185.A09 Advanced Mathematical Logic

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The Löwenheim–Skolem Theorem

Theorem (Löwenheim–Skolem Downward Theorem)**.** *Let T be a consistent theory in the language L of cardinality λ. Then T has a model of cardinality $\kappa \leq \max(\aleph_0, \lambda)$.*

Proof. By inspection of the proof of the Completeness Theorem, we can find a bound on the cardinality of the canonical structure for L. (We use ZFC as our metatheory. Recall that in ZFC, $\kappa + \lambda = \max(\kappa, \lambda) = \kappa \times \lambda$ if $\kappa \geq \aleph_0$ or $\lambda \geq \aleph_0$.)

The set of symbols that can occur in L-formulae consists of L (of cardinality λ) plus countably many individual variables plus a finite set of connectives, quantifiers, and parentheses; its cardinality is therefore $\lambda' = \lambda + \aleph_0 = \max(\lambda, \aleph_0)$. L-formulae are finite sequences of such symbols, so its cardinality is at most $\lambda' \times \aleph_0 = \max(\lambda', \aleph_0) = \max(\max(\lambda, \aleph_0), \aleph_0) = \max(\lambda, \aleph_0) = \lambda'$. Henkin constants are added in $\aleph_0$ steps; in each step, new Henkin constants correspond uniquely to the formulae of the current language, which by induction has the cardinality at most λ' (as there are at most $\lambda' \times \aleph_0 = \lambda'$ formulae in the new language at each step). The cardinality of the language L' of the Henkin completion of T is therefore at most λ'. Closed terms of L' are finite sequences of symbols from L' plus countably many logical symbols, so again there are at most $\lambda' + \aleph_0 = \lambda'$ of them. The cardinality of the universe of the canonical structure of T is therefore at most $\lambda' = \max(\lambda, \aleph_0)$. □

Corollary. *Any consistent theory in a countable language has a countable model.*

Example. Zermelo–Fraenkel set theory ZF can be formulated in the language $\{\in\}$. Consequently, if ZF is consistent, it has a countable model.

The apparent contradiction of the existence of a countable model of ZF with Cantor's theorem (entailing the existence of uncountable sets) is called *Skolem's paradox.* Its solution consists in an appropriate distinction between the object level and metalevel: at the object level, ZF indeed proves the existence of uncountable sets; however, its *metalevel* models need not be uncountable.

The discrepancy between the object-level uncountability and metalevel countability of a set a in a countable model M of ZF can be accounted for as follows: internally (i.e., as regards formulae true in M), a is uncountable, since in M there is no individual f that (in M) would be an injective function mapping the elements (in the sense of M) of a to the elements of the set ω (in M). Thus $M \models$ 'a is uncountable', even though a has only countably many elements from the metalevel point of view. Informally speaking, there are no injective functions *in M* that would make a countable in M.

The Compactness Theorem

Theorem. *For any theory T and formula φ,*

$$T \models \varphi \quad \textit{iff} \quad \text{there is a finite } T' \subseteq T \text{ such that } T' \models \varphi.$$

Proof. Left to right: By the Completeness Theorem, if $T \models \varphi$ then $T \vdash \varphi$. Since the proof of φ in T can use only finitely many axioms of T, there is a finite $T' \subseteq T$ such that $T' \vdash \varphi$. Then by the Soundness Theorem $T' \models \varphi$.

Right to left: If $T' \models \varphi$, then $T' \vdash \varphi$ by the Completeness Theorem. Since $T' \subseteq T$, trivially $T \vdash \varphi$, too. Thus by the Soundness Theorem, $T \models \varphi$. □

Similarly as in the Soundness and Completeness Theorems, there is an alternative formulation of the Compactness Theorem connecting models and consistency of theories:

Theorem. *A theory T has a model if and only if every finite theory $T' \subseteq T$ has a model.*

Proof. If every finite theory $T' \subseteq T$ has a model, then (by the Soundness Theorem) every such theory T' is consistent. Consequently, T is consistent (as any proof of a contradiction would only use finitely many axioms of T, and so would be a proof in some finite $T' \subseteq T$). Thus by the Completeness Theorem, T has a model. The converse direction is trivial. □

True arithmetic

Definition. The *language of arithmetic* consists of the binary functions $+$ and $\cdot$, the constant 0, the unary function s (the *successor*), and the binary predicates $\leq, <$.

The *numerals* are the closed terms of the language of arithmetic arising by iteration of the successor function on the constant 0. For each metamathematical natural number n we define the numeral $\overline{n}$ by induction as follows: $\overline{0}$ is the term 0; $\overline{n+1}$ is the term $s(\overline{n})$. (Thus $\overline{1}$ is the term $s(0)$; $\overline{2}$ is the term $s(s(0))$; $\overline{3}$ is the term $s(s(s(0)))$; etc.)

Definition. Let M be a structure for the language L. The set $\mathrm{Th}(M) =_{\mathrm{df}} \{\varphi \text{ a sentence} \mid M \models \varphi\}$ of the sentences true in M is called the *theory of M*.

Definition. The structure $N = \langle \omega, +, \cdot, s, \leq, < \rangle$, where ω is the set of all (metamathematical) natural numbers, $+$ and $\cdot$ are the operations of addition and multiplication, s is the function assigning $n+1$ to each $n \in \omega$, and $\leq, <$ are the usual ordering relations on ω, is called the *standard model of arithmetic.* The theory $\mathrm{Th}(N)$ is called *true arithmetic.*

True arithmetic is the set of all sentences that are true in the standard model N. Obviously, N is a model of $\mathrm{Th}(N)$. However, it follows from the Compactness Theorem that $\mathrm{Th}(N)$ has also other models which are not isomorphic to N:

Example. Let L be the language of arithmetic expanded by a new constant c. Let T be the theory in L that extends $\mathrm{Th}(N)$ by the axioms $c > \overline{n}$ for all (metamathematical) natural numbers n. Clearly, each finite subtheory T' of T has a model—namely N expanded by c interpreted as a number larger than all numerals occurring in T'. Therefore, by the Compactness Theorem, T is also consistent and has a model.

Let us denote this model by M. Since $\mathrm{Th}(N) \subseteq T$, M is a model of true arithmetic. However, it contains a number (namely, the interpretation of the constant c) which, by the axioms of T, is larger than any numeral. Since numerals are in one-to-one correspondence with metamathematical natural numbers, the interpretation of c in the model M of true arithmetic is larger than all metamathematical natural numbers.

In models of arithmetic, interpretations of numerals are called the *standard numbers;* other elements are called *non-standard numbers;* all elements of a model of arithmetic are called just *numbers.* A model of arithmetic that contains non-standard numbers is called a *non-standard model.*

Non-standard numbers in a model of true arithmetic are, from the metamathematical point of view, 'infinitely large' (i.e., larger than any metamathematical natural number). Still, they obey all rules of ordinary (true) arithmetic expressible in the (first-order) language of arithmetic. They can be used for the construction of non-standard models of mathematical analysis (as the reciprocal values of such infinitely large numbers are infinitely small, i.e., smaller than any standard rational number, but larger than 0), restoring the original idea of *infinitesimals.* Non-standard set theories formalizing these ideas are, e.g., Vopěnka's Alternative Set Theory or Hrbáček's non-standard set theory (Google these names for more information).

Robinson and Peano arithmetic

The axioms of true arithmetic $\mathrm{Th}(N)$ are not defined by an effective procedure (being all sentences true in the standard model, including, e.g., difficult Diophantine equations). Nevertheless, it has been observed in the XIX century that many truths of arithmetic follow from a rather limited set of axioms. A collection of such sufficiently strong axioms was used by Dedekind, Peano, Frege, Peirce, and others to formalize a large part of arithmetic. In the metamathematical investigations, the following axioms have become prominent:

1. $s(x) = s(y) \to x = y$
2. $s(x) \neq 0$
3. $x \neq 0 \to (\exists y)(x = s(y))$
4. $x + 0 = x$
5. $x + s(y) = s(x + y)$
6. $x \cdot 0 = 0$
7. $x \cdot s(y) = x \cdot y + x$
8. $x \leq y \leftrightarrow (\exists z)(z + x = y)$
9. $x < y \leftrightarrow (\exists z)(s(z) + x = y)$
10. $\big(\varphi(0) \land (\forall x)(\varphi(x) \to \varphi(s(x))\big) \to (\forall x)\varphi(x)$,
    for all formulae φ in the language of arithmetic ('the induction schema')

The axioms 1–9 are known as *Robinson arithmetic* Q, and the axioms 1–10 as *Peano arithmetic* PA.

Similarly as $\mathrm{Th}(N)$, also Q and PA have non-standard models. Since $Q \vdash x \leq \overline{n} \lor \overline{n} \leq x$, all models of Q (and PA and $\mathrm{Th}(N)$) have an initial segment of standard numbers, possibly followed by segments of non-standard numbers. In PA (or any stronger theory, including true arithmetic), the properties of successors ensure that every non-standard number lies in a segment order-isomorphic to the set Z of (metamathematical) integer numbers; however, this need not be so in Q:

Example. The following structure is a non-standard model of Q:

$\dot{M} = \{a_0, a_1, a_2, \ldots, b_0, b_1, b_2 \ldots\}$, where $a_0 < a_1 < a_2 < \cdots < b_0 < b_1 < b_2 < \ldots$.

$s(a_i) = a_{i+1}$, $s(b_i) = b_i$ (sic!) for all (metamathematical) $i \in \omega$.

$+$	a_m	b_m
a_n	a_{n+m}	b_{m+1}
b_n	b_n	b_{m+1}

$\cdot$	a_0	a_{m+1}	b_m
a_n	a_0	$a_{n\cdot(m+1)}$	b_0
b_n	a_0	b_{n+1}	b_{n+1}

Notice that $M \not\models x < s(x)$, thus $x < s(x)$ is not provable in Q (though it is provable in PA).

Observe that the set of standard numbers is not definable in PA (i.e., there is no formula $\varphi(x)$ in the language of PA such that $M \models \varphi(a)$ iff a is a standard number in a model M of PA), as the induction axiom for φ would fail in M. (Similarly, since PA $\subseteq \mathrm{Th}(N)$, standard numbers are not definable in true arithmetic.)

Exercise. *1. Prove in Robinson arithmetic that* $\overline{m+n} = \overline{m} + \overline{n}$.

2. Prove in Robinson arithmetic that $x \leq \overline{n} \to (x = \overline{0} \vee x = \overline{1} \vee \ldots \vee x = \overline{n})$.

3. Prove in Peano arithmetic that $x < s(x)$.

4. Prove that PA $\vdash (\exists x)\varphi(x) \to (\exists x)(\varphi(x) \wedge (\forall y)(y < x \to \neg\varphi(y)))$, *for each formula* φ *of the language of arithmetic (the* least number principle*).*