# 185.A09 Advanced Mathematical Logic

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## The Soundness Theorem

**Theorem.** *For every theory $T$ and every formula $\varphi$ in the language of $T$:*

$$\text{if} \quad T \vdash \varphi \quad \text{then} \quad T \models \varphi.$$

*(In particular, $\models \varphi$ only if $\vdash \varphi$.)*

*Proof.* First we shall prove that for any model $M$, any $M$-evaluation $e$, and any axiom $\varphi$ of predicate logic, $M \models \varphi[e]$:

- By Tarski's conditions, $M \models (\varphi \to (\psi \to \varphi))[e]$ iff: $M \models \varphi[e]$ implies that $M \models \psi[e]$ implies $M \models \varphi[e]$, which is a (metamathematically) valid statement. (We assume that classical propositional logic is part of our semantic metatheory.) Similarly for other propositional axioms.

- By Tarski's conditions, $M \models (\forall x)\varphi \to \varphi(t/x)[e]$ iff:

  $$M \models (\forall x)\varphi[e] \quad \text{implies} \quad M \models \varphi(t/x)[e]$$

  Clearly $M \models \varphi(t/x)[e]$ iff $M \models \varphi\big[e[x \mapsto \|t\|_{M,e}]\big]$. By Tarski's conditions, $M \models (\forall x)\varphi[e]$ iff $M \models \varphi(x)[e[x \mapsto a]]$ for all $a \in \dot{M}$, so in particular for $a = \|t\|_{M,e}$. Therefore the displayed implication, and so the specification axiom, holds in $M, e$. The distribution axiom is proved in a similar way.

Second we prove the soundness of the inference rules of modus ponens and generalization:

- Let $M \models \varphi[e]$ and $M \models \varphi \to \psi[e]$. By Tarski's conditions, the latter is equivalent to the implication: if $M \models \varphi[e]$ then $M \models \psi[e]$. Applying modus ponens (on the metalevel), we obtain $M \models \psi[e]$.

- Let $M \models \varphi$. Then $M \models \varphi[e]$ for every $e$, so $M \models \varphi\big[e[x \mapsto a]\big]$ for all $a \in \dot{M}$, and so $M \models (\forall x)\varphi[e]$ for every $e$.

Finally, let $\varphi_1, \ldots, \varphi_n$ be a proof of $\varphi$ in $T$ and let $M$ be a model of $T$. The previous cases prove all induction steps needed for showing that $M \models \psi_i[e]$ for each $i \leq n$ and each $M$-evaluation $e$. $\square$

Notice that while the rule of modus ponens preserves validity under a given evaluation $e$, the rule of generalization only preserves validity under *all* evaluations in a model (i.e., $M \models \varphi$ implies $M \models (\forall x)\varphi$; counterexamples to $M \models \varphi[e]$ implies $M \models (\forall x)\varphi[e]$ are easy to find).

Observe also that all of the proof steps for the axioms and rules utilized the same axiom or rule at the metalevel. This is not a circular reasoning, since the proof is not intended as justification of the rules and axioms of predicate logic. On the contrary—they are *assumed* to be valid principles of logic: they are part of our metatheory. The aim of the soundness proof is rather to show that our syntactic formalization of these logical principles (as axioms and rules of a formal language of first-order logic) is sound w.r.t. the semantics we use.

**Corollary.** *If $M \not\models \varphi[e]$ for some model $M$ of $T$ and an $M$-evaluation $e$, then $T \nvdash \varphi$.*

*Proof.* Trivial by contraposition of the definition of $T \models \varphi$. □

**Corollary.** *If a theory is inconsistent, than it has no model.*

*Proof.* An inconsistent theory proves all formulae of the language, so also a pair $\varphi$ and $\neg\varphi$. Any model of an inconsistent theory would therefore have to validate both $\varphi$ and $\neg\varphi$, which is absurd due to the Tarski condition for negation. □

**Exercise.** *Prove that for all* closed *formulae $\varphi$ in the language of $T$:*

$$T \vdash \varphi \quad \textit{iff} \quad \textit{the theory } T, \neg\varphi \textit{ is inconsistent.}$$

## Some useful metatheorems on provability

**Observation** (Closure Theorem)**.** *For any formula $\varphi$, variable $x$, and theory $T$,*

$$T \vdash \varphi \quad \textit{iff} \quad T \vdash (\forall x)\varphi.$$

*Proof.* Left to right: append $(\forall x)\varphi$ (by generalization) to the proof of $\varphi$ in $T$. Right to left: append, to the proof of $(\forall x)\varphi$ in $T$, the formulae $(\forall x)\varphi \to \varphi$ (an instance of the specification axiom) and $\varphi$ (obtained by modus ponens from the preceding two formulae). □

**Corollary.** *$T$ proves $\varphi$ iff $T$ proves the universal closure of $\varphi$.*

Note that since a semantic counterexample invalidating $\varphi \to (\forall x)\varphi$ can easily be found, the Closure Theorem cannot be strengthened to $T \vdash \varphi \leftrightarrow (\forall x)\varphi$. This exemplifies the difference between *provable equivalence* and (mere) *equiprovability.* After we prove the Completeness Theorem, we will be able to give the following semantic characterization of these notions:

- *Provable equivalence* ($T \vdash \varphi \leftrightarrow \psi$): in every model of $T$, the formulae $\varphi$ and $\psi$ have the same truth value under every evaluation of object variables.
- *Equiprovability* ($T \vdash \varphi$ iff $T \vdash \psi$): $\varphi$ is true in all models of $T$ under every evaluation iff so is $\psi$.

**Theorem** (of Equivalence)**.** *Let $\varphi[\theta'_1/\theta_1, \ldots, \theta'_k/\theta_k]$ be a formula that results from replacing* some *occurrences of the subformulae $\theta_1, \ldots, \theta_k$ in $\varphi$ by the formulae $\theta'_1, \ldots, \theta'_k$, resp. Then*

$$\theta_1 \leftrightarrow \theta'_1, \ldots, \theta_k \leftrightarrow \theta'_k \vdash \varphi \leftrightarrow \varphi[\theta'_1/\theta_1, \ldots, \theta'_k/\theta_k].$$

In the proof, we will take the provability of certain propositional tautologies for granted (their provability follows from the Weak Completeness of our propositional axioms, which can be proved separately, or by elaborating their direct proofs in our axiomatic system).

*Proof.* By induction on the complexity of $\varphi$:

Let $\psi$ be a subformula of $\varphi$. Denote $\psi[\theta'_1/\theta_1, \dots, \theta'_k/\theta_k]$ by $\widetilde{\psi}$ and $\{\theta_1 \leftrightarrow \theta'_1, \dots, \theta_k \leftrightarrow \theta'_k\}$ by $T$. We distinguish the following cases:

(a) *$\psi$ is $\theta_i$ for some $i \leq k$.* Then $\widetilde{\psi}$ is $\theta'_i$ and the equivalence $\psi \leftrightarrow \widetilde{\psi}$ is an axiom of $T$.

(b) *$\psi$ is an atomic formula other than all $\theta_i$.* Then $\widetilde{\psi}$ is $\psi$, and so $\psi \leftrightarrow \widetilde{\psi}$ is an instance of the propositional tautology $p \leftrightarrow p$.

(c) *$\psi$ is $\neg\chi$.* Then $\widetilde{\psi}$ is $\neg\widetilde{\chi}$. By the induction hypothesis, $T \vdash \chi \leftrightarrow \widetilde{\chi}$; so also $T \vdash \neg\chi \leftrightarrow \neg\widetilde{\chi}$. (To prove the latter step, use the instance $(\chi \leftrightarrow \widetilde{\chi}) \to (\neg\widetilde{\chi} \leftrightarrow \neg\chi)$ of the propositional tautology $(p \leftrightarrow q) \to (\neg p \leftrightarrow \neg q)$ and modus ponens.)

(d) *$\psi$ is $\chi \to \chi'$.* Then $\widetilde{\psi}$ is $\widetilde{\chi} \to \widetilde{\chi}'$. By the induction hypothesis, $T \vdash \chi \leftrightarrow \widetilde{\chi}$ and $T \vdash \chi' \leftrightarrow \widetilde{\chi}'$, so $T \vdash (\chi \to \chi') \leftrightarrow (\widetilde{\chi} \to \widetilde{\chi}')$. (To prove the latter step, use the propositional tautology $(p \leftrightarrow q) \to ((r \leftrightarrow s) \to ((p \to r) \leftrightarrow (q \to s)))$ and twice modus ponens.)

(e) *$\psi$ is $(\forall x)\chi$.* Then $\widetilde{\psi}$ is $(\forall x)\widetilde{\chi}$. By the induction hypothesis, $T \vdash \chi \leftrightarrow \widetilde{\chi}$, so $T \vdash \chi \to \widetilde{\chi}$, so $T \vdash (\forall x)(\chi \to \widetilde{\chi})$ by generalization, so $T \vdash (\forall x)\chi \to (\forall x)\widetilde{\chi}$ by the theorem of classical first-order logic, $(\forall x)(\zeta \to \eta) \to ((\forall x)\zeta \to (\forall x)\eta)$, whose proof is left as an exercise.

Analogously, $T \vdash (\forall x)\widetilde{\chi} \to (\forall x)\chi$, so $T \vdash (\forall x)\chi \leftrightarrow (\forall x)\widetilde{\chi}$ (by the propositional tautology $(p \to q) \to ((q \to p) \to (p \leftrightarrow q))$. □

Note that the theorem cannot be strengthened to

$$\vdash (\theta_1 \leftrightarrow \theta'_1) \land \ldots \land (\theta_k \leftrightarrow \theta'_k) \to (\varphi \leftrightarrow \varphi[\theta'_1/\theta_1, \dots, \theta'_k/\theta_k]),$$

which is refuted by the following counterexample:

*Example.* Let $M$ be a structure for the language $\{P, Q\}$, where $P, Q$ are unary predicates, with $\dot{M} = \{\mathrm{a}, \mathrm{b}\}$, $\|P\|_M = \{\mathrm{a}\}$, $\|Q\|_M = \{\mathrm{a}, \mathrm{b}\}$, and $e$ be an evaluation of individual variables such that $e(x) = \mathrm{a}$. Then clearly $M \models Px \leftrightarrow Qx[e]$, but $M \not\models (\forall x)Px \leftrightarrow (\forall x)Qx[e]$. Thus $\not\models (Px \leftrightarrow Qx) \to ((\forall x)Px \leftrightarrow (\forall x)Px[Qx/Px])$.

The counterexample exemplifies the difference between *provable implication* and (mere) *provability* for non-closed formulae. After we prove the Completeness Theorem, we will be able to give the following semantic characterization of these notions:

- *Provable implication* ($T \vdash \varphi \to \psi$): in every model $M$ of $T$ and every evaluation $e$ in $M$, whenever $\varphi$ is true in $e$, so is $\psi$.
- *Provability* ($T, \varphi \vdash \psi$): in every model $M$ of $T$ in which $\varphi$ is true under every evaluation, $\psi$ is true under every evaluation, too.

For closed formulae, both concepts are equivalent due to the Deduction Theorem (notice the necessity of the closedness assumption for $\varphi$ in the Deduction Theorem).

**Exercise.** *Prove the following metatheorems on proof methods:*

1. *$T \vdash \varphi$ iff $T, \neg\varphi \vdash \neg(\varphi \to \varphi)$ ("reductio ad absurdum")*
2. *$T, \varphi \vee \psi \vdash \chi$ iff ($T, \varphi \vdash \chi$ and $T, \psi \vdash \chi$) ("proof by cases")*
3. *$T \vdash \psi$ iff ($T, \varphi \vdash \psi$ and $T, \neg\varphi \vdash \psi$) ("neutral formula")*

## The proof of Gödel's completeness theorem

The Completeness Theorem for classical first-order logic was proved by Kurt Gödel in the end of the 1920's (it comprised his 1929 doctoral dissertation at the University of Vienna and was published in 1930). Leon Henkin's 1949 simplification of the proof by means of Henkin completion of theories, being easier to survey, has become a standard version of the proof in textbooks. An outline of Gödel's original proof (translated into modern notation) can be found in English Wikipedia, entry *Original proof of Gödel's completeness theorem.*

Gödel's Completeness Theorem is the converse of the Soundness Theorem, and can be formulated as the claim that each consistent theory has a model. The idea of the proof is to construct a model of a consistent theory as a structure over its language (in particular, over the set of its closed terms), in which the realization of predicates directly corresponds to the provability of atomic formulae in the theory (the *canonical structure*). For the construction to work well, the theory needs to be complete and have constants witnessing all existential claims (the *Henkin constants*). The first part of the completeness proof thus consists of demonstrating that these assumptions can be made without loss of generality. The canonical structure of a complete Henkin extension of the theory is then a model of the theory.

For simplicity, we shall work with theories without equality. The proof for theories with the equality predicate additionally involves factorization of the canonical structure by the equivalence relation of provable equality of closed terms.

## Henkin constants

**Definition 1.** Let $\varphi(x)$ be a formula with a *single* free variable $x$ and $c_\varphi$ a constant (called the *Henkin constant* corresponding to the formula $\varphi$). The formula $(\exists x)\varphi \to \varphi(c_\varphi/x)$ is called the *Henkin axiom* corresponding to the formula $\varphi$.

**Theorem** (of Henkin constants)**.** *Let $\varphi(x)$ be a formula in the language of a theory $T$. Let $S$ extend $T$ by a new Henkin constant $c_\varphi$ and the Henkin axiom corresponding to $\varphi$. Then $S$ extends $T$ conservatively.*

*Proof.* Let $T'$ extend $T$ just by adding the Henkin constant $c_\varphi$, and let $\psi$ be a formula in the language of $T$. Assume $S \vdash \psi$, i.e., $T', (\exists x)\varphi \to \varphi(c_\varphi/x) \vdash \psi$. Let the variable $y$ occur neither in $\varphi$ nor $\psi$. Then $y$ is free for $x$ in $\varphi$ and the formula $\big(((\exists x)\varphi \to \varphi(y/x)) \to \psi\big)(c_\varphi/y)$ is the formula $((\exists x)\varphi \to \varphi(c_\varphi/x)) \to \psi$. The proof is done as follows:

| | |
|---|---|
| $T' \vdash ((\exists x)\varphi \to \varphi(c_\varphi/x)) \to \psi$ | by the Deduction Theorem |
| $T \vdash ((\exists x)\varphi \to \varphi(y/x)) \to \psi$ | by the Theorem of Constants |
| $T \vdash (\forall y)((\exists x)\varphi \to \varphi(y/x) \to \psi)$ | by generalization |
| $T \vdash (\exists y)((\exists x)\varphi \to \varphi(y/x)) \to \psi$ | by $\vdash (\forall y)(\chi \to \psi) \leftrightarrow ((\exists y)\chi \to \psi)$, if $y$ is not free in $\psi$ |
| $T \vdash ((\exists x)\varphi \to (\exists y)\varphi(y/x)) \to \psi$ | by $\vdash (\exists y)(\chi \to \zeta) \leftrightarrow (\chi \to (\exists y)\zeta)$, if $y$ is not free in $\chi$ |
| $T \vdash \psi$ | by $\vdash (\exists x)\varphi \leftrightarrow (\exists y)\varphi(y/x)$, if $y$ is not free in $\varphi$ |

$\square$

**Exercise.** *Prove (syntactically) the theorems needed for Theorem of Henkin Constants:*

1. $\vdash (\forall y)(\chi \to \psi) \leftrightarrow ((\exists y)\chi \to \psi)$, *if $y$ is not free in $\psi$*
2. $\vdash (\exists y)(\chi \to \zeta) \leftrightarrow (\chi \to (\exists y)\zeta)$, *if $y$ is not free in $\chi$*
3. $\vdash (\exists x)\varphi \leftrightarrow (\exists y)\varphi(y/x)$, *if $y$ is not free in $\varphi$*

## Henkin extension

**Definition 2.** A theory $T$ in a language $L$ is a *Henkin theory* if for each formula $\varphi(x)$ of $L$ with a *single* free variable $x$ there is a constant $c_\varphi$ in $L$ such that $T \vdash (\exists x)\varphi \to \varphi(c_\varphi/x)$.

**Theorem.** *For each theory $T$ there is a Henkin theory $S$ which extends $T$ conservatively.*

*Proof.* Let $T_0 = T$. For each (metamathematical) natural number $n$ we define $T_{n+1}$ as the theory extending $T_n$ by the Henkin constants and Henkin axioms corresponding to all formulae in the language of $T_n$ with a single free variable.

$T_{n+1}$ is a conservative extension of $T_n$, since every $T_{n+1}$-proof of any formula $\varphi$ contains only finitely many Henkin axioms not in $T_n$; consequently, a finite number of applications of the Theorem of Henkin Constants shows the provability of $\varphi$ in $T_n$.

Even though no $T_n$ need be Henkin, the union $S = \bigcup_{n=0}^{\infty} T_n$ is, since each formula in the language of $S$ is a formula in the language of some $T_n$, and so $T_{n+1}$ contains its Henkin constant and axiom. Moreover, $S$ extends $T$ conservatively, since every proof in $S$ contains only finitely many axioms, and so is in fact a proof in some $T_n$. $\square$