185.A09 Advanced Mathematical Logic

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The Axiom of Choice and Transfinite Induction

To prove the Completeness Theorem in its full strength (i.e., for aribitrary languages), we will use certain set-theoretic principles in the metatheory (see also the notes following the proof of the Completeness Theorem). In our version of the proof, we will use the Well-Ordering Principle and Transfinite Induction:

Definition. Let A be a set and $\leq$ a partial order on A. An element a is the least element of the set $X \subseteq A$ (w.r.t. $\leq$) if $a \in X$ and $(\forall x \in X)(a \leq x)$. An ordering $\leq$ on a set A is a *well-ordering* if every subset of A has the least element w.r.t. $\leq$.

Well-Ordering Principle: Every set can be well-ordered. (I.e., for every set A there exists a well-ordering $\leq$ on A.) In Zermelo–Fraenkel set theory ZF, the Well-Ordering Principle is equivalent to the Axiom of Choice.

Theorem (Transfinite induction)**.** *Let $\varphi(x)$ be a formula in the language of set theory and $\leq$ a well-ordering of a set A. Then ZF proves:* $(\forall a \in A)\big(\big((\forall x \leq a)\varphi(x)\big) \to \varphi(a)\big) \to (\forall a \in A)\varphi(a)$.

Completion of theories

Theorem. *Each consistent theory T has a consistent complete extension T' in the same language.*

Proof. Let $\preceq$ be a well-ordering of the closed formulae of the language L of T. For each sentence φ of L, we shall construct by transfinite recursion a theory T_φ in L in such a way that either φ or $\neg\varphi$ is an axiom of T_φ. The induction hypothesis is that T_φ is a consistent extension of T and for each $\psi \prec \varphi$, the theory T_φ is an extension of T_ψ.

To construct the theory T_φ from the theories T_ψ for $\psi \prec \varphi$, we first define an auxiliary theory S_φ in the language L:

- If φ is the first element in $\preceq$, then let $S_\varphi = T$. (S_φ is obviously consistent.)
- Otherwise let $S_\varphi = \bigcup_{\psi \prec \varphi} T_\psi$. The theory S_φ is consistent, since any proof of a contradiction could only contain finitely many formulae, and therefore there would be theories $T_{\psi_1}, \ldots, T_{\psi_k}$ for $\psi_1, \ldots, \psi_k \prec \varphi$ containing all of the axioms of S_φ needed for the proof of the contradiction. Since k is finite and $\preceq$ is linear, some theory T_{ψ_j} for $j \leq k$ contains

all $T_{\psi_1}, \ldots, T_{\psi_k}$, and so the contradiction proof is a proof in T_{ψ_j}. This, however, is a (metamathematical) contradiction with the induction hypothesis that all T_ψ for $\psi \prec \varphi$ are consistent.

S_φ extends T and all T_ψ for $\psi \prec \varphi$. We define the theory T_φ as follows:

- If $S_\varphi \cup \{\varphi\}$ is consistent, let $T_\varphi = S_\varphi \cup \{\varphi\}$. The induction hypothesis then trivially holds for T_φ.
- If $S_\varphi \cup \{\varphi\}$ is inconsistent, let $T_\varphi = S_\varphi \cup \{\neg\varphi\}$. The consistency of T_φ (and so the induction hypothesis) follows from the fact that since $S \cup \{\varphi\}$ is inconsistent and φ is closed, we have $S \vdash \neg\varphi$. Thus, since S_φ is consistent, so is $S \cup \{\neg\varphi\}$.

Finally, define $T' = \bigcup_{\psi \in L} T_\psi$ and prove its consistency analogously as in the case of S_φ. □

Canonical structures

Definition. Let T be a theory whose language $L = \langle \text{Pred}, \text{Func} \rangle$ contains at least one constant. Then its *canonical structure* (or *term model*) is $M_T = \langle \dot{M}_T, (P_{M_T})_{P \in \text{Pred}}, (F_{M_T})_{F \in \text{Func}} \rangle$, where $\dot{M}_T$ is the set of closed terms of L, P_{M_T} is the set of all $\langle t_1, \ldots, t_k \rangle \in \dot{M}_T$ such that $T \vdash P(t_1, \ldots, t_k)$; and F_{M_T} assigns the term $F(t_1, \ldots, t_k)$ to each $\langle t_1, \ldots, t_k \rangle \in \dot{M}_T$.

Theorem (of canonical structure)**.** *The canonical structure of a complete Henkin theory (without equality) is its model.*

Proof. Let T be a complete Henkin theory in the language L without equality. Without loss of generality (as $T \vdash \varphi$ iff $T \vdash (\forall x)\varphi$) assume that T only contains closed formulae. For each closed instance $\tilde{\varphi}$ of a formula φ of L we shall show that

$$T \vdash \tilde{\varphi} \quad \text{iff} \quad M_T \models \tilde{\varphi}.$$

We proceed by induction on the complexity of φ:

- The step for closed instances of atomic formulae is trivial due to the definition of M_T.
- If φ has the form $\neg\psi$, then $\tilde{\varphi}$ has the form $\neg\tilde{\psi}$. Now $T \vdash \neg\tilde{\psi}$ iff $T \nvdash \tilde{\psi}$ by the completeness of T (as $\tilde{\psi}$ is closed), which is equivalent to $M_T \not\models \tilde{\psi}$ by induction hypothesis, and the latter is equivalent to $M_T \models \neg\tilde{\psi}$ by Tarski conditions.
- If φ has the form $\psi \to \chi$, then $\tilde{\varphi}$ has the form $\tilde{\psi} \to \tilde{\chi}$, where $\tilde{\psi}, \tilde{\chi}$ are, respectively, closed instances of ψ, χ.

  By Tarski conditions, $M_T \models \tilde{\psi} \to \tilde{\chi}$ iff ($M_T \models \tilde{\psi}$ implies $M_T \models \tilde{\chi}$), which by induction hypothesis is equivalent to ($T \vdash \tilde{\psi}$ implies $T \vdash \tilde{\chi}$). Since $T \vdash \tilde{\psi} \to \tilde{\chi}$ implies ($T \vdash \tilde{\psi}$ implies $T \vdash \tilde{\chi}$), it is sufficient to prove the converse contrapositively, i.e., if $T \nvdash \tilde{\psi} \to \tilde{\chi}$ then ($T \vdash \tilde{\psi}$ and $T \nvdash \tilde{\chi}$):

  If $T \nvdash \tilde{\psi} \to \tilde{\chi}$, then $T \vdash \neg(\tilde{\psi} \to \tilde{\chi})$, i.e., $T \vdash \tilde{\psi} \wedge \neg\tilde{\chi}$, by the completeness of T (as $\tilde{\psi} \to \tilde{\chi}$ is closed). Thus $T \vdash \tilde{\psi}$ and $T \vdash \neg\tilde{\chi}$, so $T \nvdash \tilde{\chi}$ by the consistency of T, q.e.d.
- In the induction step for $\forall$ we can assume that ψ does not contain free variables other than x (otherwise $(\exists x)\psi$ would not be closed) and that by induction hypothesis the claim holds for $\neg\psi$.

If $T \vdash (\exists x)\psi$, then there is a constant c such that $T \vdash \psi(c/x)$, since T is Henkin. Thus by induction hypothesis, $M_T \models \psi(c/x)$, so by Tarski conditions $M_T \models (\exists x)\psi$.

If $T \nvdash (\exists x)\psi$, then $T \vdash (\forall x)\neg\psi$ by the completeness of T. By specification, $T \vdash \neg\psi(t/x)$ for each closed term t of L. Thus by induction hypothesis, $M_T \models \neg\psi(t/x)$. Since the universe of M_T comprises *only* closed terms of L, by Tarski conditions we obtain $M_T \models \neg\psi[e]$ for each M_T-evaluation e, and so $M_T \models (\forall x)\neg\psi$, which is equivalent to $M_T \models \neg(\exists x)\psi$. $\square$

The Completeness Theorem

Theorem. *Each consistent theory has a model.*

Proof. Let T be a consistent theory. Then there is a consistent Henkin extension T' of T and a completion T'' of T' in the same language. Since T' is Henkin, so is T''. Therefore its canonical structure is a model of T'', and so of T. $\square$

Corollary. *For each theory T and each formula φ:*

$$\textit{If} \quad T \models \varphi \quad \textit{then} \quad T \vdash \varphi.$$

Proof. We shall prove the contrapositive claim. Let $T \nvdash \varphi$ and let $\bar{\varphi}$ be the universal closure of φ. Then $T \cup \{\neg\bar{\varphi}\}$ is consistent, and therefore has a model M. Since $M \models \neg\bar{\varphi}$, and since a formula is true in a model iff its universal closure is, we also have $M \models \neg\varphi$. Consequently, $T \not\models \varphi$. $\square$

Observe that the Completeness Theorem reduces an essentially 'infinite' problem (the truth of a formula in a possibly infinite class of non-isomorphic models) to a 'finite' one (the existence of a finite proof in an axiomatic system). This reduction is possible due to the limited expressive power of first-order logic (e.g., it cannot express quantification over *subsets* of the domain of a model).

Observe also that the metatheory needed for the Completeness proof is rather strong: it needs to contain some form of the Axiom of Choice (AC). In our proof, we used the well-ordering of the set of formulae, which is equivalent to the Axiom of Choice in the general case. For particular theories, the restriction of AC to the cardinality of the language is sufficient: e.g., the axiom of Countable Choice for theories in countable languages. For theories in finite languages it is sufficient to assume the well-ordering of the set of variables; since the latter set is usually assumed to be countable, so well-ordered by definition, no AC is needed for the Completeness Theorem in finite languages.

The assumption of AC in the general case is, however, unnecessarily strong: the Completeness Theorem is in fact equivalent to a strictly weaker form of AC known as the Ultrafilter Lemma (or the Boolean Prime Ideal Theorem), stating the existence of an ultrafilter extending any filter in a Boolean algebra (which is a reformulation of the theorem of completion of theories used in our proof).