24) hold. The definitional correction, rather than independent changes to those laws, is required.

The blanket transfer of first-order rules remains false independently of the measure correction. Propositional inference and generalization are sound, while term instantiation requires the restrictions described in Finding 4. Propositions III.3 and III.5 retain their statements; in particular, the latter uses the actual rigidity condition, not the false actuality characterization. The rigidity definition and the collapse laws (24)–(25) use their intended meaning with the variable convention and measure condition corrected.

The corrections concerning earlier work and expressivity restrict the introductory comparisons; they do not require changes to the two-layered syntax.

2 *A degree-theoretic framework for feasible knowledge* (Běhounek, 2020)

This section records errors found in [17], a conference paper with some proofs omitted. The review was performed by GPT-6.1 Sol.

2.1 Summary

Two substantial findings concern the missing link between accessibility and derivability in Definition 1 and Section 7 and the domain of the closure operator in Observation 1. Two moderate findings concern the proposed consistency condition for actual beliefs and the scope of the length-cutoff claim in Section 8. Four minor issues concern the attribution of a delimitation of fuzzy logics, the notation for mixed formulas, and a variable in the static-facts condition.

2.2 Findings in more detail

1. Substantial: *Accessibility in Definition 1 can assign positive feasibility to a nonconsequence.*

   Definition 1 imposes no condition relating positive transitions to the inferences of $\mathcal{E}$. Take classical propositional $\mathcal{E}$, two states w, v with $A^w = \{p\}$ and $A^v = \{p, q\}$ for distinct atoms p, q, and a crisp R with reflexive loops and the edge from w to v. The model satisfies conditions 1–3 of Section 7 and can use a constant evaluation making both atoms true. It gives $\|\mathrm{K}q\|_w = 1$ although $p \nvdash_{\mathcal{E}} q$. This conflicts with the account of feasible knowledge as a fuzzy subset of potential knowledge on p. 7.

   The same lack of an inference structure affects the last constraint on p. 13, even after correcting its set-builder variable to $[w]_n = \{w' \in W : \mathrm{Card}(A^{w'} \setminus A^w) = n\}$. With the same frame and $A^v = \{p, p \vee q, p \vee r\}$ for distinct atoms p, q, r, the transition is sound with respect to classical derivability. Nevertheless, $[w]_2 = \{v\}$ and $[w]_1 = \emptyset$. The inequality at $n = 1$ reads $1 \leq 0$, even if $\mathcal{E}$ has a single-conclusion calculus.

   For the inclusion in potential knowledge it suffices to require $A^w \vdash_{\mathcal{E}} \varphi$ for every $\varphi \in A^{w'}$ whenever $Rww' > 0$. A condition that also gives the inequality on p. 13 is to take R to be the supremum of weights of finite derivation paths, as in Figure 1, with each step retaining the current knowledge and adding at most one formula by an allowed $\mathcal{E}$-inference. Any path ending in $[w]_{n+1}$ then has a prefix ending in $[w]_n$. Fusion in an integral residuated lattice makes the weight of the full path at most that of the prefix. Taking suprema gives the inequality whenever its joins exist.

2. Substantial: *Observation 1 can lack a total fuzzy preimage operator.*

   The closure assertion quantifies over all $X \colon W \to \boldsymbol{L}$. The safety condition in footnote 14 of the paper requires the joins used to evaluate $\mathrm{K}\varphi$. For example, let $W = \mathbb{N}$, let $\boldsymbol{L}$ be the rational Gödel chain, set $Rww' = 1$ everywhere and take the same A^w in every state. All modal joins are either 0 or 1, and conditions 1–3 hold. For a rational-valued X whose values increase to an irrational in $(0, 1)$, $(R^{\leftarrow} X)(w)$ has no value in $\boldsymbol{L}$.

   The assertion on all fuzzy sets holds over complete truth-value algebras, and also for finite W. Over a general safe model, the operator can be restricted to those X for which $Y = R^{\leftarrow} X$ exists at every state. For each w, w', transitivity and preservation of existing joins by fusion give $Rww' \otimes Y(w') \leq Y(w)$. The term with $w' = w$ attains $Y(w)$ by reflexivity. Thus $R^{\leftarrow} lY$ exists and equals Y, and the operator is extensive and monotone on this domain. The fuzzy image identity (I19) of Theorem 4.6 in [19], applied to the converse relation, gives the same idempotence when the requisite images exist. Modal safety places each A_φ in the restricted domain.

3. Moderate: *The doxastic condition on p. 13 quantifies over evaluations too strongly.*

   Pointwise inclusion in every admissible $\mathcal{E}$-evaluation e' requires every actual belief to be valid under all admissible evaluations at that state. For classical propositional $\mathcal{E}$, this excludes even the consistent belief set $A^w = \{p\}$ because an evaluation can make p false.

   The intended semantic consistency condition is that there be an admissible evaluation at each w making all of A^w true to degree 1. Equivalently, there must be an admissible e' with $A(w, \varphi) \leq e'(w, \varphi)$ at all states and formulas, with e' allowed to differ from the model's evaluation e.

4. Moderate: *The conclusion of Section 8 overstates the effect of derivation length on feasibility.*

   The concluding paragraph on p. 14 can be read as claiming that unrestricted deductive closure of feasible knowledge fails as a general law. With degree 1 designated, this reading is correct. The stronger assertion that sufficiently long derivations eventually become infeasible needs further conditions. Axioms (K′) and (4′) give lower bounds on the degrees of the conclusions. The appeal to accumulation of rule costs to explain avoidance of logical omniscience is inherited from [11, Sect. 5]. Decreasing degrees along a particular path still allow higher degrees along other paths.

   For a countermodel to unrestricted deductive closure, let the states be the finite knowledge stocks reached by modus ponens and positive introspection from $\{p, p \to q\}$, with p, q distinct atoms. Retain premises at each step, assign every elementary step degree $a \in (0, 1)$, and use the path semantics of Finding 1 over a standard truth-value algebra. Interpreting the rule-cost constants by a validates (K′) and (4′). For (K′), the two premise derivations can be run in sequence, omitting redundant steps, before applying modus ponens. For (4′), one application of introspection suffices after the premise derivation. Taking suprema gives the two bounds. At the initial state, Kp and K$(p \to q)$ both have degree 1, whereas Kq has degree a. Thus Kp, K$(p \to q) \not\models$ Kq.

   The same construction over standard Gödel logic gives degree a to every newly derivable formula, including iterates of inner positive introspection that require arbitrarily many steps. The degree-1 metarule can therefore fail even when every derivable consequence has positive feasibility. A sufficient condition for a cutoff at degree 0 is to bound every elementary step degree by some $b < 1$ whose N-fold fusion is 0. Then every path of length at least N has degree 0, so a consequence that requires at least N steps has degree 0 as well. For standard Łukasiewicz fusion one can take $N = \lceil 1/(1-b) \rceil$. With standard product fusion and a constant elementary step degree $b \in (0, 1)$, a formula whose shortest derivation has length n has feasible knowledge degree b^n; these degrees remain positive at every finite length and tend to 0 as $n \to \infty$.

5. Minor issues:

   (a) Footnote 3 on p. 4 attributes a delimitation in terms of residuated lattices to [27]. Cintula's delimitation concerns chain completeness within weakly implicative logics and applies also to languages without a residuated-lattice signature. The general chain-based proposal is stated in [22].

   (b) The disambiguation convention on p. 8 addresses the two occurrences of K, but the two layers of $\mathcal{L}_{\mathrm{K}}(\mathcal{E})$ also need to be distinguished for other shared connectives in Definition 1. With Gödel $\mathcal{E}$, Łukasiewicz $\mathcal{L}$, and atomic degrees 3/4 and 1/2, the two clauses give different values to an unmarked implication, namely 1/2 for an embedded $\mathcal{E}$-formula and 3/4 for the outer connective. Marking embedded $\mathcal{E}$-formulas as units of the outer grammar makes the two clauses unambiguous.

   (c) The static-facts condition on p. 13 leaves w' unbound. The equality $e(w, \varphi) = e(w', \varphi)$ is required for all $w, w' \in W$ and $\varphi \in \mathit{Form}_{\mathcal{E}}$.

2.3 Status of the results

The path calculation in Figure 1 is valid. With accessibility generated by allowed single-formula inference steps, positive feasibility entails derivability, and the state-size inequality on p. 13 holds when its joins exist. The cost axioms admit models in which feasible knowledge at

degree 1 fails deductive closure. A cutoff at degree 0 follows under the additional path-weight condition in Finding 4.

In Observation 1, the identity $\|\mathrm{K}\varphi\| = R^{\leftarrow} A_{\varphi}$ holds in the stated safe models. The associated closure acts on all fuzzy sets when the preimage joins exist everywhere, and on the restricted domain in Finding 2 otherwise. The consistency condition on p. 13 can be used with an existential choice of an admissible evaluation.

3 *Determinate truth in fuzzy plurivaluationism* (Běhounek, 2017)

This section records errors identified in [15], a conference paper with some proofs omitted. The review was performed by GPT-6.1 Sol.

3.1 Summary

Four moderate findings concern borderline cases in Section 1.2, axiomatizability in footnote 10, the formula-length estimate in Section 4, and the limit at full truth in the discussion of (12). One repairable finding concerns the argument in Section 5 that the determinacy jolt sentence fails to be supertrue. Five minor issues concern sorites indexing, the description of precisifications in Section 1.4, the borderline-case axiom in footnote 13, and two minor errors in references.

3.2 Findings in more detail

1. Moderate: *A jolt in every classical precisification does not rule out borderline cases.*

   The inference on p. 4 from the supertruth of a jolt to the absence of borderline cases conflates two levels of evaluation. A person of height 175 cm is tall on a precisification with a threshold of 170 cm and is not tall on one with a threshold of 180 cm. Both precisifications satisfy (i)–(iv) on p. 2. Thus neither predication is supertrue, and the person is a borderline case in the sense of p. 1 when determinacy is understood as supertruth.

   The jolt concerns the transition within each precisification. Its location can vary across precisifications, allowing cases that are neither supertrue nor superfalse.

2. Moderate: *Standard first-order validity in Łukasiewicz logic is not recursively axiomatizable.*

   Footnote 10, following the definitions of standard first-order Łukasiewicz logic's tautologicity and consequence on p. 8, assigns finite axiomatizability to tautologicity and finitary entailment. The standard first-order tautologies of Łukasiewicz logic are already non-recursively enumerable [31]. Consequently there is no effective finitary axiomatization even for empty-premise entailments in that semantics.

   The finite-axiomatizability assertion applies to the standard propositional case. The usual first-order calculus is complete for the general semantics over safe MV-chains [29, Chapter 5]. This semantics uses a larger class of models than the one on p. 8.

3. Moderate: *The estimate in Section 4 undercounts the occurrences of the input variable.*

   The bound $n\varepsilon$ on p. 11 for formulas with n connectives is false. Compare two evaluations of the same formula in which the truth degree assigned to the propositional variable p changes by at most ε and all other variable assignments are fixed. In $p \,\&\, p$, changing the degree of p from 1 to 0.9 changes the formula's degree from 1 to 0.8, although the formula has only one connective.

Each binary connective in (1)–(5) changes in value by at most the sum of the changes in its two arguments, and negation preserves the size of a change. Induction on the formula therefore gives the bound $k\varepsilon$, where k is the number of occurrences of p. A formula with n connectives has at most $n+1$ such occurrences, giving the corrected bound $(n+1)\varepsilon$.

To separate $1-\varepsilon$ from 1 by giving one evaluation degree 0 and the other degree 1 requires at least $\lceil 1/\varepsilon \rceil$ occurrences of p. Iterated strong conjunction attains this minimum using $\lceil 1/\varepsilon \rceil - 1$ connectives. Thus the example at 0.9 requires ten copies of p joined by nine strong conjunctions, and the minimum number of conjuncts used in (11) is correct.

4. Moderate: *The stipulated value at full truth need not be a limit.*

   The paragraph following (12) on p. 13 calls $\| \operatorname{Det} \varphi \| = 1$ at $\|\varphi\| = 1$ a limit case. Let us write $\ell = \| \operatorname{Large} \|$ for the membership function interpreting Large in the given model of arithmetic, as on p. 9. For inputs below 1, the ceiling in (12) selects a standard positive integer n, interpreted in the model by its numeral $\bar{n}$. The left limit of the truth function at 1 is therefore $\sup_{n \geq 1} \ell(\bar{n})$, whereas the supremum required by (9) ranges over the entire arithmetic domain. In a nonstandard model, ℓ can be zero on all standard numerals and one on all nonstandard elements. This satisfies (7)–(9), yet (12) with the stipulated endpoint then gives the truth function of $\triangle$.

   The endpoint is well defined by stipulation: at $\|\varphi\| = 1$, the denominator in (12) vanishes; the paper therefore defines $\| \operatorname{Det} \varphi \| = 1$ separately at that input, regardless of the left limit. It agrees with the left limit exactly when the values of ℓ on the standard numerals have supremum 1. This holds when arithmetic is interpreted in its standard model, and it can also be imposed as a semantic condition on the allowed expansions of nonstandard models.

5. Repairable: *The determinacy jolt indeed fails to be supertrue, but the successor jolt for* Large *does not establish this.*

   At the end of Section 5, p. 13, the paper infers that the determinacy jolt sentence fails to be supertrue from the corresponding fact about the successor jolt for Large. The successor jolt for Large tests consecutive integers, whereas the distinguishing lengths of two adjacent members of the sorites series can skip an integer.

   For any $m \geq 1$, take the standard arithmetic model and write ℓ for the membership function interpreting Large. Set $\ell(n) = 0$ for $n \leq m$, $\ell(m+1) = 1/2$, and $\ell(n) = 1$ for $n \geq m+2$. The successor jolt for Large then has degree $1/2$. Suppose two adjacent members x_i and x_{i+1} of the sorites series have truth degrees $\|Px_i\| = 1 - 1/(m+2)$ and $\|Px_{i+1}\| = 1 - 1/m$, respectively. By (12), their distinguishing lengths are $m+2$ and m. The chosen interpretation of Large therefore gives $\| \operatorname{Det} Px_i \| = 1$ and $\| \operatorname{Det} Px_{i+1} \| = 0$, even though the original truth degrees differ by only $2/(m(m+2))$, which can be made arbitrarily small. Thus a model in which the successor jolt has degree below 1 can still give the determinacy jolt degree 1.

   The conclusion rather follows by exhibiting an admissible model in which the determinacy jolt has degree below 1. Since (7)–(9) constrain Large independently of P, one may take $\ell(n) = n/(n+1)$ for $n \geq 0$ in the standard arithmetic model. Every distinguishing length is at least 1, and the stipulated value at full truth is 1. Hence all Det values are at least $1/2$, so the determinacy jolt sentence has degree at most $1/2$.

   In the latter model, if Px_i has truth degree 0, then $\operatorname{Det} Px_i$ has truth degree $1/2$, so Det sends input degree 0 to $1/2$, so the operator Det does not preserve full falsity. For a *finite* sorites series, one can also keep the determinacy jolt below degree 1 while preserving full falsity and full truth. This is possible whenever the original series has no adjacent pair

x_i, x_{i+1} with $\|Px_i\| = 1$ and $\|Px_{i+1}\| = 0$. With $\ell(0) = 0$ and $\ell(n) = 1 - 1/n$ for $n \geq 1$, Det sends 0 to 0 and 1 to 1, and sends every intermediate degree to an intermediate degree. A full determinacy jolt would therefore require an adjacent pair with original degrees 1 and 0, in that order. Each adjacent pair contributes a value below 1, and the maximum of these finitely many values is below 1.

6. Minor issues:

   (a) On p. 1, the sorites series should start at x_0 to include the clearly positive endpoint specified in the same sentence.

   (b) On p. 6, Section 1.4 describes the precisifications of *tall* as gradual, although the stated postulates (i)–(iv) also admit a bivalent threshold at 175 cm. The intended point is that the class *admits* gradual models in which (10) has degree below 1, as in Section 3.

   (c) The proposed borderline-case axiom in footnote 13 has degree at most 1/2, since $(\forall n)(\mathrm{Large}(n) \vee \neg\,\mathrm{Large}(n))$ has degree at least 1/2. The intended condition is expressed by $\neg(\forall n)\triangle(\mathrm{Large}(n) \vee \neg\,\mathrm{Large}(n))$. However, this axiom uses $\triangle$, contrary to the recommendation in Section 6 to avoid that connective; its admissibility in the auxiliary machinery for vagueness is left implicit in that recommendation. The same condition can nevertheless be imposed semantically by requiring $0 < \ell(n) < 1$ for some n, where $\ell = \|\,\mathrm{Large}\,\|$. A stronger $\triangle$-free axiom is $\neg(A \,\&\, A)$ for $A \coloneqq (\forall n)(\mathrm{Large}(n) \vee \neg\,\mathrm{Large}(n))$, which forces $\|A\| = 1/2$ and hence ensures a borderline case. It is compatible with (7)–(9), as witnessed by $\ell(n) = n/(n+1)$ in the standard arithmetic model.

   (d) In the reference to Běhounek (2011), the book's subtitle is *Logical, Philosophical, and Linguistic Perspectives*. In the reference to *Vagueness and Degrees of Truth*, Smith's initials are N. J. J.

3.3 Status of the results

A jolt occurs in each classical precisification, and borderline cases arise where the precisifications disagree. The postulates for fuzzy *tall* admit gradual models in which (10) has degree below 1. The semantic identification of supertruth with consequence over the stipulated model class is independent of effective axiomatizability. Axioms (7)–(9) admit models with borderline cases, and the corrected axiom for footnote 13 requires their existence.

The corrected estimate in Finding 3 establishes that iterated strong conjunction uses the minimum number of conjuncts required in (11). For an input of degree $1 - \varepsilon$, the exact minimum connective count is $\lceil 1/\varepsilon \rceil - 1$, one less than the number of conjuncts.

The stipulated value $\|\,\mathrm{Det}\,\varphi\| = 1$ at $\|\varphi\| = 1$ agrees with the left limit in the standard arithmetic model and in exactly those nonstandard expansions in which the largeness degrees of the standard numerals have supremum 1, as explained in Finding 4. Finally, the determinacy jolt sentence does indeed fail to be supertrue, by the corrected argument in Finding 5; individual admissible precisifications can still exhibit full jolts.

4 *A minimalistic many-valued theory of types* (Běhounek, 2017)

This section records errors in [16]. The entire paper, including the references, was checked as part of a systematic error-checking effort covering several selected papers. The review concerns

mathematical correctness and incorrect references, rather than stylistic matters or routine omitted proof steps. The error checks for this article were carried out by GPT-5.6 Sol and written up by GPT-6 Astra.

4.1 Summary

The review identified one substantial gap: the claimed equivalence between extensionality and the η-axiom is not established by the printed proof. The proposed alternative axiomatization can be repaired by adding abstraction congruence (but the necessity of such an addition has not been established).

One moderate issue concerns an incomplete characterization of the models in the Łukasiewicz example. Two further issues are repairable: typing and proof errors in the congruence theorem and a premature use of functionality in the canonical-model construction. There are also several minor formula and reference errors.

No fundamental error was found; the soundness and completeness results for the original system and for its three sample extensions remain valid after the local corrections below.

4.2 Findings in more detail

1. Substantial: *The equivalence claimed in Theorem 2.13 and the proposed alternative axiomatization are not established.*

   The converse proof in Theorem 2.13 derives $\lambda x(Fx) \equiv \lambda x(Gx)$ from $Fx \equiv Gx$ and reflexivity by replacing Fx with Gx under λx. However, x is free in Gx and becomes bound after this replacement, so the substitutability condition of Definition 2.6 fails. Consequently the proof does not establish that (eq), (λ-abs), and (η-ax) suffice to derive extensionality. This also affects the equivalence claimed in Remark 2.9 and the alternative axiomatization proposed immediately before Theorem 2.13 (p. 1311).

   This error is reported and corrected in [25]. A valid alternative axiomatization is obtained by supplementing the proposed system of (subst), (eq), (λ-abs), and (η-ax) with the abstraction-congruence rule

   $$A \equiv B \;/\; \lambda xA \equiv \lambda xB.$$

   This rule supplies the missing inference and is itself derivable in the original system using substitution, λ-abstraction, and extensionality. Alternatively, one can simply retain (ext) in the axiomatization.

   The amendment establishes the equivalence for the supplemented axiomatization; it does not establish that the additional rule is necessary. The original converse implication should therefore be regarded as unproved rather than refuted.

2. Moderate: *Example 3.4 omits a condition on truth-value equality in its characterization of models.*

   The models of TT(Ł) are said to be exactly the $\mathrm{TT}_{\to}$-models whose truth-value domain, equipped with the interpretations of $\to$ and $\neg$, is an MV-algebra. This omits the additional axiom identifying $\equiv_{ooo}$ with Łukasiewicz equivalence. For example, take the standard MV-algebra $[0,1]_{Ł}$, $D = \{1\}$, full function spaces, and crisp equality at every type. This is a $\mathrm{TT}_{\to}$-model, but the identification axiom fails at $a = 0$, $b = \frac{1}{2}$, since $\mathrm{Eq}_o(a,b) = 0$ whereas $(a \leftrightarrow b) = \frac{1}{2}$.

   The characterization becomes correct upon adding the requirement that $\mathrm{Eq}_o(a,b) = (a \leftrightarrow b)$ for all truth values a, b. The equality condition in Definition 2.19 then forces $D = \{1\}$, since $\mathrm{Eq}_o(1,b) = (1 \leftrightarrow b) = b$ is designated exactly when $b = 1$. Every

MV-algebra still extends to such a model by interpreting truth-value equality as MV-equivalence and choosing suitable equalities at the other types. The conclusion about the propositional fragment is unaffected.

3. Repairable: *Theorem 2.41 has overly restrictive typing and an inadmissible replacement in the proof of part (3).*

   The theorem assumes that A, A', B, B' all have the same type α. In part (2), however, the containing formula B must be allowed to have a type different from that of A, A'; this generality is already needed in Theorem 2.43. In part (3), the type of B must agree with that of x, independently of the type of A, A'. Moreover, the proof of part (3) replaces A by A' inside $(\lambda x A)B$, which is forbidden by Definition 2.6 when x is free in A'.

   These issues can be corrected by stating the typing conditions separately and replacing the proof of part (3). Specifically, A, A' have the same type, which is o in part (1); B in part (2) has arbitrary type, subject to substitutability; and B, x in part (3) have the same type, with both substitutions defined. Part (3) then follows by a direct application of (subst) to $T \vdash A \equiv A'$:

   $$T \vdash (A \equiv A')[B/x], \qquad \text{i.e.,} \qquad T \vdash A[B/x] \equiv A'[B/x].$$

4. Repairable: *The proof of Theorem 2.43 uses functionality before it has been established.*

   In the left-to-right direction, equality of the graph-sets $V(A_{\gamma\beta})$ and $V(A'_{\gamma\beta})$ is taken to imply $V(A_{\gamma\beta}B_\beta) = V(A'_{\gamma\beta}B_\beta)$ for each closed B_β. Equality of these sets alone does not justify this inference; their functionality is established only in Theorem 2.44.

   The inference can be justified within the existing induction. For a fixed closed B_β, the pair $\langle V(B_\beta), V(A_{\gamma\beta}B_\beta)\rangle \in V(A_{\gamma\beta})$ belongs also to $V(A'_{\gamma\beta})$. By the definition of this graph-set, it therefore equals $\langle V(C_\beta), V(A'_{\gamma\beta}C_\beta)\rangle$ for some closed C_β, so

   $$V(C_\beta) = V(B_\beta), \qquad V(A'_{\gamma\beta}C_\beta) = V(A_{\gamma\beta}B_\beta).$$

   The induction hypothesis at β yields $C_\beta \sim_T B_\beta$. The corrected Theorem 2.41(2) and the induction hypothesis at γ then give $V(A'_{\gamma\beta}C_\beta) = V(A'_{\gamma\beta}B_\beta)$. The required pointwise equality follows, and the rest of the proof is unchanged.

5. Minor issues:

   (a) The opening paragraph of Section 2.2 (p. 1313) admits any positive cardinality for M_o and an arbitrary proper designated subset. Definition 2.19 forces $D \neq \emptyset$ by reflexivity, and hence $|M_o| \geq 2$ because D is proper. The appropriate formulations are "any cardinality at least 2" and "an arbitrary nonempty proper subset", respectively.

   (b) Footnote 3 to Definition 2.21 reverses the arguments in the currying of Eq_α, contrary to Convention 2.5 and the evaluation used in Lemma 2.26(1). The correct currying clause is $(\mathrm{Eq}'_\alpha(m))(m') = \mathrm{Eq}_\alpha(m, m')$. The reversal does not affect designatedness, but it changes the truth-value interpretation.

   (c) In the proof of Theorem 2.44, the displayed congruence uses B_β, C_β as arguments of $A_{\beta\alpha}$, although the chosen formulas are B_α, C_α. The intended congruence is $A_{\beta\alpha}B_\alpha \sim_T A_{\beta\alpha}C_\alpha$.

   (d) In Section 3.1, in the amendment to Definition 2.8 (p. 1325), the concise form of modus ponens has conclusion A. The correct form is $A, A \to B \;/\; B$.

   (e) The right-to-left proof of Corollary 3.11 assumes $F(m_\alpha) \in D$ for all $m_\alpha \in D$; the latter formula should read $m_\alpha \in M_\alpha$.

(f) Remark 3.16 refers to Sections 3.1 and 3.2 for the joint amendments defining $\mathrm{TT}_{\to\triangle}$. The reference should be to Sections 3.1 and 3.3.

(g) Reference 3 assigns the chapter *Introduction to mathematical fuzzy logic* to volumes 37–38 of *Studies in Logic*, although it occurs in volume 37 only.

(h) Reference 6 dates *On fuzzy type theory* to 2004, but the publication year is 2005.

4.3 Status of the results

The substantial gap concerns the proposed elimination of extensionality in Theorem 2.13. Adding abstraction congruence yields a valid amended alternative axiomatization, but it remains unestablished whether extensionality is derivable from the alternative system as printed. This issue does not affect the original axiomatization of TT_0, which retains extensionality.

Example 3.4 requires the explicit condition on truth-value equality, but its intended construction and conclusion about the propositional fragment remain valid. The local corrections to Theorems 2.41 and 2.43 preserve the canonical-model construction and the completeness theorems. No fundamental defect was found in the paper's main results.

5 *Maxima and minima in fuzzified linear orderings* (Běhounek, 2016)

This section records errors in [14]. The entire paper, including the references, was checked as part of a systematic error-checking effort covering several selected papers. The review concerns mathematical correctness and incorrect references, rather than stylistic matters or routine omitted proof steps. The checks were carried out and written up by GPT-6 Astra.

5.1 Summary

The principal existence theorem, Theorem 3.2, is correct. One substantial error was identified: Theorem 2.24(1) is false for arbitrary reflexive relations and requires the additional assumption of symmetry. Four further findings are moderate: overstatements concerning connectedness and minima in truth-value chains; an invalid example on the extended real line; an incorrect suggestion of a Łukasiewicz counterexample for strong-intersection maxima; and a wrong use of duality in the proof of Corollary 3.8, together with an overstatement about minima following Corollary 3.9. There are also three minor reference issues. No fundamental error was found. In particular, the weak lattice-completeness conclusions of Corollaries 3.8–3.9 remain valid under their printed assumptions.

5.2 Findings in more detail

1. Substantial: *Theorem 2.24(1) is false without symmetry of E.*

   Theorem 2.24(1) asserts $E_{L_E} = E$ for every reflexive fuzzy relation E. However, e.g., if $X = \{x, y\}$ with $x \lhd y$ and E is the crisp order $\trianglelefteq$, then E is reflexive but $E_{L_E} = \mathrm{Id} \neq E$. The cited proof in [12, Thm. 3.5(1)] implicitly assumes symmetry when replacing E^{-1} by E.

   The identity is valid if E is both reflexive and symmetric. This additional hypothesis restricts the statement rather than establishing it in its printed generality. It is already satisfied when E is a similarity, so the correspondence used in the paper's principal existence results is unaffected.

2. Moderate: *The connectedness claims in Example 2.17 and the conclusion about minima in Example 3.6 are too general.*

   Example 2.17 asserts that $\Leftrightarrow$ and $\Rightarrow$ are left-connected at every nonzero element and right-connected at every element other than 1 of an arbitrary complete residuated chain. Both assertions fail in the two-element Boolean algebra. The right-connectedness assertion also fails at 0 in the standard Gödel algebra: $y \Rightarrow 0 = 0$ for every $y > 0$.

   The latter defect invalidates the conclusion in the last paragraph of Example 3.6 that every nonempty fuzzy set of membership degrees has a nonempty minimum. Indeed, take $\mathbf{L} = [0,1]_{\mathrm{G}}$, $X = [0,1]$, $Lxy = (x \Rightarrow y)$, and the crisp set $A = (0,1]$. Then $A^{\downarrow L}(0) = 1$, whereas, for $x > 0$,

$$A^{\downarrow L}(x) = \inf_{0<y<x} (x \Rightarrow y) = \inf_{0<y<x} y = 0.$$

   Thus $\operatorname{Min}_L A$ is empty, although L is left-connected at every nonminimal element and right-connected at every interior element of X.

   The connectedness claims can be corrected as follows. Left-connectedness at x follows whenever some $0 < y < x$ exists; at a least positive element c, it holds precisely when $c \Rightarrow 0 > 0$. Right-connectedness holds at every $0 < x < 1$, and at 0 precisely when some $y > 0$ satisfies $y \Rightarrow 0 > 0$. These qualifications apply to both $\Rightarrow$ and $\Leftrightarrow$. The claim about minima in Example 3.6 is valid, for example, with the additional assumption of right-connectedness at 0. The claim about maxima is valid for every complete residuated chain: if $Ax > 0$ for some $x > 0$, then $A^{\uparrow L}(x) \geq x > 0$; if the support is $\{0\}$, then 0 itself is a maximum to positive degree.

3. Moderate: *Adjoining infinities does not make the Euclidean example satisfy the maximum-existence hypotheses.*

   Example 3.6 asserts that every nonempty fuzzy subset of $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ has a nonempty maximum in L_{Eucl} because the extended order is complete. The required left-connectedness at $+\infty$ is missing. In fact, any compatible extension E of the specified Euclidean similarity must satisfy, for finite $x < y$ with $y - x \geq 1$,

$$E(x, +\infty) \leq E(x, y) = 0.$$

   Thus $+\infty$ cannot be left-connected in such an extension. For the crisp set $A = \mathbb{R} \subseteq \overline{\mathbb{R}}$, $A^{\uparrow L_E}(x) = 0$ at every finite x, since there is a $y \in \mathbb{R}$ with $y \geq x + 1$. Neither added endpoint belongs to A, so $\operatorname{Max}_{L_E} A$ is empty.

   The assertion can be restricted to fuzzy subsets of $\mathbb{R}$ with support bounded above, for maxima, and bounded below, for minima. The subsequent statement in Example 3.6 about bounded supports is correct. An assertion covering all fuzzy subsets of the extended real line requires a different similarity, with the appropriate connectedness at the added endpoints; merely extending E_{Eucl} cannot provide this.

4. Moderate: *The suggested Łukasiewicz counterexample for strong-intersection maxima cannot exist.*

   The paragraph preceding Corollary 3.7 (p. 91) suggests constructing a counterexample to the $*$-based version of Theorem 3.2 in the standard MV-algebra, using $0.5 *_{\mathrm{Ł}} 0.5 = 0$. In fact, the $*$-based version of that theorem is valid over $[0,1]_{\mathrm{Ł}}$. To see this, use Lemma 3.1 to choose $x_0 \in \operatorname{Supp} A$ with $c = E(a, x_0) > 0$, where $a = \sup_{\trianglelefteq}(\operatorname{Supp} A)$, and put

$$b = \sup\{Ay : x_0 \trianglelefteq y \trianglelefteq a\}.$$

There is an x in this interval with $Ax > \max(0, b - c)$. Compatibility gives $E(y, x) \geq c$ whenever $x \lhd y \trianglelefteq a$, whence

$$A^{\uparrow L_E}(x) \geq \min(1, 1 - b + c), \qquad Ax *_{\mathrm{L}} A^{\uparrow L_E}(x) > 0.$$

The general warning about $*$-based maxima is nevertheless correct and can be supported by a nilpotent-minimum example instead. Let $\mathbf{L} = [0, 1]_{\mathrm{NM}}$, where $u *_{\mathrm{NM}} v = \min(u, v)$ if $u + v > 1$ and is 0 otherwise. On $X = [0, 1]$, let $E(x, x) = 1$ and $E(x, y) = 1/4$ for $x \neq y$; let $Ax = 3/4$ for $x < 1$ and $A1 = 0$. The similarity E is compatible and left-connected at 1, but for every $x < 1$,

$$A^{\uparrow L_E}(x) = (3/4 \Rightarrow_{\mathrm{NM}} 1/4) = 1/4, \qquad Ax *_{\mathrm{NM}} A^{\uparrow L_E}(x) = 0.$$

Thus $\mathrm{Max}^*_{L_E} A$ is empty. Corollary 3.7 itself is correct.

5. Moderate: *The proof of Corollary 3.8 uses the wrong direction of connectedness, and the subsequent claim about minima needs an additional assumption.*

   The proof applies the dual of Corollary 3.5 to $A^{\uparrow L}$, although that dual requires right-connectedness at the infimum of its support, whereas the assumptions provide left-connectedness. Showing that $A^{\uparrow L}$ is nonempty does not supply the missing hypothesis. Corollary 3.8, and hence Corollary 3.9, nevertheless remain true as stated.

   A proof under the printed assumptions follows by applying the maximum-existence result to lower cones. If $\bot$ is the least element of X, then $L(\bot, y) = 1$ for every y, and hence $B^{\downarrow L}(\bot) = 1$ for every fuzzy set B. Corollary 3.5 together with Remark 3.3 therefore gives

   $$\mathrm{Inf}_L B = \mathrm{Max}_L(B^{\downarrow L}) \neq \emptyset.$$

   Taking $B = A^{\uparrow L}$ and using Theorem 2.30(2) yields $\mathrm{Sup}_L A \neq \emptyset$. This also establishes the claimed weak lattice completeness without any right-connectedness assumption.

   The paragraph following Corollary 3.9 (p. 91) goes further and claims nonempty maxima *and minima* for every nonempty fuzzy set under these same assumptions. Its assertion about minima is false, as witnessed by the Gödel example in item 2 above. The missing assumption concerns this prose assertion, not the statements of Corollaries 3.8–3.9.

   That assertion becomes valid upon additionally requiring right-connectedness at every nonmaximal element. Left-connectedness alone supplies the maxima and weak lattice completeness, but not the minima of arbitrary nonempty fuzzy sets.

6. Minor issues:

   (a) The paragraph following Definition 2.28 (p. 89) inaccurately refers to the unchanged properties of $*$-based maxima in [19, Thm. 5.12]: the corresponding claims actually proved in [12, Thm. 4.6] use $Lxy \wedge Lyx$ instead of $Lxy * Lyx$ in part (2), and additionally assume linearity of L for the bound by Exy in part (3). The correct reference is to these modified claims in the technical report, already cited for the proofs and discussion, rather than to an unchanged transfer of the earlier properties.

   (b) Reference 5 (p. 92), Höhle's *Quotients with respect to similarity relations*, gives the page range as 31–34; the correct range is 31–44.

   (c) Reference 26 (p. 93), Novák's *On fuzzy type theory*, gives the publication year as 2004; the correct year is 2005.

5.3 Status of the results

Theorem 2.24(1) is false as stated; adding symmetry restricts its scope to a class on which the identity is valid.

The incorrect claims about connectedness, maxima, and minima in arbitrary truth-value chains and the extended Euclidean line in Examples 2.17 and 3.6 must be restricted as indicated above. The suggested Łukasiewicz counterexample for strong-intersection minima (p. 91) is unavailable, but a nilpotent-minimum counterexample supports the intended general warning. Corollaries 3.8–3.9 are also valid under their printed, merely left-connectedness assumptions, after the proof correction above.

Theorem 3.2 and its maximum-existence corollaries are valid under their stated assumptions; their minimum-existence duals require the corresponding right-connectedness assumptions. The reference concerning the two notions of maxima needs to identify the modified properties already proved in the technical report. These corrections do not undermine the paper's principal existence results.

6 *In which sense is fuzzy logic a logic for vagueness?* (Běhounek, 2014)

The following corrections address errors found in [13], a conference paper that sketches the argument for a consequence-based account of vagueness. GPT-6.1 Sol conducted the review. Page numbers follow the article's own printed numbering (1–13); page numbers in the proceedings table of contents (26–3á) are offset by 25.

6.1 Summary

Two substantial findings concern the omitted restriction to a fixed height structure in the formalization of *tall* and the passage from standard first-order Łukasiewicz consequence to provability. Two moderate findings concern the scope of standard completeness for MTL and Gödel logic and the claim about calculating truth degrees in Section 10. Ten minor issues concern the quantifier calculus, two connective formulas, the historical attribution in Section 10, and the bibliography.

6.2 Findings in more detail

1. Substantial: *The model-class equalities for* tall *leave the fixed height structure implicit.*

   Section 4 specifies $\widetilde{T}\colon \mathbb{R}^+ \to [0,1]$, gives the values of a and b, and treats comparison of heights as crisp (footnote 3). Footnote 9 carries the same reading into Section 7. On this reading, height variables range over the length sort $\mathbb{R}^+$ in centimeters, a and b name 163 and 196, and $\geq$ is the usual crisp order on that sort. The interpretation of the unary fuzzy predicate T varies over the admissible interpretations on this fixed background. In the classical coding, $\widetilde{T}$ is a function into a separate sort of degrees with the fixed ordered domain $[0,1]$.

   The model-class equalities on pp. 5 and 8 leave this restriction implicit. Section 6 defines $\mathrm{Mod}(\mathcal{T})$ over all models, so specifying a height sort without fixing its domain and order would still allow unintended interpretations. For example, the height domain $\{163, 196\}$ with its usual restricted order, the intended values of a, b, and $\widetilde{T}(a) = 0$ and $\widetilde{T}(b) = 1$ satisfies (15)–(17). It falsifies $(\exists x)(\neg(a \geq x)\ \&\ \neg(x \geq b))$, which is true throughout the intended $\mathcal{P}$. In the classical formulation, even the full first-order theory of the fixed ordered background with named endpoints allows countable models by downward Löwenheim–Skolem.

Let $\mathcal{K}$ consist of the structures with the indicated fixed background. The intended model-class equalities in Sections 4 and 7 then read $\mathcal{P} = \mathrm{Mod}_{\mathcal{K}}(\mathcal{T})$, with $\mathrm{Mod}_{\mathcal{K}}(\mathcal{T})$ the models of $\mathcal{T}$ in $\mathcal{K}$. Over that background, (15) has full truth exactly when $\widetilde{T}$ is nondecreasing, and (16)–(17) fix the endpoint degrees as in (5)–(6). The semantic equivalences in Sections 5 and 8 then identify supertruth with consequence relative to this same background. The ensuing equivalence with derivability requires completeness for the relative consequence relation.

2. Substantial: *Standard first-order Łukasiewicz consequence and finitary provability diverge.*

   The passage to derivability on p. 9 invokes the alternatives in Section 6. Enlarging the class of truth-degree algebras, as proposed in footnotes 7 and 10, changes the consequence relation. Section 9 (p. 10) likewise treats nonstandard models as dispensable for Łukasiewicz consequence, with a reference back to (8)–(9). For example, let p abbreviate $P(a)$, and let powers denote iteration of &. Each premise in $\mathcal{T} = \{p \to (\neg p)^n \mid n \geq 1\}$ requires the real degree of p to be at most $1/(n+1)$. Consequently $\mathcal{T}$ entails $\neg p$ over the standard algebra, whereas every finite subset has a standard model with p of positive degree. Finitary derivability therefore misses this consequence. Moreover, even standard first-order validity is not recursively enumerable [31].

   With $\exists$ defined by duality, the finitary predicate calculus is strongly complete over safe MV-chains [23, Theorem 5.2.3]. This gives a derivability characterization when the admissible models range over that semantics. To retain the standard truth-degree algebra and arbitrary object domains in a countable language, a sound, strongly complete option is the first-order rational Pavelka calculus with the rule from all $\bar{r} \to \varphi$ for rational $0 < r < 1$ to φ. Pavelka completeness yields every such premise when φ is a standard consequence, and the rule preserves full truth [23, Section 5.2.2]. The fixed height structure still calls for relative consequence as in Finding 1.

3. Moderate: *The strong standard completeness assertion for* MTL *and Gödel logic omits a language restriction.*

   Footnotes 8, 10, and 11 invoke standard completeness for consequence without restricting the predicate language. This omission is inherited from the formulation of Theorem 5.2.5 in [23], the paper's reference 2. Let p_α (for $\alpha < \omega_1$) and q abbreviate distinct unary predicates applied to a fixed constant, and take

   $$\mathcal{T} = \{p_\alpha \to q \mid \alpha < \omega_1\} \cup \{(p_\beta \to p_\alpha) \to p_\alpha \mid \alpha < \beta < \omega_1\}.$$

   In a standard model with q below 1, the premises force all p_α below 1 and strictly increasing with α. This is impossible in $[0, 1]$. Every finite subset, however, has a standard Gödel model with q below 1, obtained by assigning the finitely many relevant p_α increasing values below the value of q. These models work for MTL as well. Thus q is a standard consequence of $\mathcal{T}$ in both logics, but has no finitary derivation from it.

   The strong standard completeness claims hold for countable predicate languages, with the standard MTL models ranging over the left-continuous t-norms and their residua.

4. Moderate: *A plurivaluation can determine exact degrees.*

   The practical discussion on p. 12 rules out calculating any truth degrees or membership functions on the grounds that they vary across admissible models. But (4)–(6) already fix $\widetilde{T}(h) = 0$ for $h \leq a$ and $\widetilde{T}(h) = 1$ for $h \geq b$ in every model of the example.

   For $a < h < b$, every degree in $[0, 1]$ is possible. For any such degree, a nondecreasing membership function can take that constant value throughout (a, b) and the stipulated

values outside that interval. Thus the postulates determine some degrees exactly and leave others underdetermined. They leave the entire membership function underdetermined in this example.

5. Minor issues:

   (a) The axiom list on p. 7 needs the syntactic convention $(\exists x)\varphi \equiv_{\mathrm{df}} \neg(\forall x)\neg\varphi$, consistent with (11). If $\exists$ is primitive, the printed calculus is sound even under a constant-zero interpretation of $\exists$, which makes $(\exists x)(P(x) \to P(x))$ unprovable. With the convention made explicit, the two displayed universal schemes suffice for the intended finitary Łukasiewicz calculus, as in Table 6 of the paper's reference 2.

   (b) Footnote 6 repeats the definiens of the strong conjunction (12) for $\oplus$. The intended strong disjunction is $\varphi \oplus \psi \equiv_{\mathrm{df}} (\neg\varphi \to \psi)$.

   (c) The explanation of (16) on p. 8 interchanges the truth-degree endpoints. The required equivalence is $\|\neg\varphi\|_{\mathbf{M},v} = 1$ iff $\|\varphi\|_{\mathbf{M},v} = 0$, which indeed holds for residual negation in every t-norm logic and gives $\widetilde{T}(a) = 0$ from the full truth of $\neg Ta$. The printed equivalence does not correspond to (16) and only holds in IMTL or stronger logics.

   (d) The historical remark on p. 12 dates consequence-based fuzzy description logic to reference 8. For fuzzy description logics, Straccia had already defined entailment over a class of models and given a calculus for it [32, Sections 3.4 and 5]. Reference 8 extends this line of work to residuated continuous-t-norm semantics.

   (e) The book subtitle in reference 1 is *Logical, Philosophical, and Linguistic Perspectives*.
   The article title in reference 5 is *The Logic of Inexact Concepts*.
   The place of publication in reference 6 is Dordrecht, not "Dordercht".
   In reference 11 the chapter title is *Two Theories about Adjectives* and the book title is *Formal Semantics of Natural Language*.
   The publication year in reference 18 is 1999. Smith's initials in reference 19 are N. J. J.

6.3 Status of the results

Over the fixed height structure specified in the example, the axioms for *tall* define the intended plurivaluation. Supertruth coincides with consequence relative to that structure. The passage to derivability requires completeness for that relative consequence relation. With the existential convention made explicit, the finitary Łukasiewicz calculus covers the general MV-chain semantics. An infinitary calculus can cover standard Łukasiewicz semantics with arbitrary object domains. The MTL and Gödel completeness claims apply to standard semantics with unrestricted object domains and countable predicate languages. The *tall* example still exhibits indeterminacy in the middle range, alongside exact degrees determined by the meaning postulates.

7 *Graded properties of unary and binary fuzzy connectives* (Běhounek, 2012)

This section records errors identified in [10]. The review was performed by GPT-6.1 Sol.

7.1 Summary

One substantial finding concerns the lower bound for null elements in Theorem 5.1(N2). One moderate finding concerns the multiplicity estimate in Remark 3.5. One repairable finding concerns the proof of Theorem 3.20(U33). There are also minor errors in definitions, example values, multiplicity indices, notation, and references.

7.2 Findings in more detail

1. Substantial: *The asserted lower bound for arbitrary null elements is false.*

   Theorem 5.1(N2) asserts

   $$\mathrm{Hgt}(\mathbf{c}) \to \mathrm{Plt}(\mathbf{c}) \;\Rightarrow\; \mathrm{LNull}(\mathbf{c}, \eta)$$

   for arbitrary η. For the constant connective $\mathbf{c}(\alpha, \beta) = 0$, its antecedent is 1, whereas $\mathrm{LNull}(\mathbf{c}, 1) = 0$. The defect is not confined to connectives of zero height: in standard Łukasiewicz logic, the constant connective $\mathbf{c} = 0.2$ has antecedent 1 and $\mathrm{LNull}(\mathbf{c}, 1) = 0.2$. The assertion that the proof is analogous to that of (N1) does not establish (N2), since the proposed bound does not control the distance of η from the range of $\mathbf{c}$.

   A valid bound, retaining the dependence on the proposed null, is

   $$\big(\mathrm{Hgt}(\mathbf{c}) \to \eta\big) \wedge \big(\eta \to \mathrm{Plt}(\mathbf{c})\big) \;\Rightarrow\; \mathrm{LNull}(\mathbf{c}, \eta).$$

   Indeed, the two conjuncts imply respectively $(\eta\, \mathbf{c}\, \alpha) \to \eta$ and $\eta \to (\eta\, \mathbf{c}\, \alpha)$ for every α. In particular, the uniform bound in (N1) also applies to nulls:

   $$\mathrm{Plt}(\mathbf{c}) \wedge \neg\, \mathrm{Hgt}(\mathbf{c}) \;\Rightarrow\; \mathrm{LNull}(\mathbf{c}, \eta).$$

   The printed bound is valid with the additional restriction $\mathrm{Plt}(\mathbf{c}) \le \eta \le \mathrm{Hgt}(\mathbf{c})$. All these amendments also hold for right nulls.

2. Moderate: *The compensation of a consequent multiplicity rounds in the wrong direction.*

   Remark 3.5 asserts $\mathrm{Cng}^m_{\lceil n/m \rceil}(\mathbf{u}) \Rightarrow \mathrm{Cng}_{n,m}(\mathbf{u})$. This is false when m does not divide n. For example, in standard Łukasiewicz logic take $\mathbf{u} = \mathbf{w}_{\mathrm{sq}}$, $n = 3$, and $m = 2$. Then $\mathrm{Cng}_2(\mathbf{u}) = 1$, so the asserted antecedent is 1. At $\alpha = 3/4$, $\beta = 1$, however, the defining implication for $\mathrm{Cng}_{3,2}(\mathbf{u})$ has value

   $$(\alpha \leftrightarrow \beta)^3 \to (\mathbf{u}\alpha \leftrightarrow \mathbf{u}\beta)^2 = \tfrac{1}{4} \to \tfrac{1}{8} = \tfrac{7}{8} < 1.$$

   The valid estimate, for $m \ge 1$, uses the floor:

   $$\mathrm{Cng}^m_{\lfloor n/m \rfloor}(\mathbf{u}) \;\Rightarrow\; \mathrm{Cng}_{n,m}(\mathbf{u}).$$

   Writing $k = \lfloor n/m \rfloor$, m copies of the defining implication give $(\alpha \leftrightarrow \beta)^{mk} \to (\mathbf{u}\alpha \leftrightarrow \mathbf{u}\beta)^m$. Since $mk \le n$, the antecedent $(\alpha \leftrightarrow \beta)^n$ implies $(\alpha \leftrightarrow \beta)^{mk}$, as required. The same replacement is needed in the following sentence of the remark.

3. Repairable: *The proof of* (U33) *incorrectly treats composition as preserving the same congruence index.*

   The proof of Theorem 3.20(U33) claims that $\mathrm{Cng}_n(\mathbf{u}), \mathrm{Cng}_n(\mathbf{v}) \Rightarrow \mathrm{Cng}_n(\mathbf{uv})$ is an instance of (U10). It is not: (U10) multiplies the indices. For instance, with $n = 2$ and $\mathbf{u} = \mathbf{v} = \mathbf{w}_{\mathrm{sq}}$ in standard Łukasiewicz logic, both premises are 1, but $\mathrm{Cng}_2(\mathbf{uv}) = 2 - 3(1/2)^{4/3} < 1$.

The stated (U33) nevertheless holds. Put $D = (\mathbf{v} \approx \mathbf{id})$. Transitivity of equivalence gives

$$D^2 \,\&\, (\alpha \leftrightarrow \beta) \;\Rightarrow\; (\mathbf{v}\alpha \leftrightarrow \mathbf{v}\beta).$$

Taking n copies and then using $\mathrm{Cng}_n(\mathbf{u})$ yields

$$\mathrm{Cng}_n(\mathbf{u}), D^{2n}, (\alpha \leftrightarrow \beta)^n \;\Rightarrow\; (\mathbf{uv}\alpha \leftrightarrow \mathbf{uv}\beta).$$

Residuation and generalization give precisely (U33), including $n = 0$.

4. Minor issues:

   (a) In the table of proof steps on p. 4, the quantifier shift labelled "of $\exists$ to the antecedent" has the wrong implication inside the existential quantifier. Its correct form is $(\exists x)(\varphi \to \nu) \Rightarrow ((\forall x)\varphi \to \nu)$, with x not free in ν.

   (b) The explanation of inclusion of binary connectives on p. 6 omits the quantifier over β. The correct formula is $\mathbf{c} \subseteq \mathbf{d} \equiv (\forall\alpha\beta)(\alpha\,\mathbf{c}\,\beta \to \alpha\,\mathbf{d}\,\beta)$, as correctly given in footnote 12.

   (c) Equation (8) omits clipping to the unit interval, so its printed $\mathbf{w}_{\mathrm{ns}}$ need not be a fuzzy connective and compositions such as $\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{ns}}$ need not be defined. The intended reading is $\mathbf{w}_{\mathrm{ns}}(\alpha) = ((\alpha + (\mathrm{rand}(\alpha) - 0.5)/100) \vee 0) \wedge 1$, as in the corresponding noise construction of [20, Ex. 3.5]. This restores the intended bounded noise and the subsequent uses of this connective.

   (d) The formula for bounded addition on p. 8 has $\ldots \vee 0$ instead of $\ldots \wedge 1$. The correct identity is $1 - (\alpha \,\&_{\mathrm{Ł}}\, \beta) = ((1 - \alpha) + (1 - \beta)) \wedge 1$.

   (e) Example 3.1 incorrectly assigns zero plinth and antitony to $\mathbf{w}_{\mathrm{eps}}$. Since its range is $[0.01, 1]$, the values of Plt, Ant, and AntCng in its row on p. 9 are all 0.01; the Cng_0 entry in Example 3.4 is likewise 0.01. Consequently, Example 3.10 should not say that only $\mathbf{w}_{\mathrm{N}}$ has nonzero plinth.

   (f) The opening formula of Remark 3.14 should be $\mathrm{Surj}_n(\mathbf{v}) \equiv (\forall\beta)(\exists\alpha)(\beta \leftrightarrow \mathbf{v}\alpha)^n$. It prints $\mathbf{u}$ on the left and omits the exponent on the right; the definition in Theorem 3.13(U14) is correct.

   (g) In Example 3.15, the bounds 0.528, 0.809, and 0.956 for $\mathbf{w}_{\mathrm{N}}\mathbf{w}_{\mathrm{sq}}$ were taken from the row for $\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{sq}}$ rather than that for $\mathbf{w}_{\mathrm{sq}}$. The displayed application of (U10) gives the stronger bounds $\mathrm{Cng} \geq 0.75$ and $\mathrm{Cng}_n = 1$ for $n \geq 2$, and the same correction applies to the bounds for $\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{N}}$. In particular, the assertion that all the former bounds except the last are inexact must be adjusted.

   (h) The final reference in the proof of Theorem 3.20(U35) should be to (U7), which transfers monotony, rather than to (U6).

   (i) Observation 4.3(C5) uses implication between unary connectives where it needs inclusion. The correct right-hand side is $(\forall\gamma)(\mathbf{c}(\mathbf{id}, \underline{\gamma}) \subseteq \mathbf{d}(\mathbf{id}, \underline{\gamma}))$.

   (j) The conclusion of Theorem 4.5(C12) should be $\mathrm{LMon}_{i\cdot j}(\mathbf{uc})$, rather than $\mathrm{Mon}_{i\cdot j}(\mathbf{uc})$. Its proof establishes the left-argument property, as required by its premise; the right-argument counterpart is analogous.

   (k) In Theorem 4.6(C26), the assertion $\neg\,\mathrm{LAnt}(\mathbf{p}_k)$ is valid only for $k = 1$, and its right-argument counterpart only for $k = 2$. In the unused coordinate the projection is constant, so $\mathrm{LAnt}(\mathbf{p}_2) = \mathrm{RAnt}(\mathbf{p}_1) = 1$.

(l) The first premise of Theorem 4.14(C39) omits its argument $\mathbf{c}$. It should read $\mathrm{MonCng}_{il,jr}(\mathbf{c})$.

(m) In the proof of Theorem 4.15(C40), the occurrences of $\alpha\,\mathbf{c}\,\alpha'$ and $\beta\,\mathbf{c}\,\beta'$ in the premises should be $\alpha \leftrightarrow \alpha'$ and $\beta \leftrightarrow \beta'$. In the proof of (C42), both lines involving the indices l_b, r_b should use $\mathbf{b}$ throughout, rather than $\mathbf{a}$.

(n) The conclusion of Theorem 4.15(C46) omits the factor n in both indices: it should be $\mathrm{MonCng}_{nhl,nkr}(\mathbf{uc})$. For example, with $n = 2$, $i = h = k = l = r = 1$, $\mathbf{u} = \mathbf{w}_{\mathrm{sq}}$, and $\mathbf{c} = \mathbf{p}_1$ in standard Łukasiewicz logic, both printed premises have value 1, but the printed conclusion has value 0.75. The corrected indices follow by taking n copies of the defining implication for the inner connective, just as in (C40).

(o) In the last paragraph of Example 4.19, the two claims of a 0.99 monotony bound from (C43) and (C80) concern $\mathbf{w}_{\mathrm{ns}}\mathbf{t}_{\Pi}$, rather than $\mathbf{w}_{\mathrm{rd}}\mathbf{t}_{\Pi}$, whose monotony is 1. The final 0.98 bound for $\mathbf{t}_{\Pi}(\mathbf{w}_{\mathrm{ns}}\mathbf{p}_1, \mathbf{w}_{\mathrm{ns}}\mathbf{p}_2)$ follows from (C81), rather than (C80).

(p) In the recalled formulas on p. 28, (12)–(13) omit the candidate unit or null η from both sides. The last formula in the same list should conclude $\mathrm{LNull}(\mathbf{c}, \zeta)$, rather than $\mathrm{LNull}(\mathbf{c}, \eta)$; Theorem 5.2(N4) gives the correct generalization.

(q) The last displayed estimate in Example 5.8 should have $\mathrm{LNull}(\mathbf{w}_{\mathrm{ns}}\mathbf{t}_{\Pi}, 0)$ as its first factor on the right, rather than the corresponding null degree at 1. The stated lower bound 0.975 then follows as intended.

(r) The quantifier in the definition of Ass on p. 30 must enclose the entire equivalence: $\mathrm{Ass}(\mathbf{c}) \equiv (\forall\alpha\beta\gamma)(((\alpha\,\mathbf{c}\,\beta)\,\mathbf{c}\,\gamma) \leftrightarrow (\alpha\,\mathbf{c}\,(\beta\,\mathbf{c}\,\gamma)))$. This quantifier-scope error is inherited from the definition of associativity in [20, Def. 3.16]; the subsequent arguments use the intended fully quantified formula.

(s) In Theorem 6.7(A16)–(A18), the arguments of SIdem_i and Idem should be $\mathbf{c}(\mathbf{up}_1, \mathbf{vp}_2)$, rather than the unary connective $\mathbf{c}(\mathbf{u}, \mathbf{v})$. Equivalently, the conclusions can be written as $\mathbf{id} \subseteq^i \mathbf{c}(\mathbf{u}, \mathbf{v})$ or $\mathbf{id} \approx \mathbf{c}(\mathbf{u}, \mathbf{v})$; its value at α is precisely the diagonal value used in the proofs.

(t) The reference to (U72) in the proof of Theorem 6.12(A41) should be to (C72).

(u) The definition of $\mathrm{LC}^{\triangle}(\mathbf{t})$ in Definition 7.3 leaves α free. A universal quantifier $(\forall\alpha)$ is needed to make it the stated property of $\mathbf{t}$ alone.

(v) In the proof of Theorem 7.7(V17), the asserted equivalence for $\triangle\varphi \to \psi$ omits a negation. The correct equivalence is $(\triangle\varphi \to \psi) \leftrightarrow (\neg\triangle\varphi \vee \psi)$, as correctly stated in (V7).

(w) In reference 12, the page range is 73–78 rather than 73–76, and the conference title has "Multiple-Valued Logic", rather than "Multiple-Valued Logics". In reference 14, Beliakov, Pradera, and Calvo are authors, not editors, so "(Eds.)" should be removed. Reference 17 omits the publication year, 2007.

7.3 Status of the results

Theorem 5.1(N2) is false as stated. The bounds in Finding 1 give valid replacements; the printed bound holds when the proposed null lies between the plinth and height of the connective. The preservation and transfer results of Theorems 5.2, 5.4, and 5.7 do not use (N2) and remain valid.

The multiplicity comparison in Remark 3.5 requires the floor in place of the ceiling. The subsequent theory uses the single-parameter congruence properties and does not rely on the erroneous comparison. Theorem 3.20(U33) remains valid with the corrected proof in Finding 3.

The missing multiplicities in (C46), the projection exceptions in (C26), and the notation and quantifier corrections restore the intended formal statements. The noise examples require the bounded version of $\mathbf{w}_{\mathrm{ns}}$; their quoted lower bounds are retained with that correction. The changes to the shift example and the composition estimates are local numerical corrections. With these amendments, the general preservation estimates and the ground-logic comparisons of Section 7 are retained.

8 *Extending Cantor–Łukasiewicz set theory with classes* (Běhounek, 2010)

This section records errors found in [9], an extended conference abstract with some proofs omitted. The review was performed by GPT-6.1 Sol. Page numbers refer to the preprint, with its first page counted as 1.

8.1 Summary

One fundamental finding concerns Theorem 1. The proposed model expansion can leave quantified formulas without truth values, and the consistency claim depends on White's defective proof. One moderate finding concerns the availability of the class FN after Definition 4 and in Section 3. Three minor issues concern the scope of the introductory diagonal argument, the logical language on p. 2, and a bound variable in Definition 4.

8.2 Findings in more detail

1. Fundamental: *The expansion in Theorem 1 can be unsafe, and the consistency claim rests on an unproved premise.*

   The proof takes all functions from the set domain D to the truth-value chain L as classes. Safety of the CŁ-model guarantees the bounds needed by its set formulas. The expanded formula $(\exists x)(x \in A)$ requires $\sup f[D]$ for every function $f\colon D \to L$. This bound need not exist in L.

   The obstruction occurs among CŁ-models if the base theory has a model. Indeed, set comprehension applied to $\neg((n-1)(x \in x))$ gives a set whose self-membership value a_n solves $a_n = \neg((n-1)a_n)$ for each $n \geq 2$, where multiples denote repeated MV-sums. In the ordered-group representation of L with unit u, this forces $na_n = u$. The distinct values a_n also force the domain D of the original CŁ-model to be infinite. In a nonprincipal ultrapower of the model, the element $\varepsilon = [(a_n)_{n\geq 2}]$ is positive and every finite multiple of ε is strictly below 1. The ultrapower is safe in the set language. This follows from Łoś's theorem applied to the classical many-sorted presentation with a truth-value sort and formula evaluations, in which the quantifier clauses assert least-upper-bound and greatest-lower-bound conditions. The set $\{n\varepsilon \mid n \geq 1\}$ has no supremum in the ultrapower chain. If b were its supremum, $b - \varepsilon$ would be a smaller upper bound, since $(n+1)\varepsilon \leq b$ for every n. A class with this range therefore makes $(\exists x)(x \in A)$ undefined. Strong completeness in the paper's reference 6 uses safe chain models [30].

   Independently, the assertion on p. 1 that CŁ is consistent is inherited from [34]. This is the premise of the consistency conclusion in Theorem 1. However, White's proof contains a flaw, and consistency of unrestricted CŁ thus remains open [33].

Over a complete MV-chain, in particular the standard chain, the full class expansion is safe. Class comprehension assigns to each instance its membership function, and equality of functions validates the class extensionality scheme. Set formulas retain their values. This yields equality of set-language semantic consequences over complete chains and gives consistency of CŁC conditional on the existence of such a base model. Syntactic conservativity remains unresolved.

2. Moderate: *Class comprehension need not supply the external class of standard numerals.*

   The example after Definition 4 and the discussion on p. 4 treat FN as a class available in the theory. Yatabe's overspill argument concerns set membership and does not provide a class object. In fact, if CŁC has a model, a countably saturated elementary extension of its classical many-sorted presentation gives a model with no such class. For a class A containing every standard numeral to degree 1, the conditions $\|x \in A\| = 1$ and $\|x = \bar{n}\| = 0$ for every standard n are finitely satisfiable by distinct standard numerals. Saturation realizes the conditions, giving a member of A outside the standard numerals even modulo Leibniz equality.

   The example is valid in full standard-valued expansions (i.e., in models of CŁ valued in the standard MV-algebra on $[0, 1]$ and expanded by *all* functions $D \to [0, 1]$ as classes), where the characteristic function of the standard numerals is available and overspill rules out its representation by a set [36]. More generally, whenever this characteristic function is represented by a class, the class is indeed proper. To see this, let θ be Yatabe's recursive relation. On standard numerals its defining clauses give $\|\langle\bar{n}, z\rangle \in \theta\| = (n + 1)\neg\|z \in z\|$. This calculation holds in any safe MV-chain model, since each standard successor has a unique predecessor up to Leibniz equality. If a set s represented the characteristic function, the set

   $$r = \{x \mid (\exists n)(n \in s \,\&\, \langle n, x\rangle \in \theta)\}$$

   would have self-membership value p equal to $\sup_{n\geq 1} n\neg p$. If $p = 1$, every multiple has value 0. If $p < 1$, every multiple is at most p, and $p - \neg p$ is a smaller upper bound, as in Finding 1. Both cases are impossible. Thus, whenever FN is represented by a class, it is proper, even over truth-value chains other than the standard MV-algebra. What fails is the general claim that every model has such a class: if CŁC has a model, the saturation argument in the previous paragraph produces a model in which FN is absent.

   For an internal example of a proper class in any model of CŁC, the diagonal class $[x \mid \neg\triangle(x \in x)]$ suffices. Its membership value at each set differs from that set's self-membership value, so it cannot be fully co-extensional with any set.

3. Minor issues:

   (a) The introductory claim on p. 1 about any bivalent or finitely valued operator includes constant truth functions such as $\varphi \mapsto 0$, which are already definable in the language. A more accurate formulation is that admitting a $\{0, 1\}$-valued connective with a nonconstant restriction to Boolean inputs, such as $\triangle$, into unrestricted set comprehension yields a contradiction. A general definition of kernels as sets in CŁ would make $\triangle$ definable, so this is indeed a basic construction of fuzzy set theory that is unavailable in the set language of CŁ on pain of contradiction.

   (b) The paragraph immediately after Theorem 1 names Ł$\forall$ when describing the Henkin-style class axioms. Their stated extensionality scheme uses $\triangle$, so the appropriate base logic for this description is Ł$\forall_{\triangle}$.

   (c) In Definition 4, the condition for an improper class should be $(\exists x)\triangle(x \approx A)$. The printed a is unbound.

8.3 Status of the results

The argument for Theorem 1 gives a safe expansion over *complete* truth-value chains, as explained in Finding 1. The general conservativity claim and the consistency of both unrestricted theories remain open.

Theorem 2 and the internal diagonal example of a proper class follow from the class axioms independently of Theorem 1. The kernel of a set x is available by class comprehension as $[q \mid \triangle(q \in x)]$. The use of FN as a class carrying standard arithmetic presupposes its representation in the class domain, as in Finding 2.

9 *Number-free mathematics based on t-norm fuzzy logic* (Běhounek, 2009)

This section records errors found in [8]. The review was performed by GPT-6.1 Sol. The paper is a conference treatment presenting preliminary constructions with some proofs omitted.

9.1 Summary

Two substantial findings concern Observation 5.1: the complement inequality in claim 3 and the lattice identities in claim 4. One moderate finding concerns the characterization of closedness following that observation. One repairable finding concerns the proof of Theorem 7.2. There are also five minor formula corrections in Sections 6–7 and Table 1.

9.2 Findings in more detail

1. Substantial: *The second complement inequality in Observation 5.1(3) is reversed.*

   The asserted inequality

   $$\mathrm{LimSup}(\setminus A, x_0) \geq \neg\, \mathrm{LimInf}(A, x_0)$$

   fails in standard product logic. Let $x_0 = 0$ and let the membership function of A be

   $$A(x) = \min(1, |x|).$$

   Then $\mathrm{LimInf}(A, 0) = 0$. Product negation is zero at every positive truth value, so $\setminus A$ is the crisp singleton $\{0\}$ and $\mathrm{LimSup}(\setminus A, 0) = 0$. The printed inequality therefore requires $0 \geq 1$. The same example disproves both one-sided versions covered by Footnote 8.

   The correct inequality has the opposite direction:

   $$\mathrm{LimSup}(\setminus A, x_0) \leq \neg\, \mathrm{LimInf}(A, x_0).$$

   For example, for any punctured neighborhood U of x_0, $\sup_U \neg A \leq \neg \inf_U A$. Taking the infimum over neighborhoods and using $\neg \sup_i a_i = \inf_i \neg a_i$ gives the corrected inequality. These are the usual residuated-negation and quantifier laws; the corresponding argument applies to one-sided limits as well. The first inequality in claim 3 remains valid, and both become equalities for involutive negation, as stated in the paper.

2. Substantial: LimInf *does not preserve max-unions, and* LimSup *does not preserve min-intersections; only inequalities hold.*

   Observation 5.1(4) asserts that LimInf commutes with both $\sqcap$ and $\sqcup$, and adds the analogous assertions for LimSup. The union identity for LimInf and the intersection

identity for LimSup are false even for crisp sets, independently of the chosen t-norm. Take $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$. At every $x_0 \in \mathbb{R}$,

$$\begin{aligned} \operatorname{LimInf}(A, x_0) &= \operatorname{LimInf}(B, x_0) = 0, \\ \operatorname{LimSup}(A, x_0) &= \operatorname{LimSup}(B, x_0) = 1. \end{aligned}$$

However, $A \sqcup B = \mathbb{R}$ and $A \sqcap B = \emptyset$, so their LimInf and LimSup are respectively 1 and 0. The examples also work for either one-sided limit. The interpretation of Observation 5.1(4) on p. 452 consequently gives a false criterion for being a limit point of an intersection, and its dual assertion is false as well.

The valid identities are

$$\begin{aligned} \operatorname{LimInf}(A \sqcap B, x_0) &= \operatorname{LimInf}(A, x_0) \wedge \operatorname{LimInf}(B, x_0), \\ \operatorname{LimSup}(A \sqcup B, x_0) &= \operatorname{LimSup}(A, x_0) \vee \operatorname{LimSup}(B, x_0). \end{aligned}$$

The other two identities must be weakened to

$$\begin{aligned} \operatorname{LimInf}(A \sqcup B, x_0) &\geq \operatorname{LimInf}(A, x_0) \vee \operatorname{LimInf}(B, x_0), \\ \operatorname{LimSup}(A \sqcap B, x_0) &\leq \operatorname{LimSup}(A, x_0) \wedge \operatorname{LimSup}(B, x_0). \end{aligned}$$

Monotonicity gives these inequalities. The identities follow by passing to a common smaller punctured interval, using the linearity of truth values and the usual quantifier shifts. They hold for one-sided limits too.

3. Moderate: *The proposed closedness condition describes absence of isolated points instead.*

   The paragraph following Observation 5.1 (p. 451) identifies

   $$(\forall x_0 \in A) \operatorname{LimSup}(A, x_0)$$

   with closedness. For crisp A, the function $x_0 \mapsto \operatorname{LimSup}(A, x_0)$ is the characteristic function of the derived set A', since the defining intervals exclude x_0. Thus the displayed condition says $A \subseteq A'$: every point of A is an accumulation point. For example, $A = \mathbb{Q}$ satisfies it but is not closed, while the closed set $A = \{0\}$ does not satisfy it.

   The condition reducing to classical closedness is the reverse inclusion,

   $$(\forall x_0)(\operatorname{LimSup}(A, x_0) \to Ax_0).$$

   The printed openness condition, expressed using LimInf, does have the stated crisp interpretation.

4. Repairable: *The proof of Theorem 7.2 drops the restriction on a tail index.*

   The step said to follow propositionally is

   $$\big((n > n_0) \to Cx_n x\big) \,\&\, \big((n > n_0) \to Cx_n x'\big) \to Cxx'.$$

   It is invalid when $n \leq n_0$: both factors are then 1, regardless of Cxx'. This failure occurs even for the crisp equality similarity and distinct x, x'; thus full symmetry and transitivity do not justify the step. Additionally, after the symmetry step, the tail formulas involve $Cx_n x$ and $Cx_n x'$, whereas the two limit predicates are defined using Cxx_n and $Cx'x_n$. Thus the final identification with the stated limit predicates is not justified as written.

   A direct replacement proves the theorem with its stated graded bound, using the usual symmetry predicate corrected in Finding 5e. Put

   $$U_m \coloneqq (\forall n > m) Cxx_n, \qquad V_k \coloneqq (\forall n > k) Cx'x_n.$$

For arbitrary cutoffs m, k, the index $r = \max(m, k) + 1$ lies in both tails. Universal specification, symmetry on $Cx'x_r$, and transitivity through x_r give

$$\operatorname{Sym} C \,\&\, \operatorname{Trans} C \,\&\, U_m \,\&\, V_k \to Cxx'.$$

Each graded hypothesis is used once. Generalization on the two cutoffs and existential quantifier shifts yield

$$\operatorname{Sym} C \,\&\, \operatorname{Trans} C \,\&\, \operatorname{Lim}_C(\vec{x}, x) \,\&\, \operatorname{Lim}_C(\vec{x}, x') \to Cxx',$$

which is precisely Theorem 7.2.

5. Minor issues:

   (a) In Section 6 (p. 452), an inner truth value $\alpha \in \mathrm{L}$ is said to represent the truth value of $\emptyset \in \alpha$. For $\mathrm{L} = \operatorname{Ker}\operatorname{Pow}\{a\}$, the correct atom is $a \in \alpha$, as in the following displayed correspondence.

   (b) The class operation in the next paragraph of Section 6 (p. 452) incorrectly restricts its bound variable by $x \in \mathrm{L}$. For $X_1, \ldots, X_n \in \mathrm{L}$, the correct expression is $\overline{c}(X_1, \ldots, X_n) = \{x \in \{a\} \mid c(x \in X_1, \ldots, x \in X_n)\}$; the resulting class must be supported on the fixed singleton.

   (c) In Section 7 (p. 452), the axiom expressing separation is printed as $Cxy \leftrightarrow x = y$. Its correct form is $\Delta Cxy \leftrightarrow x = y$, which expresses the preceding condition $c(x, y) = 1$ iff $x = y$. Without Δ, the displayed axiom forces $Cxy = 0$ whenever $x \neq y$.

   (d) The Łukasiewicz normalization in Section 7 (p. 452) is printed as $c(x, y) = (1 - d(x, y))/d_{\max}$. The correct formula is $c(x, y) = 1 - d(x, y)/d_{\max}$, where $d_{\max} > 0$ is a finite upper bound on the distances. In particular, the printed expression gives $c(x, x) = 1/d_{\max}$ rather than 1.

   (e) In Table 1, the definition of $\operatorname{Sym} R$ has Rzx in the consequent, leaving z free. The correct definition is $(\forall xy)(Rxy \to Ryx)$.

9.3 Status of the results

Observation 5.1(3) requires reversal of its second inequality; its involutive-negation specialization remains valid. In claim 4, the liminf–intersection and limsup–union identities remain valid, while the other two equalities require the weaker inequalities in Finding 2. These corrections also amend the associated topological explanation. The separate closedness formula requires reversal of its implication, without changing the definitions of LimSup or LimInf.

Theorem 7.2 retains its original statement and graded bound with the replacement argument in Finding 4 and the intended definition of symmetry restored. Theorem 7.1 and its metric-convergence interpretation do not depend on the false parts of Observation 5.1. The local corrections in Sections 6–7 restore the stated inner-truth-value construction, metric separation, and bounded-metric normalization. None of these findings invalidates the probability or distribution-function representations in Sections 3–4.

10 *Modeling costs of program runs in fuzzified propositional dynamic logic* (Běhounek, 2008)

This section records errors found in the proceedings version of [7]. The paper is a preliminary conference treatment. The review was performed by GPT-6.1 Sol. Page references follow the proceedings pagination, 11–18.

10.1 Summary

One substantial finding concerns the clause for correct feasible runs in Section 6, which can combine degrees from different runs. A repairable finding concerns a missing premise in the proof of soundness of the induction rule. There are also minor corrections to formulas, cost conventions, the wording of the claim about feasibility tests, citations, and bibliographic information.

10.2 Findings in more detail

1. Substantial: *The clause for correct feasible runs can combine the degrees of different runs.*

   The second modal clause near the end of p. 17 in the printed version (p. 9 in the corrected preprint) in Section 6 is intended to give the degree to which there is a run that is both admissible and feasible. With the choice operation of Section 5 applied separately to $\widetilde{R}$ and $\widetilde{C}$, the two degrees for a fixed pair of states can come from different branches. In product logic, let $\alpha = p? \cup q?$ on a single state w. Let the admissibility and feasibility degrees of the two tests be $(1, 1/2)$ and $(1/2, 1)$, respectively. Both runs have joint degree $1/2$, whereas $\widetilde{R}_\alpha ww = \widetilde{C}_\alpha ww = 1$. With $\widetilde{V}_\varphi w = 1$, the printed clause gives degree 1. The same two marginal values would arise from a single run with both degrees equal to 1, whose joint degree really would be 1.

   A joint relation can be defined by

   $$\widetilde{J}_\alpha ww' = \sup_\pi (r_\pi * c_\pi),$$

   where π ranges over the runs of α from w to w' and r_π, c_π are the admissibility and feasibility degrees of that same run. Using $\widetilde{J}_\alpha$ as the transition relation in (17) then gives the intended joint degree. If both components accumulate by $*$ along a run, the joint relation composes by the sup-$*$ operations of Section 5. Commutativity and associativity give the joint degree of a concatenation as the $*$-product of the two joint degrees, and preservation of suprema by $*$ gives the corresponding relational composition. Choice and finite iteration likewise take suprema of joint degrees.

2. Repairable: *The induction step in the proof of soundness of the induction rule drops one of its two premises.*

   In the proof of soundness of the induction rule (p. 17), write $A = \widetilde{V}_\varphi$ and let $F(B) = \widetilde{R}_\alpha{}^{\rightarrow} B$ be the forward image. The two assumptions give $F(A) \subseteq A$ and $F^n(A) \subseteq A$, respectively, by the image–preimage adjunction and the composition law. The printed translation of the induction step retains only $F(A) \subseteq A$ and does not use the translated induction hypothesis in the monotonicity argument.

   Using both inclusions gives $F^{n+1}(A) = F(F^n(A)) \subseteq F(A) \subseteq A$. By adjunction this gives the required inclusion into the subproduct preimage under $\widetilde{R}_\alpha^{n+1}$. Taking the pointwise intersection over the powers gives the conclusion for $\widetilde{R}_\alpha^*$ by Corollary 5.16 of reference 11.

   The author's corrected preprint (p. 8) already corrects this issue by supplying the missing induction hypothesis in the monotonicity argument.

3. Minor issues:

   (a) Equation (7) intersects a binary relation with a unary set. The intended relation is the identity restricted to V_φ, namely $R_{\varphi?} = \{(w, w) \mid w \in V_\varphi\}$. The fuzzy counterpart on p. 17 similarly has value $\widetilde{V}_\varphi w$ at (w, w) and zero off the diagonal.

(b) In the explanation preceding (13), the run of α_i goes from w_{i-1} to w_i. The printed target w_1 is correct only for the first leg.

(c) Equation (13) ranges over all intermediate states, including ones for which a leg is impossible. For the printed infimum, the cost combiner f must be defined to return $+\infty$ whenever either leg has cost $+\infty$. Equivalently, the infimum can be taken over intermediate states where both legs are possible, with the empty infimum equal to $+\infty$.

(d) The citations to reference 10 in the discussion of fuzzy composition and reflexive transitive closure (pp. 15–16) only cover the max–min case. It is Definition 4.1 and Theorem 4.2 of reference 11 that give the general sup-$*$ composition and the union law used for the closure construction.

(e) On p. 15, the parenthetical discussion following (17) and the paragraph introducing the fuzzy preimage notation refer to (16). Both passages actually use the fuzzy clause (17).

(f) The paragraph following (19) names the unary predicates V_α and $\widetilde{V}_\alpha$. The predicates in the displayed preimages are V_φ and $\widetilde{V}_\varphi$.

(g) In the proof of (9), the second factor of the min-intersection should be $\widetilde{V}_{[\beta]\varphi}$.

(h) The equality introducing the proof of (10) omits φ from the left-hand subscript. The correct left-hand side is $\widetilde{V}_{[\alpha^*]\varphi}$.

(i) The reference to Lemma B.8(L5) of reference 14 in the proof of soundness of the induction rule should point to the intersection law, Lemma B.8(L6).

(j) The qualification about feasibility tests in the penultimate paragraph of Section 6 (p. 18) is ambiguous about their scope. Under the diagonal test clause, allowing unrestricted feasibility tests permits noncrisp admissibility whenever a test condition has an intermediate truth degree. Crisp admissibility can be retained by requiring crisp test conditions, for example by using $(\Delta\psi)?$ for a graded feasibility formula ψ.

(k) The place of publication in reference 2 is Dordrecht, not Dordercht. The surname of the third author in reference 7 is Tiuryn.

10.3 Status of the results

The argument in Finding 2 establishes soundness of the induction rule as stated. The soundness arguments for (8)–(10), modus ponens, and necessitation retain their conclusions under the single-relation semantics of Section 5.

For Section 6, the modality for correct runs can still use the admissibility relation alone. The modality for correct feasible runs requires a joint score when alternative runs can have different admissibility and feasibility degrees. Crisp admissibility can coexist with tests of feasibility under the restriction in Finding 3j.

11 *Relative interpretations over first-order fuzzy logic* (Běhounek, 2006)

The errors identified in [6] are described below. The review was conducted by GPT-6.1 Sol. The paper is a conference treatment that presents the applications in outline.

11.1 Summary

Five substantial findings concern capture of variables in Definition 4, ill-formed comparisons in Definition 7, failure of closure under composition in Observation 11, failure of proof transfer in Theorem 12, and the criterion for conservative function definitions in Remark 16. Three moderate findings concern the axiomatization of crisp identity on p. 5 and the relativization and Cartesian-product claims in Example 17. Three minor findings concern the binders in Lemmas 1–2 and the arity and equality clauses of Definition 4.

11.2 Findings in more detail

1. Substantial: *The variable translation in Definition 4 can capture free variables.*

   The assignment of x_i^s to $F_s^\star(x_i^{s\star})$ reuses the same target variable when two source sorts have the same carrier sort. For example, let distinct sorts s, t both map to a single target sort and let both encoding functions be the identity. The formula $(\forall x_1^t)(x_1^s = x_1^t)$ then becomes $(\forall x_1)(x_1 = x_1)$, losing its free variable. Renaming the source binder to x_3^t before translating gives a formula with a different truth value in any two-element target model. In particular, the hypothesis of Lemma 10 is satisfied by identity encodings, but the translated intersubstitutivity axiom with premise $x_1^s = x_2^s$ and matrix $(\forall x_1^t)(x_1^s = x_1^t)$ is false when $x_1 = x_2$ in that model. Its first translated comparison formula is true and its second, $(\forall x_1)(x_2 = x_1)$, is false.

   An interpretation can instead use a fixed injective assignment of source variables to target variables, respecting the prescribed carrier sorts. Each quantifier then binds the assigned variable. This restores the distinction between free and bound occurrences and preserves renaming of bound variables. For Lemma 10, the equality of the two encoded terms then licenses replacement in the translated context by target identity congruence, including under quantifiers whose binders are distinct from the variables of those terms.

2. Substantial: *Definition 7 admits ill-formed comparisons.*

   Sharing nonlogical symbols does not ensure that φ or $\varphi(x_1^\star, \ldots, x_k^\star)$ is a formula of $\mathbf{L}'$. For example, let P have argument sort s in both languages, let r be a target sort incomparable with s, and send the source sort s to r using the identity encoding and P to a predicate of argument sort r. Then $\varphi = P(x^s)$ meets the stated common-symbol condition, but $P(x^\star)$ is ill-formed. Bound variables can also use source sorts absent from $\mathbf{L}'$. For invariance, the mere presence of the variable sorts in $\mathbf{L}'$ does not ensure that source subsumptions used in typing φ hold there.

   Definition 7 can be corrected by adding two well-formedness conditions. In the definition of absoluteness, the result $\varphi(x_1^\star, \ldots, x_k^\star)$ of capture-free substitution for the free variables must be a formula of $\mathbf{L}'$. In the definition of invariance, φ itself must be a formula of $\mathbf{L}'$. With both comparisons well-defined, Observation 8 follows by transitivity of equivalence.

3. Substantial: *Composition can fall outside Definition 4.*

   Already with one sort, if the first translation encodes x by $F_s^\star(x)$ and the second encodes variables by G, the composite sends x to $F_s^{\star\diamond}(G(x))$. Definition 4 requires one application of a unary function symbol to a variable. The composite need not have that form or have a name in the target language. The categorical assertion following Observation 11 and the identical interpretation in Example 14 have a related identity issue when the language lacks an identity-function symbol. Naming composites also leaves a separate obstacle for theories. The second translation can fail to preserve the first translation's derivations, by Finding 4. Indeed, take the first map to be the identical interpretation of a theory with the axiom $(\exists x)(x = c)$ into the empty theory in the language with c.

The second map in that finding sends this axiom to an unprovable sentence, even if the identity and composite encodings are named by definitions.

Unary terms in the designated target variable can serve as encodings, with the variable itself serving as the identity encoding. Equivalently, their values can be named by conservative definitions as in Remark 16. Composition of the resulting formula translations preserves theorem transfer when both factors preserve it.

4. Substantial: *Definition 9 does not ensure the proof transfer asserted in Theorem 12.*

   In the translated quantifier matrix the variable occurs through $F_s^\star$, whereas the translation of an admissible substitution term can lie outside its range. For a one-sorted counterexample, take $\mathbf{T}$ to be the empty theory in a language with a constant c, and let $\mathbf{T}'$ have c and a unary $F_s^\star$ with the injectivity axiom. Map c to itself and x to $F_s^\star(x)$. The translated identity axioms hold by identity congruence in the target language, so this is an interpretation by Definition 9. The source theorem $(\exists x)(x = c)$ translates to $(\exists x)(F_s^\star(x) = c)$, which fails in the two-valued model on $\mathbb{N}$ with $c = 0$ and $F_s^\star(n) = n + 1$. The proof's appeal to substitutability misses the distinction between substituting $t^\star$ into a translated context and substituting it for the variable *inside* $F_s^\star$.

   A sufficient amendment for the first-order calculus is to use the variable assignment of Finding 1 and require, for each source term t of sort $w \preceq s$,

   $$\mathbf{T}' \vdash (\exists z^{s\star})(F_s^\star(z^{s\star}) = t^\star),$$

   with $z^{s\star}$ fresh. Writing $B(v)$ for the translated matrix with the encoded argument replaced by a fresh variable, identity congruence and quantifier shifting give

   $$(\exists z^{s\star})(F_s^\star(z^{s\star}) = t^\star) \to \big((\forall z^{s\star})B(F_s^\star(z^{s\star})) \to B(t^\star)\big).$$

   This proves the translated specification axiom. The same witness condition yields the translated existential-introduction axiom. Injective variable assignment preserves the side conditions on generalization and quantifier shifts. The other proof steps then transfer by the original induction. The range condition is also necessary for full theorem transfer, since its instances translate logical theorems $(\exists x^s)(x^s = t)$ with fresh x^s. Sort renamings with identity encodings satisfy it when they preserve subsumption.

   Lemma 15 survives independently of unrestricted proof transfer. Its invocation of Corollary 3 does not by itself justify translating the equivalences used to flatten terms. A direct induction on t of principal sort w gives $\mathbf{T}' \vdash t^\star = F_w^\star(t)$. In fact, the identity instance of (3), evaluated on a term of sort $v \preceq u$ in both argument positions, equates the v- and u-encodings of that term. One then instantiates (4) with the arguments $t_1, \ldots, t_k$ and with $F(t_1, \ldots, t_k)$ in place of y^s, using identity congruence and the induction hypotheses on the arguments. Equation (3) now handles arbitrary atomic formulas. The rest of the formula induction and the conservativity argument in Lemma 15 apply.

5. Substantial: *The existence condition in Remark 16 can give a nonconservative function extension.*

   In a logic with Δ, provability of $(\exists y)\varphi(y)$ may express a supremum of 1 without any value attaining 1. For example, over first-order ŁΠ take the theory with axioms $(\exists y)P(y)$ and $(\forall y)\neg\Delta P(y)$. It has a standard model on the positive integers with $P(n) = 1 - 1/(n+1)$. Adding a constant c with defining axiom $P(c)$ gives $\Delta P(c)$ by necessitation and $\neg\Delta P(c)$ by specification. The extended theory is inconsistent.

   For logics with Δ, the sufficient existence premise is $(\exists y)\Delta\varphi(\vec{x}, y)$, with the parameters universally closed, as in Theorem 28 of [21], the paper's reference 1. With a unique

witness of truth degree 1, the crisp graph $\Delta\varphi$ also gives elimination of the new symbol. The amended existence premise is necessary for conservativity here because the defining axiom, Δ-necessitation and existential introduction already give it in the extended theory. The ordinary existence premise does suffice for conservative Skolemization over the standard Δ-free core fuzzy calculi [23, Theorems 5.1.6–5.1.7].

6. Moderate: *The proposed identity axioms on p. 5 do not force crispness over all the stated base logics.*

   Over the three-element Gödel chain $0 < a < 1$, take a two-element domain and interpret $=$ by 1 on the diagonal and by a off it, in a language with no other predicates. The quotient of the chain identifying a with 1 commutes with the connectives and the finite quantifier operations. In the quotient, all values of $x = y$ are 1, so every formula has a value independent of the assignment there. Consequently, the values of $\varphi(x)$ and $\varphi(y)$ agree in the quotient and their equivalence has degree at least a. When $x = y$ has degree 1, the two assignments coincide. Thus reflexivity and the displayed intersubstitutivity scheme both hold in this model of graded equality.

   The usual crispness axiom $(x = y) \vee \neg(x = y)$ can be added, with the quotient by equality at degree 1 giving genuine identity. In the Δ-setting the strengthened intersubstitutivity axiom with Δ applied to the equivalence, as in Definition 23 of the paper's reference 1, already yields crispness.

7. Moderate: *The relativization in Example 17 requires conditions on the restricted domains.*

   The relativization item allows any crisp class as the domain of discourse. On the empty crisp class the relativization of the logical theorem $(\exists x)(x = x)$ is false. When the source includes the tuple apparatus of Section 4 of the paper's reference 1, there is also a nonempty obstruction. If the whole object sort is restricted to a two-element crisp class, the interpreted pairing axioms require an injection of its square into that same class.

   For the tuple-free comprehension and extensionality theory, a nonempty crisp base class suffices. At each higher sort one uses the represented fuzzy subclasses supported on the preceding restricted domain. Comprehension in the ambient theory gives their relative comprehension by cutting the defining formula to that domain. Extensionality holds because membership is zero outside the domain. For tuple-bearing versions the interpreted domains must also admit the required tuple structure, for example by being closed under the ambient tuple operations and projections with the indicated sorts.

8. Moderate: *The projection identities in the transposition item of Example 17 fail for arbitrary factors.*

   With existential projection and the usual product for the chosen t-norm $*$, in standard semantics we have

   $$\mathrm{dom}(A \times B)(x) = A(x) * \sup_y B(y), \qquad \mathrm{rng}(A \times B)(y) = \big(\sup_x A(x)\big) * B(y).$$

   For instance, if A has membership 1 somewhere and B has constant membership $1/2$, the first identity asserted in the example fails there. Taking $B = \emptyset$ gives a crisp counterexample.

   The asserted domain identity holds when $\sup_y B(y) = 1$, and the asserted range identity holds when $\sup_x A(x) = 1$. Transposition interchanges these two qualified laws after interchanging the factors.

9. Minor issues:

   (a) The binders in Lemma 1 and equations (1)–(2) of Lemma 2 can capture variables of t. For instance, with $t = f(x)$ and f interchanging two objects, the Lemma 1 instance for $\varphi(x, y)$ given by $x = y$ has true antecedent and false consequent. With $\varphi(x) = P(x)$, the same t refutes equation (1) when P is always false and equation (2) when P is always true. The displayed quantifiers should bind a fresh variable z, replacing the occurrences abstracted from the formula while leaving t untouched.

   (b) The predicate and function arity conditions in Definition 4 are written as implications whose antecedents already assume matching argument counts. Matching counts must be required for each symbol, including nullary symbols.

   (c) The equality clause in Definition 4 is written only for $x = y$. It should assign $t^\star = u^\star$ to $t = u$ for arbitrary terms t, u, as required for the use of identity in the proofs.

11.3 Status of the results

Definition 4 needs a capture-free variable assignment. With this correction Lemma 10 holds, and with the additional well-formedness conditions on Definition 7, Observation 8 holds as stated. Theorem 12 is false under the printed Definition 9 even with a single sort and an injective variable encoding; the range condition in Finding 4 suffices for proof transfer. The identical interpretation of Example 14 retains its conservativity characterization with identity encodings given by terms or conservative names. Such encodings also allow composition of the translations, and theorem-preserving interpretations compose. Lemma 15 holds by the direct term argument. Corollary 3 holds with fresh variables in the decomposition.

In the presence of Δ, the function introduction in Remark 16 requires the amended existence premise. The stated criteria for predicate definitions and nonempty crisp subsorts remain valid.

For Example 17 the relativization holds in the tuple-free setting over nonempty crisp domains and needs compatible tuple structure when tuples are present. Transposition still exchanges domains and ranges, while the displayed product projections require the respective qualifications in Finding 8. The map $x \mapsto \langle x, 0\rangle$ continues to give a representation of fuzzy classes by column relations. For the class fragment, the target sorts can be taken to be the image of the object map and the classes supported on that image. Taking the 0-section is inverse to $A \mapsto A \times \{0\}$ on those classes. Comprehension supplies both maps in the ambient theory, and membership and equality are transported by the resulting correspondence on the restricted sorts [24, Section 3].

12 *Extensionality in graded properties of fuzzy relations* (Běhounek, 2006)

This section records errors in [5]. The paper is a concise conference treatment, with some proofs deliberately abbreviated or omitted. The checks and this report were carried out by GPT-6 Astra. Page references below count the pages from 1, since the checked copy had no printed page numbers.

12.1 Summary

Three substantial errors were identified. Proposition 3.5 and Remark 2.5 incorrectly identify full similarity with full self-relative similarity without requiring reflexivity. Lemma 3.8(1) gives a false bound for the strong conjunction of extensionality and reflexivity. Observation 4.2

contains false equivalences whose forward implications also fail with the printed exponents. Counterexamples and verified restricted or weaker versions are given below. There is also one minor bibliographic error.

12.2 Findings in more detail

1. Substantial: *Self-relative similarity does not imply ordinary similarity in Remark 2.5 and Proposition 3.5.*

   Proposition 3.5 asserts $\Delta\mathrm{Sim}\, E \leftrightarrow \Delta\mathrm{Sim}_E E$ without assumptions on E. Definition 2.2 explicitly permits arbitrary E, and $\mathrm{Refl}_E E$ is identically true; it cannot ensure ordinary reflexivity. For example, on a one-element universe in standard product logic, let $E(a,a) = 1/2$. Then

   $$\mathrm{Sim}\, E = \tfrac{1}{2}, \qquad \mathrm{Refl}_E E = \mathrm{Sym}_E E = \mathrm{Trans}_E E = 1.$$

   Thus the two sides of the claimed equivalence have values 0 and 1. The proof cannot apply Corollary 3.4 in the reverse direction without first establishing its proximity hypothesis.

   The forward implication is correct. A verified characterization is

   $$\Delta\mathrm{Sim}\, E \leftrightarrow \Delta\mathrm{Refl}\, E \;\&\; \Delta\mathrm{Sim}_E E.$$

   Indeed, under full reflexivity, specializing the primed variables to the unprimed ones in $\mathrm{Sym}_E E$ and $\mathrm{Trans}_E E$ gives ordinary symmetry and transitivity. The other direction follows from Lemma 3.6 and Corollary 3.4. Consequently Proposition 3.5 is valid on the restricted class of fully reflexive relations, but is false in its printed generality.

   Remark 2.5 repeats the false equivalence and suggests $\Delta\mathrm{Sim}_E E$ as an alternative hypothesis for Lemma 2.3 and Corollary 2.4. That replacement alone is invalid: in the same one-point product example, equation (2) gives $E'(a,a) = 1/8 \neq E(a,a)$. If also $R(a,a) = 1/8$, then $\mathrm{Refl}_{2E} R = 1$ whereas $\mathrm{Refl}_{1E} R = 1/4$. The original hypothesis $\Delta\mathrm{Sim}\, E$ of those results can be retained; an equivalent self-relative formulation includes $\Delta\mathrm{Refl}\, E$ as above.

2. Substantial: *The strong-conjunction conclusion of Lemma 3.8(1) requires an additional graded resource.*

   Lemma 3.8(1) asserts

   $$\mathrm{Prox}\, E \;\&\; \mathrm{Trans}^2 R \to (\mathrm{Refl}_E^2 R \to \mathrm{Ext}_E R \;\&\; \mathrm{Refl}\, R).$$

   This is false even when E is a full similarity and R is fully transitive. Over the standard Łukasiewicz algebra, on four elements in the displayed order, take

   $$E = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} .8 & .8 & .8 & .6 \\ .8 & 1 & 1 & .8 \\ 0 & 0 & 1 & .8 \\ 0 & 0 & .8 & 1 \end{pmatrix}.$$

   Direct calculation gives

   $$\mathrm{Prox}\, E = \mathrm{Trans}\, R = 1, \quad \mathrm{Refl}_E R = \mathrm{Refl}\, R = .8, \quad \mathrm{Ext}_E R = .6.$$

   For example, $R \circ R \leq R$, and the maximum loss in transporting an entry of R within the two E-classes is .4. The antecedent $\mathrm{Refl}_E^2 R$ therefore has value .6, while $\mathrm{Ext}_E R \;\&\; \mathrm{Refl}\, R$ has value .4.

Two verified amendments are available. With the printed antecedent, the conclusion can use weak conjunction $\wedge$ in place of strong conjunction $\&$. Alternatively, the strong conjunction is justified by $\mathrm{Refl}_E^3 R$ in place of $\mathrm{Refl}_E^2 R$. To see the resource count, write $a = \mathrm{Refl}_E R$, $r = \mathrm{Refl}\, E$, $s = \mathrm{Sym}\, E$, and $t = \mathrm{Trans}\, R$, using juxtaposition for the t-norm product. The pointwise definitions give

$$st^2a^2 \le \mathrm{Ext}_E R, \qquad ra \le \mathrm{Refl}\, R.$$

The first inequality follows by transporting along $x'\ R\ x\ R\ y\ R\ y'$; the second follows on the diagonal. Hence rst^2a^2 bounds both conclusions separately, whereas rst^2a^3 bounds their strong conjunction. These are weaker statements, not repairs preserving the printed graded bound. Full reflexivity of R is also a sufficient additional assumption for the original bound; it is not claimed necessary.

Theorem 3.7 nevertheless remains true under its original assumptions. At truth degree 1, either corrected version of the lemma yields the required implication. Equivalently, if E is fully reflexive and symmetric, R is fully transitive, and $E \subseteq R$ holds fully, the two pointwise inequalities above give both full reflexivity and full E-extensionality of R. The converse follows from Lemma 3.8(2), which is valid as stated.

3. Substantial: *Observation 4.2 has false converses and insufficient reflexivity exponents.*

   The three displayed equivalences are false, and merely replacing their outer $\leftrightarrow$ by $\rightarrow$ does not suffice. The following counterexamples use standard Łukasiewicz logic; unspecified entries of R are zero. For reflexivity, on one element take $E(a,a) = R(a,a) = .9$. Then $\mathrm{Refl}\, E = .9$, $\mathrm{Refl}_E R = 1$, but $\mathrm{Refl}_E^{\exists} R = .8$. For symmetry, on two elements take $E = .9I$ and $R_{12} = 1$, $R_{21} = .8$. Then $\mathrm{Refl}^2 E = .8$, $\mathrm{Sym}_E R = 1$, but $\mathrm{Sym}_E^{\exists} R = .6$. For transitivity, on three elements take $E = .9I$ and $R_{12} = R_{23} = 1$, $R_{13} = .7$. Then $\mathrm{Refl}^2 E = .8$, $\mathrm{Trans}_E R = 1$, but $\mathrm{Trans}_E^{\exists} R = .6$. Thus all three proposed forward bounds fail.

   The converses fail independently, even with crisp relations: on two elements let $E_{12} = E_{21} = 1$, $E_{11} = E_{22} = 0$, and let R be the universal relation. All six relative properties in Observation 4.2 have value 1, while $\mathrm{Refl}\, E = 0$.

   Verified weaker replacements are

$$\begin{aligned}
&\mathrm{Refl}^2 E \to (\mathrm{Refl}_E R \to \mathrm{Refl}_E^{\exists} R),\\
&\mathrm{Refl}^4 E \to (\mathrm{Sym}_E R \to \mathrm{Sym}_E^{\exists} R),\\
&\mathrm{Refl}^4 E \to (\mathrm{Trans}_E R \to \mathrm{Trans}_E^{\exists} R).
\end{aligned}$$

   For $r = \mathrm{Refl}\, E$, one use of r on the diagonal gives $r\,\mathrm{Refl}_E R \le Rxx$, and a second use supplies the witness $x' = x$ in Definition 4.1. For symmetry and transitivity, setting the two outer primed variables equal to the corresponding unprimed variables costs r^2; using those same elements as existential witnesses costs another r^2. Universal generalization and, in the transitivity case, residuation across the existential intermediate variable give the displayed implications. The three examples above attain these corrected bounds, so the indicated exponents cannot in general be decreased.

   In particular, the intended comparison between universal and existential relative properties is valid when E is fully reflexive. This restricted comparison survives, but the printed graded equivalences do not. Observation 4.3 itself is correct: its composition formulas follow directly by renaming bound variables and residuating across the existential quantifiers.

4. Minor issues:

   (a) Reference 5 (PDF p. 8), Esteva and Godo's *Monoidal t-norm based logic: Towards a logic for left-continuous t-norms*, gives the volume of *Fuzzy Sets and Systems* as 123. The correct volume is 124.

12.3 Status of the results

Proposition 3.5 is false as stated; full reflexivity supplies the missing restriction. Its valid forward implication remains available, and Lemma 2.3 retains its valid original full-similarity hypothesis. The displayed reflexivity case of Corollary 2.4 follows from that lemma. Remark 2.5 cannot replace this hypothesis by self-relative similarity alone.

Lemma 3.8(1) is false as a graded assertion. Its verified weaker versions suffice for Theorem 3.7, whose original conclusion and assumptions are preserved. Lemma 3.8(2) and Lemma 3.6 are valid. Observation 4.2 must lose its converses and use the larger exponents above, or be restricted to fully reflexive E. Observation 4.3 needs no correction.

The supplied proof of Theorem 3.2(4) is sound, and Example 3.3 correctly demonstrates the distinction between graded extensionality and the new relative properties. Corollary 3.4 is a valid consequence of Theorem 3.2 by Δ-necessitation and combination of the component equivalences.

13 *Towards a formal theory of fuzzy Dedekind reals* (Běhounek, 2005)

This section records errors identified in [4]. The review was performed by GPT-6.1 Sol. The paper is a conference treatment with some proofs omitted and the algebraic operations and interval theory presented in outline.

13.1 Summary

Two substantial findings concern the definition of addition in Section 5, which does not preserve Dedekind cuts, and the definition of interval addition in Section 6, which depends on the chosen endpoint representation. One moderate finding concerns the claimed extension to arbitrary crisp posets. There are also seven minor formula, explanatory, and bibliographic corrections.

13.2 Findings in more detail

1. Substantial: *Addition over rational cuts need not produce a Dedekind cut.*

   The definition of $+_*$ in Section 5 (p. 953) fails to define a binary operation on $\mathbb{R}$, even on proper crisp inputs. Let

   $$A = \{q \in \mathbb{Q} \mid q > \sqrt{2}\}, \qquad B = \{q \in \mathbb{Q} \mid q > -\sqrt{2}\}.$$

   Both are valid upper Dedekind cuts, representing opposite irrational numbers. For every choice of $*$, the displayed extension-principle formula gives

   $$A +_* B = \{q \in \mathbb{Q} \mid q > 0\}.$$

   Indeed, rational $a > \sqrt{2}$ and $b > -\sqrt{2}$ cannot sum to 0, whereas every positive rational q admits such a decomposition. The resulting open cut violates the intended continuity axiom at 0: every positive rational belongs to it, but 0 does not. Thus the claimed

extension of crisp real addition fails independently of the quantifier slip in that axiom noted below.

A cut-valued replacement can be obtained by right-closing the raw sum. If $C = A +_* B$ denotes the printed construction, set

$$A \mathbin{\widehat{+}}_* B = \{p \in \mathbb{Q} \mid (\forall q \in \mathbb{Q})(p < q \to q \in C)\}.$$

In standard semantics its membership at p is $\inf_{q>p} C(q)$. Since C is upper, density of $\mathbb{Q}$ shows that this is again upper and satisfies the continuity axiom. For crisp cuts representing finite reals r, s, it is precisely the closed upper cut representing $r + s$. In particular, it changes the sum in the counterexample to $\bar{0}$.

2. Substantial: *Interval addition is not well-defined on the represented fuzzy sets.*

   Section 6 defines $[A, B]_*$ as a fuzzy set, but its addition formula (p. 954) uses an endpoint representation that need not be unique. This problem persists over the real underlying domain stipulated there, independently of Finding 1. For $* = \mathrm{G}$, put $c = 1/2$ and define upper cuts A_t and a lower cut B by

$$A_t(q) = \begin{cases} 0 & q < 0, \\ c & 0 \le q < t, \\ 1 & q \ge t, \end{cases} \qquad B(q) = \begin{cases} 1 & q \le 1, \\ c & 1 < q \le 3, \\ 0 & q > 3. \end{cases}$$

   For $t = 2$ and $t = 5/2$, both $[A_t, B]_{\mathrm{G}}$ have membership c on $[0, 3]$ and 0 elsewhere. Thus they are the same nonempty fuzzy interval. Adding the crisp interval $[0, 1]$ by the proposed endpoint rule leaves A_t unchanged and replaces $B(q)$ by $B(q - 1)$. At $q = 2$ the resulting interval has membership 1 when $t = 2$, but membership c when $t = 5/2$. The same fuzzy set therefore has two different sums with $[0, 1]$.

   One possible amendment treats intervals as ordered pairs (A, B), retaining their endpoint data instead of identifying equal intersection membership functions. Another is to restrict the set-based construction to intervals $I = [A, B]_*$ whose kernel is a nonempty bounded interval $[a, b]$ of real numbers. The upper cut A is then recovered from I by retaining its membership values for $q \le a$ and assigning membership 1 for $q > a$; dually, B has membership 1 for $q < b$ and retains the values of I for $q \ge b$. Thus the endpoint cuts are uniquely determined by the represented fuzzy set, so the addition rule no longer depends on a choice of representation. A cut-valued endpoint addition is still required, as explained in Finding 1.

3. Moderate: *The continuity axiom does not give a completion of an arbitrary crisp poset.*

   The opening paragraph of Section 6 (p. 953) says that no special property of $\mathbb{Q}$ has been used and that the construction and results extend to any crisp poset. Even with the intended quantifier in the continuity axiom, this is false. On the two-element chain $0 < 1$, that axiom at 1 forces $A(1) = 1$, because there are no strictly greater elements. At 0 it then forces $A(0) = A(1) = 1$. The only admissible cut is the whole chain, so the construction cannot embed, let alone complete, the original two-element order.

   Density and the absence of endpoints are relevant to the order-theoretic construction. Its stated extension to the crisp real line is valid; more generally the upper-cut construction works over dense linear orders without endpoints.

4. Minor issues:

   (a) Definition 2.1 reverses the quotient in product implication, printing $\min(1, x/y)$. The correct value is 1 for $x \leq y$ and y/x for $x > y$, including $0 \to_\Pi 0 = 1$. This slip is inherited from Definition 1 of [21].

   (b) The side condition for ($\forall$1) in Definition 2.4 refers to substitutability in ψ, although the formula displayed there is φ. The correct side condition refers to φ.

   (c) Axiom 2 preceding Definition 4.1 binds p twice and leaves q free. The inner quantifier must bind q, giving

   $$(\forall p \in \mathbb{Q})\big[(\forall q \in \mathbb{Q})(q > p \to q \in A) \to p \in A\big].$$

   This is the right-continuity condition described in the text.

   (d) The example immediately after Definition 4.1 (p. 952) gives $\sqrt{2} = \{q \in \mathbb{Q} \mid q^2 > 2\}$. That set includes large negative rationals and is not upper. The intended cut is $\{q \in \mathbb{Q} \mid q > 0 \wedge q^2 > 2\}$.

   (e) The explanation immediately following the addition formula in Section 5 (p. 953) uses Aq and Bq, interpreted as $A, B \leq_* q$. The formula instead combines Aa and Bb over decompositions $a + b = q$; $A, B \leq_* q$ alone does not guarantee $A + B \leq_* q$, as the crisp example $A = B = \overline{1}$, $q = 1$ shows.

   (f) The parenthesis in Definition 6.1 calls the crisp real line a complete lattice. The intended example is the extended real line $\mathbb{R} \cup \{\pm\infty\}$, with the cut axioms imposed also at its endpoints; the stated kernel-endpoint conclusions then hold.

   (g) Reference 4 gives Cintula's paper the page range 21–34. The correct range is 289–302 [26]. The same erroneous range appears in reference 5 of [21], from which it is inherited here.

13.3 Status of the results

With the intended continuity axiom, the upper-cut construction over $\mathbb{Q}$ and its order-theoretic extension to $\mathbb{R}$ remain available. The counterexample to the arbitrary-poset claim does not affect these domains, and the arithmetic and interval-representation defects do not affect the lattice construction of Theorem 4.4.

The addition formula printed in Section 5 does not yield an operation on the defined Dedekind reals. Right-closing its output gives a cut-valued replacement and restores ordinary addition on finite crisp reals; the algebraic assertions must be formulated for the amended operation. Independently, the proposed interval addition cannot act on all represented fuzzy sets. Retaining endpoint pairs or restricting to normal intervals with finite kernel endpoints removes the demonstrated representation ambiguity.

With the cut axioms imposed on the *extended* real line, including its endpoints, the kernel-endpoint, normality, and singleton-kernel claims following Definition 6.1 hold; the corresponding kernel endpoints may be infinite.

14 *Fuzzy MacNeille and Dedekind completions of crisp dense linear orderings* (Běhounek, 2005)

The findings below concern errors identified in the preliminary conference announcement [3]. The paper was reviewed by GPT-6.1 Sol.

14.1 Summary

Two substantial findings concern the interpretation of lattice completeness in Definition 9 and the transfer of the MacNeille supremum formula to the Dedekind completion in Section 3.2. Two moderate findings concern the comparison of conjunctions and the comparison with Dubois and Prade's levelwise real-number condition in Section 5. Minor issues concern local formulations and references.

14.2 Findings in more detail

1. Substantial: *In Definition 9, the domain for computing extrema is left implicit, and degree-one completeness can lack witnesses.*

   The inference to infima after Definition 9 (p. 8) suggests that the cones used to define suprema and infima are intended to be computed within A. Section 2, however, fixes a relation $\leq$ before introducing A, and Definition 9 restricts the supremum witness to A without specifying a corresponding restriction of the cones in Definitions 5 and 7. If the cones are computed over the full domain of $\leq$, the inference fails even classically. Take that domain to be $[0,1]$ with its usual order and let $A = \{0, 1/2\}$. Every subset of A has its supremum in $[0,1]$ lying in A, but the infimum of $\emptyset$ in $[0,1]$ is 1.

   Even if A is the entire domain of $\leq$, truth of the completeness formula to degree 1 need not give a 1-true supremum. In standard Łukasiewicz semantics, let $A \coloneqq [0,1)$ be the crisp domain and interpret $x \leq y$ by $x \to y \coloneqq \min(1, 1 - x + y)$. For a fuzzy subset X, put $s \coloneqq \sup_{x \in A}(X(x) \,\&\, x)$. By residuation the two cones have membership functions $X^{\uparrow}(y) = s \to y$ and $X^{\uparrow\downarrow}(y) = y \to s$. For the second identity, the infimum defining the lower cone is attained at 0. Thus the membership of y in $\operatorname{Sup} X$ is $1 - |s - y|$. Its supremum over $y \in A$ is 1 for every X, so the completeness formula has degree 1. For $X = A$, however, $s = 1$ and no $y \in A$ has supremum degree 1.

   For a crisp A, the intended reading can be made explicit by computing both cones and both extrema in the restriction of $\leq$ to A. Writing $\operatorname{Sup}_A X$ for the resulting set of suprema, Definition 9 can be replaced by:

   $$A \text{ is lattice complete} \equiv_{\mathrm{df}} (\forall X)\big[\Delta(X \subseteq A) \to (\exists x \in A)\Delta(x \in \operatorname{Sup}_A X)\big].$$

   The first Δ restricts the condition to represented X whose inclusion in A has degree 1. The second ensures that the existential has degree 1 only if some $x \in A$ is a 1-true supremum. The dual of claim (3) of Lemma 8 then gives attained infima from attained suprema, since the lower cone computed within A is itself a subset of A. For a fuzzy A, an internal reading requires an explicit convention for relativizing the cones.

2. Substantial: *The MacNeille supremum formula gives the wrong supremum in the Dedekind completion.*

   The paragraph after Definition 16 (p. 9) extends all the properties in Theorem 14 to $\mathcal{D}(X)$, including $\sup \mathcal{A} = (\bigcup \mathcal{A})^{\uparrow\downarrow}$. In standard Gödel semantics on the crisp rational order, let $A \in \mathcal{D}(\mathbb{Q})$ be the constant cut of degree $1/2$ and take the crisp singleton family $\mathcal{A} = \{A\}$. Its supremum in $\mathcal{D}(\mathbb{Q})$ is A. But $A^{\uparrow} = \emptyset$, since for each y some rational $a > y$ belongs to A to positive degree. Hence $(\bigcup \mathcal{A})^{\uparrow\downarrow} = \mathbb{Q}$.

   The claim after Definition 16 should distinguish the supremum formulas for the two completions. The formula $\sup \mathcal{A} = (\bigcup \mathcal{A})^{\uparrow\downarrow}$ in the last sentence of Theorem 14 applies to $\mathcal{M}(X)$. In $\mathcal{D}(X)$ ordered by inclusion, the 1-true supremum of $\mathcal{A}$ is instead the right closure of $U = \bigcup \mathcal{A}$, defined by

   $$C(U) = \{x \mid (\forall y < x)(y \in U)\}.$$

The union is lower, so $U \subseteq C(U)$ to degree 1. Density gives $C(C(U)) = C(U)$, and $C(U)$ is lower. For any cut B, the inclusions $C(U) \subseteq B$ and $U \subseteq B$ have the same degree. Indeed, if the latter has degree r, then $r \,\&\, C(U)(x) \leq B(y)$ for every $y < x$, so $r \,\&\, C(U)(x) \leq B(x)$ by the cut axiom. The converse inequality follows from $U \subseteq C(U)$ to degree 1. By residuation, the degree of $U \subseteq B$ is precisely the degree to which B is an upper bound of $\mathcal{A}$. This establishes the corrected supremum formula. The infimum formula $\inf \mathcal{A} = \bigcap \mathcal{A}$ does hold in $\mathcal{D}(X)$, since the pointwise infima and residuals defining the intersection preserve the cut axiom.

3. Moderate: *Minimum has opposite effects in antisymmetry and in the supremum predicate.*

   The second paragraph of Section 5 (p. 10) characterizes the definitions using $\wedge$ in the cited works as narrower. Replacing & by $\wedge$ in the antecedent of E-antisymmetry indeed makes the condition stronger. But in the comparison of supremum and infimum predicates, replacing the strong intersection in Definition 7 by minimum gives pointwise *larger* grades. For example, in standard Łukasiewicz semantics, take two points with reflexive $\leq$ and off-diagonal degree $1/2$. For $A = \{a\}$ and the other point b, both $b \in A^{\uparrow}$ and $b \in A^{\uparrow\downarrow}$ have degree $1/2$. The minimum assigns b supremum degree $1/2$, whereas the strong intersection assigns it degree 0.

   The comparison should distinguish the stronger condition on orders from the larger membership grades of the weakly conjoined extrema. For fixed cones, either conjunction yields the same kernels, so the connective in the extrema predicates alone makes no difference to degree-one existence.

4. Moderate: *Level cuts can represent infinite endpoints in the comparison with Dubois and Prade.*

   The final paragraph of Section 5 (p. 10) asserts agreement with the cut requirement attributed to reference 9 by Dubois and Prade. However, in the real-number case, Dubois and Prade require *ordinary* real values at the levels of a fuzzy real number [28]. On $\mathbb{Q}$ the constant cut of degree $1/2$ belongs to $\mathcal{D}(\mathbb{Q})$ and, in standard Łukasiewicz semantics, to $\mathcal{M}(\mathbb{Q})$ as well, since both cone operators take this constant cut to itself. Its weak α-cut is $\mathbb{Q}$ for $0 < \alpha \leq 1/2$ and $\emptyset$ for $1/2 < \alpha \leq 1$. Both are crisp cuts of the completion, but they represent $+\infty$ and $-\infty$, respectively, rather than ordinary reals.

   The level-cut assertion is valid with values in the *extended* crisp completion. On the rational domain, finite real values at every positive level are obtained exactly for cuts with nonempty kernel and infimum of membership grades equal to 0. The first condition makes every weak α-cut nonempty, and the second makes every such cut proper. Each is right-closed by Definition 15.

5. Minor issues:

   (a) In Theorem 14, the condition $\mathcal{A} \in \mathcal{M}(X)$ should read $\mathcal{A} \subseteq \mathcal{M}(X)$. The argument of Sup is a family of stable sets.

   (b) The explanation after Definition 15 (p. 9) identifies the second axiom with left-continuity. At a least point the axiom also forces membership 1, because the quantifier ranges over the empty set. In particular, if singleton orders are admitted, their only cut is crisp, so the noncrisp-cut assertion after Theorem 18 requires more than nonemptiness.

   (c) The place of publication in reference 3 is Dordrecht [29], not Dordercht. The title in reference 5 is *Towards a formal theory of fuzzy Dedekind reals* [4].

14.3 Status of the results

For a crisp A, computing cones and extrema within A and requiring an attained supremum for every represented fuzzy subset of A justifies the inference to attained infima following Definition 9. The degree-one subsets of A in Example 2 have internal suprema and infima for represented families under inclusion. The MacNeille formulas of Theorem 14 give attained internal extrema for represented families in $\mathcal{M}(X)$.

The Dedekind completion has the same infimum formula as the MacNeille completion and has suprema given by the right closure in Finding 2. The two supremum formulas agree wherever the two completions coincide, in particular under Theorem 19. The principal cuts still embed the crisp dense order and preserve its existing crisp suprema and infima.

The distinction between conjunctions concerns graded membership and leaves the comparison of degree-one extrema intact for fixed cones. Positive weak level cuts of the completed objects in standard models are elements of the crisp completion. The comparison with ordinary real values requires the finiteness conditions in Finding 4.

15 *Fuzzification of Groenendijk–Stokhof propositional erotetic logic* (Běhounek, 2004)

This section records errors found in [2]. The checks were performed and this report was drafted by GPT-6.1 Sol. The version checked is the third (final) galley proofs, since the published article was not available for the review. Page references use the published article's pagination, 167–188.

15.1 Summary

Two substantial errors were found. The first concerns the generality of Lemma 4.4 and the adequacy theorem, Theorem 5.2. Their converse implications are false for an arbitrary extension $\mathcal{F}$ of BLΔ: full class comprehension can produce additional propositional laws. An explicit counterexample is given below. Propositional standard completeness is a sufficient restriction under which the stated correspondences hold; in particular, they remain valid for GΔ, ŁΔ, and $\Pi\Delta$.

The second substantial error occurs in Theorem 6.3, equation (11): the printed bound for strong equivalence of questions is false. Squaring the antecedent gives a verified weaker bound, and full equivalence of propositions still yields full equivalence of questions.

One moderate finding concerns the partition criterion and the reversed Łukasiewicz inequality in the discussion following Definition 6.1. There are also two minor corrections.

15.2 Findings in more detail

1. Substantial: *Class comprehension need not be propositionally conservative over an arbitrary extension of* BLΔ.

   Section 4 allows $\mathcal{F}$ to be an arbitrary extension of BLΔ. In this generality, the right-to-left implications in Lemma 4.4 and Theorem 5.2, and the analogous entailment correspondence immediately following Theorem 5.2, are false. The converse proof of Lemma 4.4 assumes that every non-theorem of $\mathcal{F}$ has a countervaluation yielding the specified full class model. Propositional standard completeness does not hold for arbitrary axiomatic extensions, and replacing the standard algebra by an arbitrary nonstandard chain does not resolve the obstruction from class quantification and comprehension.

For a concrete counterexample, put $m(p) = p \vee \neg p$ and let $\mathcal{F}_0$ be ŁΔ extended by the schema

$$\neg(m(p)^2) \to m(p)^2, \qquad p^2 = p \,\&\, p.$$

This logic does not prove $m(p)$. Indeed, it is sound in the Chang MV-chain

$$C = \Gamma(\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}, (1,0)),$$

with $\Delta c = 1$ exactly when $c = 1$. Writing $u = (1,0)$, the elements of C are $(0,n)$ and $(1,-n)$ for $n \geq 0$, and its strong conjunction is $c \,\&\, d = \max(0, c + d - u)$. For every c, $m(c) = (1,-n)$ for some $n \geq 0$; hence

$$m(c)^2 = (1,-2n), \qquad \neg(m(c)^2) = (0,2n) \leq m(c)^2.$$

The added schema is therefore valid, whereas, at $p = (0,1)$, $m(p) = (1,-1) < u$.

In contrast, the class theory over $\mathcal{F}_0$ proves $m(x \in A)$ for every object x and class A. To see this directly, abbreviate

$$\alpha(x) = (\forall X) m(x \in X).$$

The Łukasiewicz tautology $\neg m(p) \to m(q)$, universal generalization, and contraposition give $\neg\alpha(x) \to \alpha(x)$, so $m(\alpha(x))$ is equivalent to $\alpha(x)$. Instantiating the added schema at $\alpha(x)$ therefore gives

$$d(x) \to \neg d(x), \qquad d(x) = \neg(\alpha(x)^2).$$

Comprehension provides the class $D_x = \{y \mid d(x)\}$. Specializing the class quantifier in $\alpha(x)$ to D_x yields

$$\alpha(x) \to m(x \in D_x), \qquad m(d(x)) \leftrightarrow \alpha(x)^2,$$

where $m(d(x)) \leftrightarrow \neg d(x)$ follows from $d(x) \to \neg d(x)$ and involutivity. Thus $\alpha(x) \to \alpha(x)^2$ is provable. Together with $\neg\alpha(x) \to \alpha(x)$, this proves $\alpha(x)$ in Łukasiewicz logic, and specification gives $m(x \in A)$. Consequently the class theory proves

$$V \subseteq \mathrm{Op}_m(A), \qquad W \subseteq \|m(p)\|,$$

although $\mathcal{F}_0 \nvdash m(p)$. Taking $\varphi = \top$ and $\psi = m(p)$ also disproves the converse of Lemma 4.4 and the stated entailment correspondence.

A verified restricted version assumes that $\mathcal{F}$ is propositionally complete with respect to standard $[0,1]$-algebras satisfying its axioms. A countervaluation then extends to the full one-object class model used in the printed proof, establishing the converse. This sufficient condition covers the three standard logics above, as well as BLΔ with its standard class of algebras. It is not asserted to be necessary. The forward implications, obtained by substitution and generalization, remain valid for arbitrary $\mathcal{F}$. The restriction changes the generality of the adequacy theorem; the counterexample rules out a repair preserving its printed scope.

2. Substantial: *Theorem 6.3(11) overestimates the degree of strong equivalence of questions.*

   Equation (11) asserts

$$(A \equiv B) \to (?A \equiv_{\circ} ?B), \qquad \circ \in \{\mathrm{ft}, \mathrm{fft}\}.$$

The definition of $\equiv_\circ$ uses the strong conjunction of the two entailments. The argument in Appendix A establishes the bound by $A \equiv B$ for each direction separately, but does not justify combining them with only one occurrence of that graded antecedent.

For a counterexample, take a crisp three-world logical space in an intended standard Łukasiewicz model, with membership values

	w_1	w_2	w_3
A	0	1/10	1/10
B	1/10	1/10	1/5

Then $\|A \models B\| = 1$, $\|B \models A\| = 9/10$, and hence $\|A \equiv B\| = 9/10$. For both choices of $\circ$,

$$\|A \models_\circ ?B\| = \|B \models_\circ ?A\| = 9/10.$$

The valid one-directional bounds from the proof give $\|?A \models_\circ ?B\| \geq 9/10$ and $\|?B \models_\circ ?A\| \geq 9/10$. Conversely, the class A in the first defining universal quantifier, and the class B in the second, give the matching upper bounds, since each class fully answers its own question. Thus

$$\|?A \equiv_\circ ?B\| = (9/10) \,\&_{\text{Ł}}\, (9/10) = 4/5 < 9/10.$$

A verified weaker replacement is

$$(A \equiv B)^2 \to (?A \equiv_\circ ?B).$$

It follows by combining the two valid directional implications with strong conjunction. The example attains this corrected bound. Alternatively, the original assertion is valid when $A \equiv B$ is fully true, or when the underlying logic is Gödel logic, where strong conjunction is idempotent. Replacing strong equivalence of questions by the weak conjunction of the two entailments also yields the original bound, but changes the defined equivalence notion. These amendments do not establish equation (11) in its printed graded generality. Equations (10), (12)–(14) do not require the same amendment: in (12) the equivalence compares two truth degrees, rather than two independently quantified entailment degrees.

3. Moderate: *Partition refinement does not suffice for full* fft*-answerhood, and its Łukasiewicz inequality is reversed.*

   The discussion following Definition 6.1 (p. 178) says that all three answerhood notions are fully true if the truth-level partition of A refines that of B. This is correct for $\models_{\text{t}}$ and $\models_{\text{ft}}$, but false for $\models_{\text{fft}}$. For example, on two worlds take $A = (1/2, 3/5)$ and $B = (1/2, 7/10)$ in standard Łukasiewicz logic. Both truth-level partitions are discrete, yet

   $$\|A \models_{\text{fft}} ?B\| = 9/10, \qquad \|A \models_{\text{ft}} ?B\| = 1.$$

   These are also the values in the paper's own example in footnote 23. The following paragraph separately reverses the Łukasiewicz criterion, saying that the difference for A does not exceed the difference for B.

   The correct full fft-answerhood condition is

   $$(A(w) \leftrightarrow A(w')) \leq (B(w) \leftrightarrow B(w')) \quad \text{for all } w, w' \in W.$$

   In Łukasiewicz logic this is

   $$|B(w) - B(w')| \leq |A(w) - A(w')|.$$

Partition refinement remains necessary, but it does not control these graded comparisons. The common refinement criterion applies only to the t and ft notions; the fft notion additionally requires the displayed nonexpansiveness condition. The product and Gödel descriptions are most unambiguously stated using their equivalence truth functions in the first displayed inequality.

4. Minor issues:

   (a) Footnote 5 in Section 3 (p. 170) names the ordinal-sum characterization "Mostert–Shield's". The correct surname is Shields.

   (b) The description of intended models in Section 4 (p. 172) denotes the functions from U to $[0,1]$ by $U^{[0,1]}$. The correct notation is $[0,1]^U$.

15.3 Status of the results

Lemma 4.4 and Theorem 5.2 are false at their stated generality over arbitrary extensions of BLΔ. Their forward implications remain valid, and their full correspondences hold under the sufficient standard-completeness restriction above. In particular, the intended adequacy results for GΔ, ŁΔ, and $\Pi\Delta$ survive, and the restriction leaves both constructions of answerhood intact. Full comprehension and class quantification are essential to the counterexample; replacing the standard countermodel in the proof by an unspecified nonstandard algebra is insufficient.

Theorem 6.3(11) is false as a graded bound. Its squared-antecedent version is valid, while the original full-truth consequence and its Gödel specialization remain valid. The other parts of Theorem 6.3 and the connective-preservation result, Theorem 6.4, retain their statements under the standing crispness assumption on W. The correction to the partition discussion changes the explanation of full fft-answerhood, not Definition 6.1 or Theorem 6.2.

Theorem 5.3 and Theorems 6.6–6.9 retain their original statements under their printed hypotheses. Their proofs do not require the false converse of the general adequacy claim, so the failure of general propositional conservativity does not invalidate them. The distinction between truth-value answerhood and yes–no answerhood, and the yes–no entailment and equivalence construction, remain available with these corrections.

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