185.A09 Advanced Mathematical Logic

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The semantics of classical first-order logic

We will represent the two *truth values* of classical logic by the numbers 0 (false) and 1 (true).

A *structure* M for the predicate language $L = \langle \mathrm{Pred}, \mathrm{Func} \rangle$ (or an L-*structure*) has the form: $M = \langle \dot{M}, (P_M)_{P \in \mathrm{Pred}}, (F_M)_{F \in \mathrm{Func}} \rangle$, where $\dot{M}$ is a non-empty set; for each n-ary predicate symbol $P \in \mathrm{Pred}$, P_M is an n-ary relation on $\dot{M}$ (identified with an element of $\{0, 1\}$ if $n = 0$); for each n-ary function symbol $F \in \mathrm{Func}$, F_M is a function $\dot{M}^n \to \dot{M}$ (identified with an element of $\dot{M}$ if $n = 0$).

Example. A structure for the language of elementary group theory is any algebra with one binary, one unary, and one nullary operation (not necessarily a group).

Let M be a structure for L. An M-*evaluation* of the object variables is a mapping v which assigns an element from M to each object variable. Let v be an M-evaluation, x a variable, and $a \in \dot{M}$. Then by $v[x \mapsto a]$ we denote the M-evaluation such that $v[x \mapsto a](x) = a$ and $v[x \mapsto a](y) = v(y)$ for each object variable y different from x.

Let M be an L-structure and v an M-evaluation. We define the *values* of terms and the *truth values* of formulae in M for an evaluation v recursively by *Tarski's conditions:*

$$
\begin{aligned}
\|x\|_{M,v} &= v(x) \\
\|F(t_1, \ldots, t_n)\|_{M,v} &= F_M(\|t_1\|_{M,v}, \ldots, \|t_n\|_{M,v}), \text{ for every } n\text{-ary } F \in \mathrm{Func} \\
\|P(t_1, \ldots, t_n)\|_{M,v} &= P_M(\|t_1\|_{M,v}, \ldots, \|t_n\|_{M,v}), \text{ for every } n\text{-ary } P \in \mathrm{Pred} \\
\|c(\varphi_1, \ldots, \varphi_n)\|_{M,v} &= c(\|\varphi_1\|_{M,v}, \ldots, \|\varphi_n\|_{M,v}), \text{ for every } n\text{-ary propositional connective } c \\
\|(\forall x)\varphi\|_{M,v} &= \inf\{\|\varphi\|_{M,v[x \mapsto a]} \mid a \in \dot{M}\} \\
\|(\exists x)\varphi\|_{M,v} &= \sup\{\|\varphi\|_{M,v[x \mapsto a]} \mid a \in \dot{M}\}
\end{aligned}
$$

Note that in the clause for P, the characteristic function of the relation P_M is used on the right-hand side, and in the clause for c, the left-hand side refers to the connective (syntax), while the right-hand side refers to the function realizing the connective (semantics).

In classical first-order logic with equality, we additionally have the following Tarski condition for the equality predicate:

$$\|t_1 = t_2\|_{M,v} = \mathrm{id}_M(\|t_1\|_{M,v}, \|t_2\|_{M,v}),$$

where id_M is the characteristic function of the identity relation on $\dot{M}$.

We shall write:

- $\|\varphi(a_1,\dots,a_n)\|_M$ instead of $\|\varphi(x_1,\dots,x_n)\|_{M,v}$ if $v(x_i)=a_i$ for all $i\le n$
- $M\models\varphi[v]$ if $\|\varphi\|_{M,v}=1$
- $M\models\varphi$ if $M\models\varphi[v]$ for each M-evaluation v
- $\models\varphi$ if $M\models\varphi$ for each L-structure M (we also say that φ is a *tautology*).

Exercise. *Formalize in classical first-order logic: "There is a man such that if he drinks, then everybody drinks," and decide whether it is a tautology of classical first-order logic.*

Let M be an L-structure and T an L-theory. Then M is called a *model of* T (or T-model) if $M\models\varphi$ for each $\varphi\in T$. As obviously each structure is a model of the empty theory, we shall use the term *model* for both models and structures in the rest of the text.

Example. An algebra is a model of elementary group theory iff it is a group.

Caveat: $\langle V,E\rangle$, where V is the universe of sets and $E=\{\langle x,y\rangle\in V^2\mid x\in y\}$ is the membership relation, *is not* a model of set theory (not even a structure for the language of set theory), since V is not a set. However, it will be shown in this course that if set theory (say, ZFC) is consistent, then it has a (set-sized, even countable) model *within* the universe of sets. Notice the apparent *Skolem paradox* (there is a countable model of set theory, even though set theory proves the existence of uncountable sets), which is dissolved by a careful distinction between the object theory and its metatheoretical models (as will be further elucidated later in the course).

We shall write $T\models\varphi$ if $M\models\varphi$ for each model M of T, and say that T *entails* φ (or synonymously, that φ is a *consequence* of T). The relation $\models$ between theories and formulae is called the (semantic) *consequence* (or *entailment*) relation of classical first-order logic.

Local and global consequence relation: There are in fact two competing definitions of the consequence relation for predicate logic that can be found in the literature.

The definition used in this course, also called the *global* consequence relation, can be expanded as follows:

$$T\models\varphi\quad\equiv_{\mathrm{df}}\quad(\forall M)\Big((\forall v)(M\models T[v])\Rightarrow(\forall v)(M\models\varphi[v])\Big),$$

where $M\models T[v]$ abbreviates "$M\models\varphi[v]$ for all $\varphi\in T$". The alternative definition, of the *local* consequence relation, reads as follows:

$$T\models_{\mathrm{loc}}\varphi\quad\equiv_{\mathrm{df}}\quad(\forall M)(\forall v)\Big((M\models T[v])\Rightarrow(M\models\varphi[v])\Big).$$

Observe that the relations $\models$ and $\models_{\mathrm{loc}}$ differ, e.g., in the soundness of the generalization rule: $Px\models(\forall x)Px$, but $Px\not\models_{\mathrm{loc}}(\forall x)Px$. Nevertheless, the two consequence relations coincide if T is a set of *closed* formulae. In particular, the tautologies of predicate logic (i.e., the consequences of $T=\emptyset$) are the same under both definitions. The difference can therefore be neutralized by requiring theories to comprise closed formulae only (which is mostly harmless, as $M\models\varphi$ iff $M\models(\forall x)\varphi$, so formulae have the same models as their universal closures).

Semantic completeness: We say that a theory T is *semantically complete* w.r.t. a class K of T-models if it holds for all formulae φ that $T\vdash\varphi$ iff for each $M\in K$, $M\models\varphi$.

We aim at Gödel's Completeness Theorem, which says that each theory in classical first-order logic is semantically complete w.r.t. the class of *all* of its models: for example, that group theory proves exactly those identities that hold in every group. (Gödel's Incompleteness Theorem, on the other hand, asserts the semantic incompleteness of certain theories of arithmetic w.r.t. the single standard model $\mathbb{N}$ of arithmetic.)

The Deduction Theorem

Theorem. *For every* closed *formula φ in the language of the theory T:*

$$T, \varphi \vdash \psi \quad \textit{iff} \quad T \vdash \varphi \to \psi.$$

By the Deduction Theorem, the (metatheoretical) derivability relation between closed formulae is *internalized* by implication in the object language (of the formulae of classical first-order logic). The right-to-left direction is sometimes called the *Detachment Theorem,* and the equivalence the *Deduction–Detachment Theorem;* the name Deduction Theorem is then reserved for the left-to-right direction.

Proof. Right to left: Appending the formulae φ, ψ to the end of the proof of $\varphi \to \psi$ in T yields the proof of ψ in T, φ.

Left to right: Let $\chi_1, \ldots, \chi_n$ be a proof of ψ in T, φ. By induction on $i \leq n$ we prove that $T \vdash \varphi \to \chi_i$, taking the following cases:

1. If χ_i is an axiom of predicate logic or an axiom of T, then the following sequence of formulae is the proof of $\varphi \to \chi_i$ in T:

$\chi_i \to (\varphi \to \chi_i)$	instance of the axiom of propositional logic
χ_i	axiom of predicate logic or T
$\varphi \to \chi_i$	by modus ponens

2. If χ_i is the formula φ, then $T \vdash \varphi \to \chi_i$ is an instance of the theorem $\varphi \to \varphi$ of propositional logic.

3. If χ_i is derived from χ_j and χ_k by modus ponens, then χ_k is the formula $\chi_j \to \chi_i$. Furthermore, by the induction hypothesis there are proofs $\alpha_1, \ldots, \alpha_{n_j}, \varphi \to \chi_j$ and $\beta_1, \ldots, \beta_{n_k}, \varphi \to (\chi_j \to \chi_i)$ in T. Concatenating the latter two proofs and appending the following three formulae:

$(\varphi \to (\chi_j \to \chi_i)) \to ((\varphi \to \chi_j) \to (\varphi \to \chi_i))$	instance of a propositional axiom
$(\varphi \to \chi_j) \to (\varphi \to \chi_i)$	by modus ponens
$\varphi \to \chi_i$	by modus ponens

   thus yields the proof of $\varphi \to \chi_i$ in T.

4. If χ_i is derived from χ_j by generalization, then χ_i is $(\forall x)\chi_j$ for some variable x. Furthermore, by induction hypothesis there is a proof $\alpha_1, \ldots, \alpha_{n_j}, \varphi \to \chi_j$ in T. Appending to the latter proof the following three formulae:

$(\forall x)(\varphi \to \chi_j)$	by generalization
$(\forall x)(\varphi \to \chi_j) \to (\varphi \to (\forall x)\chi_j)$	the distribution axiom of predicate logic
$\varphi \to (\forall x)\chi_j$	by modus ponens

   then yields the proof of $\varphi \to \chi_i$ in T. (NB: the closedness of φ was needed in this step for the distribution axiom.) $\square$

Note that without the precondition of the closedness of φ, the Deduction Theorem would fail. For example, $Px \vdash Py$ (verify!), but $Px \to Py$ is not a theorem of classical first-order logic (as will be easily seen after we prove the Soundness Theorem).

The Theorem of Constants

The restriction of the Deduction Theorem to closed formulae can sometimes be circumvented by replacing free variables by new constants. The conservativeness of such replacements is ensured by the Theorem of Constants.

Recall that the formula $\varphi(t/x)$ results from replacing *all free occurrences* of the variable x in φ by t. The definition can be straightforwardly extended to the substitution of a term t for a *constant* c in φ, which will analogously be denoted by $\varphi(t/c)$.

Theorem. *Let T' extend T by adding the constant c to the language (not adding any new axioms or rules, though). Then for each formula φ in the language of T:*

$$T \vdash \varphi \quad \textit{iff} \quad T' \vdash \varphi(c/x).$$

(Thus in particular, T' is a conservative extension of T.)

Proof. Left to right: Appending the following three formulae to any proof of φ in T yields a proof of $\varphi(c/x)$ in T':

$(\forall x)\varphi$	by generalization
$(\forall x)\varphi \to \varphi(c/x)$	by the axiom of specification
$\varphi(c/x)$	by modus ponens

Right to left: Let $\psi_1, \ldots, \psi_n$ be a proof of $\varphi(c/x)$ in T'. Let y be a new variable, not occurring in the latter proof. Thus y is substitutable for c in each ψ_i. By induction on $i \le n$ we show that $\psi_1(y/c), \ldots, \psi_i(y/c)$ is a proof in T, taking the following cases:

1. If ψ_i is a predicate axiom, then $\psi_i(y/c)$ is also a predicate axiom (of the same form); we only need to check the preconditions of the specification and distribution axioms:
   (a) If ψ_i is a specification axiom, then it is a formula $(\forall z)\alpha \to \alpha(t/z)$ for some term t free for z in α. Then $t(y/c)$ is free for z in $\alpha(y/c)$, since $\alpha(y/c)$ contains only the quantifiers occurring in α and y does not occur in α.
   (b) If ψ_i is a distribution axiom, then it is a formula $(\forall z)(\alpha \to \beta) \to (\alpha \to (\forall z)\beta)$, where z is not free in α. Then obviously z is not free in $\alpha(y/c)$, either.
2. If ψ_i is an axiom of T, then c does not occur in ψ_i, so $\psi_i(y/c)$ is ψ_i.
3. If ψ_i is derived by modus ponens from ψ_j and ψ_k, then ψ_k is $\psi_j \to \psi_i$ and $\psi_k(y/c)$ is $\psi_j(y/c) \to \psi_i(y/c)$, so $\psi_i(y/c)$ is derived from $\psi_j(y/c)$ and $\psi_k(y/c)$ by modus ponens, too.
4. If ψ_i is derived from ψ_j by generalization, then ψ_i is $(\forall z)\psi_j$ for some variable z, and $(\forall z)\psi_j(y/c)$ is derived from $\psi_j(y/c)$ by generalization, too.

We have shown that $(\varphi(c/x))(y/c)$, i.e., $\varphi(y/x)$ is provable in T. Since x is free for y in $\varphi(y/x)$ (as y only replaced *free* occurrences of x in φ), and since $(\varphi(y/x))(x/y)$ is φ, appending the following three formulae:

$$\begin{array}{ll} (\forall y)\varphi(y/x) & \text{by generalization} \\ (\forall x)\varphi(y/x) \to (\varphi(y/x))(x/y) & \text{by specification} \\ (\varphi(y/x))(x/y) & \text{by modus ponens} \end{array}$$

to the proof of $\varphi(y/x)$ in T yields a proof of φ in T. $\square$