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Fuzzy logics among substructural logics

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Substructural logics and residuated lattices

Substructural logics

= a unifying framework for several kinds of logics:

- Linear logics
- Relevance logics
- Lambek calculus, BCK, . . .
- . . . and also fuzzy logics!

Substructural logics arise from some established calculus
(most often: LK of Boolean logic or LJ of intuitionistic logic)
by removing some of the *structural rules*

Recall: Gentzen-style calculi have

- Operational rules (for introduction of connectives)
- Structural rules
(for manipulation with premises and conclusions)

Example: The calculus LK for classical logic

(Axiom) $$\frac{}{A \Rightarrow A}$$

(Cut) $$\frac{\Gamma \Rightarrow \Pi, A \quad A, \Delta \Rightarrow \Sigma}{\Gamma, \Delta \Rightarrow \Pi, \Sigma}$$

Operational rules:

($\wedge$-L) $$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$$, ditto B

($\wedge$-R) $$\frac{\Gamma \Rightarrow \Pi, A \quad \Gamma \Rightarrow \Pi, B}{\Gamma \Rightarrow \Pi, A \wedge B}$$

($\vee$-L) $$\frac{A, \Gamma \Rightarrow \Pi \quad B, \Gamma \Rightarrow \Pi}{A \vee B, \Gamma \Rightarrow \Pi}$$

($\vee$-R) $$\frac{\Gamma \Rightarrow \Pi, A}{\Gamma \Rightarrow \Pi, A \vee B}$$, ditto B

($\rightarrow$-L) $$\frac{\Gamma \Rightarrow \Pi, A \quad B, \Delta \Rightarrow \Sigma}{A \rightarrow B, \Gamma, \Delta \Rightarrow \Pi, \Sigma}$$

($\rightarrow$-R) $$\frac{A, \Gamma \Rightarrow \Pi, B}{\Gamma \Rightarrow \Pi, A \rightarrow B}$$

($\neg$-L) $$\frac{\Gamma \Rightarrow \Pi, A}{\neg A, \Gamma \Rightarrow \Pi}$$

($\neg$-R) $$\frac{A, \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi, \neg A}$$

Structural rules:

(Exchange-L) $\dfrac{\Gamma, A, B, \Delta \Rightarrow \Pi}{\Gamma, B, A, \Delta \Rightarrow \Pi}$

(Exchange-R) $\dfrac{\Gamma \Rightarrow \Pi, A, B, \Sigma}{\Gamma \Rightarrow \Pi, B, A, \Sigma}$

(Weakening-L) $\dfrac{\Gamma \Rightarrow \Pi}{A, \Gamma \Rightarrow \Pi}$

(Weakening-R) $\dfrac{\Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi, A}$

(Contraction-L) $\dfrac{A, A, \Gamma \Rightarrow \Pi}{A, \Gamma \Rightarrow \Pi}$

(Contraction-R) $\dfrac{\Gamma \Rightarrow \Pi, A, A}{\Gamma \Rightarrow \Pi, A}$

LK is sound and complete for Boolean logic
if $\Gamma \Rightarrow \Pi$ is interpreted as $\bigwedge \Gamma \to \bigvee \Delta$

The calculus **LJ** for intuitionistic logic
= LK with all sequents $\Gamma \Rightarrow \Delta$ having $\text{Card}(\Delta) \leq 1$

Notice:
Only (E-L), (W-L), (W-R) for $\Gamma \Rightarrow$, and (C-L) occur in LJ

Motivation for dropping structural rules

(A) To block some derivations:

- Relevance logics drop W to prevent the paradoxes of material implication ($\varphi \to (\psi \to \varphi)$, $\varphi \wedge \neg\varphi \to \psi$, etc.)
- Dropping C bars Russell's paradox

(B) To model other phenomena than classical logic:

- Sequential access to premises (E)
- Temporal aspects of $\wedge$, $\vee$ in natural language (E)
- Costs or resources needed for obtaining the premises (C)
- Exhaustivity of premises (W) . . . resources must be spent
- Formulae as procedures, types in categorial grammar, etc.

Intuitionistic substructural logics

LJ − E,W,C = FL, Full Lambek calculus

- Introduced by Ono (1994)
- Lambek calculus is also known as *categorial grammar*
  (as it describes the behavior of grammatical categories)
- *Full*, since in the full language
  (Lambek calculus has just conjunction and implication)

By adding E,W,C back, we get 7 logics FL_X:
FL, FL_e, FL_w, FL_c, FL_{ec}, FL_{ew}, Int $= FL_{wc} = FL_{ewc}$

FL_{ew} is also known as Höhle's (1995) monoidal logic
$FL_{e(w)}$ = multiplicative–additive intuitionistic (affine) linear logic

All FL_X except FL_c have cut-elimination
FL, FL_e, FL_w, and FL_{ew} are decidable

Classical substructural logics

LK − E,W,C = CFL = FL + the law of double negation

By adding E,W,C back we get 7 logics $\mathrm{CFL}_{\mathrm{x}}$:

CFL, $\mathrm{CFL}_{\mathrm{c}}$, $\mathrm{CFL}_{\mathrm{w}}$,

$\mathrm{CFL}_{\mathrm{e}}$ = MALL = multiplicative–additive linear logic without exponentials (Girard 1987)

$\mathrm{CFL}_{\mathrm{ew}}$ = AMALL = affine MALL = Grishin's (1974) contraction-free logic

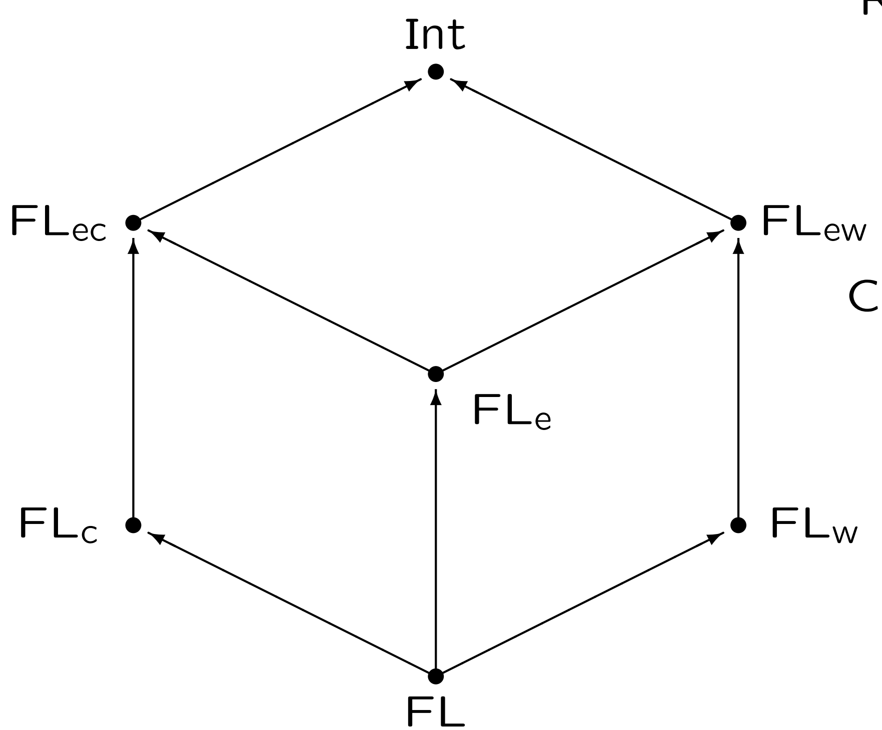
$\mathrm{CFL}_{\mathrm{ec}}$ = R^{ND} = R minus distribution (Meyer 1966)

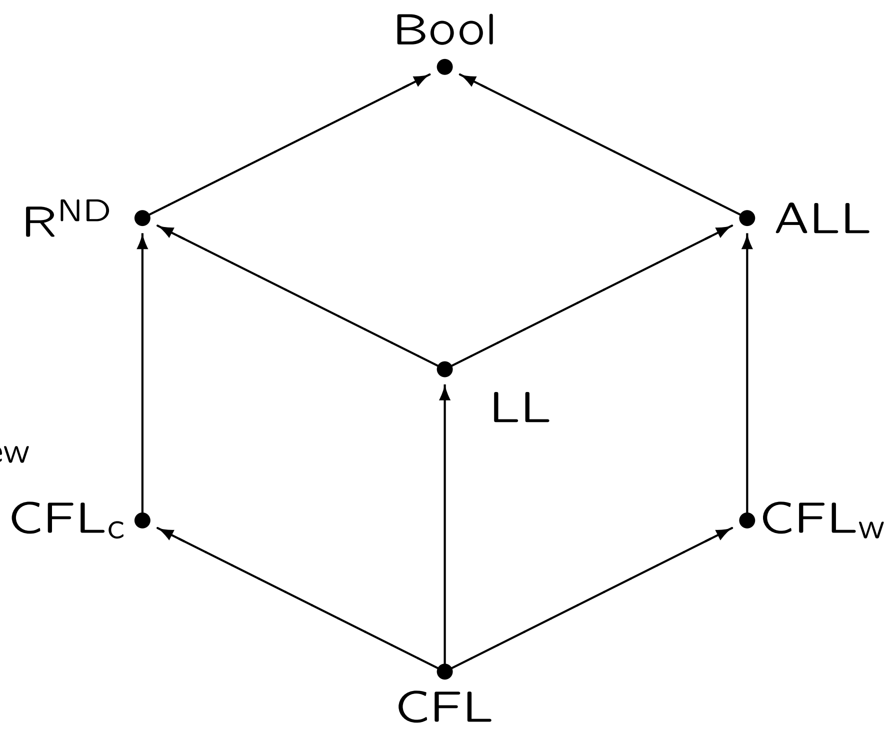
$\mathrm{CFL}_{\mathrm{wc}}$ = $\mathrm{CFL}_{\mathrm{ewc}}$ = Bool

FL_{x} and $\mathrm{CFL}_{\mathrm{x}}$ can further be modified by adding axioms

$\Rightarrow$ a rich family of substructural logics

Basic intuitionistic and classical substructural logics:





Lattice and group connectives

Recall Gentzen's rules for $\wedge$, $\vee$:

$$(\wedge\text{-L})\ \frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ ,\ \text{ditto } B \qquad (\wedge\text{-R})\ \frac{\Gamma \Rightarrow \Pi, A \quad \Gamma \Rightarrow \Pi, B}{\Gamma \Rightarrow \Pi, A \wedge B}$$

$$(\vee\text{-L})\ \frac{A, \Gamma \Rightarrow \Pi \quad B, \Gamma \Rightarrow \Pi}{A \vee B, \Gamma \Rightarrow \Pi} \qquad (\vee\text{-R})\ \frac{\Gamma \Rightarrow \Pi, A}{\Gamma \Rightarrow \Pi, A \vee B}\ ,\ \text{ditto } B$$

Compare them to Ketonen's (1944) and Curry's (1960) variants:

$$(\wedge\text{-L})'\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad (\wedge\text{-R})'\ \frac{\Gamma \Rightarrow \Pi, A \quad \Delta \Rightarrow \Sigma, B}{\Gamma, \Delta \Rightarrow \Pi, \Sigma, A \wedge B}$$

$$(\vee\text{-L})'\ \frac{A, \Gamma \Rightarrow \Pi \quad B, \Delta \Rightarrow \Sigma}{A \vee B, \Gamma, \Delta \Rightarrow \Pi, \Sigma} \qquad (\vee\text{-R})'\ \frac{\Gamma \Rightarrow \Pi, A, B}{\Gamma \Rightarrow \Pi, A \vee B}$$

They are equivalent in the presence of structural rules, but define different connectives in their absence

$\Rightarrow$ Without structural rules, there are two sets of connectives:

($\wedge$-L), ($\wedge$-R) . . . lattice conjunction $\wedge$
(also: weak, additive, comparative, extensional)
($\wedge$-L)$'$, ($\wedge$-R)$'$ . . . group conjunction & (or *fusion*, $\otimes$)
(also: strong, multiplicative, parallel, intensional)

Similarly for disjunction (lattice $\vee$, group $\veebar$ or $\oplus$)

They have different inferential properties (ergo, meaning):

In FL, $(\varphi \,\&\, \psi \to \chi) \;\dashv\vdash\; (\psi \to (\varphi \to \chi))$ . . . using *both* conjuncts
while $(\varphi \wedge \psi \to \chi) \;\dashv\vdash\; (\varphi \to \chi) \vee (\psi \to \chi)$
. . . using *any* of the conjuncts

Without exchange, $\to$ and $\neg$ split as well
Without weakening, unit and null elements of $\wedge$/& differ, too
($\bot$ vs. 0, and $\top$ vs. 1)

Hilbert-style presentation

The calculi can be translated into the Hilbert style

We shall mostly use $\mathsf{FL}_{\mathsf{ew}}$:

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \qquad \text{(B)}$$
$$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)) \qquad \text{(C)}$$
$$\varphi \to (\psi \to \chi) \qquad \text{(K)}$$
$$\varphi \to \varphi \vee \psi,\ \psi \to \varphi \vee \psi,\ (\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$$
$$\varphi \wedge \psi \to \varphi,\ \varphi \wedge \psi \to \psi,\ (\chi \to \varphi) \wedge (\chi \to \psi) \to (\chi \to \varphi \wedge \psi)$$
$$\varphi \to (\psi \to \varphi \wedge \psi)$$
$$\varphi \to (\psi \to \varphi \mathbin{\&} \psi),\ \varphi \to (\psi \to \chi) \to (\varphi \mathbin{\&} \psi \to \chi)$$
$$\bot \to \varphi,\ \varphi \to \top$$

Axioms corresponding to structural rules:

Weakening . . . $\varphi \mathbin{\&} \psi \to \varphi$

Exchange . . . $\varphi \mathbin{\&} \psi \to \psi \mathbin{\&} \varphi$

Contraction . . . $\varphi \to \varphi \mathbin{\&} \varphi$

Semantics of substructural logics

Lattice connectives ensure the lattice structure
Additionally there must be a monoidal operation for &
and its (so-called) *residuum* for implication
$\Rightarrow$ Residuated lattices

Definition: $\mathbf{L} = (L, \wedge, \vee, *, /, \backslash, 1)$ is a residuated lattice iff

- $(L, \wedge, \vee)$ is a lattice
- $(L, *, 1)$ is a monoid
- $x * y \leq z$ iff $x \leq z/y$ iff $y \leq z \backslash x$ holds ("residuation")

If $*$ is commutative, then $/$ equals $\backslash$ and is usually written as $\Rightarrow$

Definition: A residuated lattice is

- Commutative iff $*$ is commutative
- Bounded iff it has the top $\top$ and bottom $\bot$
- Integral iff $1 = \top$
- Increasing idempotent iff $x \leq x * x$ holds

Theorem (soundness and completeness):

FL = the logic of all residuated lattices with a fixed element 0

E . . . commutative

W . . . bounded integral

C . . . increasing idempotent

Algebras of these varieties can also be called (C)FL_{X}-algebras

$\mathsf{FL}_{\mathsf{WC}}$-algebras = Heyting algebras

$\mathsf{CFL}_{\mathsf{WC}}$-algebras = Boolean algebras

Ono (2003) delimits substructural logics as

the logics of (classes of) residuated structures

Fuzzy logic: motivation, history, a basic picture

The Sorites paradox

Sorites (the paradox of the heap)
= a paradox ascribed to Eubulides of Miletus
(Greek $\sigma\omega\rho\iota\tau\eta\varsigma$ = an adjective to o $\sigma\omega\rho\acute{o}\varsigma$ = heap)

Falakros (the paradox of the bald man)
= a variant of the Sorites paradox, also ascribed to Eubulides
(Greek $\varphi\alpha\lambda\alpha\kappa\rho o\varsigma$ = bald)

Eubulides = a Greek philosopher of the Megarian school
4th century BCE (contemporary with Aristotle)
attributed several paradoxes (incl. the Liar)

The paradox:

A myriad grains form a heap

If a single grain is removed from a heap, it remains a heap

$\Rightarrow$ *(after* 10.000 *steps)* 0 *grains form a heap*

Formally:

Px_0 and $(\forall n)(Px_n \to Px_{n+1})$, but $\neg Px_N$ for some large N

(in a situation where both premises seem to be plausible)

The paradox can be formulated in various forms:

(independently of the logical means employed)

- Propositionally:

  p_0 and $p_0 \to p_1$ and $\ldots$ and $p_{N-1} \to p_N$, but $\neg p_N$

  (where all $N + 1$ premises look plausible)

- Without implication:

  p_0 and not $p_0 \,\&\, \neg p_1$ and not $p_1 \,\&\, \neg p_2$ etc.

- Without connectives:

  Is a man with no hair bald? Is a man with 1 hair bald?
  Is a man with 2 hairs bald? . . . (etc. ad infinitum)

  (there is no exact point where we should say No)

$\Rightarrow$ not a paradox of material implication or any logical constant

The phenomenon underlying the Sorites is called *vagueness*

Sometimes used as an informal definition:

A predicate P is *vague* iff

one can construct a Sorites series for P

A *Sorites series* for P is a series $x_0, \ldots, x_N$ of individuals s.t.

Px_0 and $Px_n \rightarrow Px_{n+1}$ for all n and $\neg Px_N$ all seem plausible

(usually due to indistinguishability of x_n and x_{n+1} w.r.t. P)

Example:

A series of 1000 people successively differing

by +0.5 mm in height is a Sorites series for *tall*

Further examples of vagueness:

- Colors (the hues of *green* do not have a sharp boundary)
- Geographical features (where exactly do *mountains* end?)
- Predicates of natural language like *tall*, *warm*, *young*, . . .
  (hard to find a non-vague natural-language predicate)

Virtually all real-world notions are vague in a sufficiently fine-grained scale (in which nanosecond a woman gets *pregnant*?)

Exact (sharp) notions can be found in mathematics
(the property $n < 5$ is sharp)
but in the real world (or natural language) they are just idealizations

The phenomenon of vagueness is abundant
(as shown by the examples)

Under vagueness, classical reasoning is problematic
(as shown by the Sorites paradox)

$\Rightarrow$ New ways of logical modeling of vagueness are required

We do not attempt at finding the “true nature”
or the “right theory” of vagueness

Rather, we construct a useful model
(a good compromise between precision + generality
vs. complexity)

The philosophy of vagueness

Logic of vagueness . . . the laws of inference under vagueness
Philosophy of vagueness . . . requires a fuller account
(e.g., *why* things are such and so)

Different philosophies of vagueness can share the same logic
Example: epistemicism and supervaluationism

Most influential philosophies of vagueness:

- Epistemicism . . . uses Boolean logic
- Supervaluationism . . . uses Boolean or modal logic
- Degree theories . . . use certain many-valued logics

Fuzzy logic is only a *logic* of vagueness (a model, a tool)
No philosophical assumptions on the nature of truth are made
(interpretations other than vagueness are possible)

Fuzzy logic as a field of study

Fuzzy logic in broad sense

- A toolbox of engineering methods
- Generalization of truth-tables to $[0, 1]$
- Often mathematically poor

Fuzzy logic in narrow sense

- Logical systems aimed at the formalization of approximate reasoning

Mathematical fuzzy logic

- Building upon the tradition of classical and many-valued logic
- Formal calculi and formal semantics
- Metamathematical study of the systems of fuzzy logic
- Analogues of classical metatheorems

(completeness, deduction, . . .)

The history of mathematical fuzzy logic

1917–20	Łukasiewicz logic
1932	Gödel logic (implicitly)
1968	Goguen: "fuzzy logic"
1978	Pavelka logic
1993	Gottwald's monograph
1998	Hájek's monograph (BL, Π)
1999	ŁΠ (Esteva, Godo, Montagna)
2001	MTL (Esteva, Godo)
2004–6	uninorm logic (Metcalfe)
	weakly implicative fuzzy logics (Cintula)
	higher-order fuzzy logic (Novák; Běhounek, Cintula)

Logics suitable for graded reasoning: a qualified guess

Observe:

(1) Vagueness is all-pervading, and yet

(2) we do quite well with classical reasoning

(most times—apart from the Sorites)

⇒ we can hope that fuzzy logic need not be much different from classical logic

⇒ Strategy: Omit from Bool just what is necessary

(uniformly, not adaptively, for simplicity)

What should be dropped from Bool?

(1) The law of double negation ($\neg\neg$):

In some contexts implausible for vague notions

Sometimes it is reasonable to model $\neg$ as following the ($\neg\neg$) law

eg, measure $\neg$*tall* by missing cm's, then $\neg\neg$*tall* $=$ *tall*

but sometimes not:

- *guilty* has degrees

  (murdering $>$ stealing $>$ crossing on the red light)
- *not guilty* has no degrees (no crime at all, however grave)

  LEM holds for not-guilty $\Rightarrow$ *not-not-guilty* $\neq$ *guilty*

  (similarly for *luminous, having money,* etc.)

The most prominent logic without ($\neg\neg$) is Int

$\Rightarrow$ we conjecture that in the presence of vagueness we should in general reason "intuitionistically" rather than classically

$=$ take LJ rather than LK as the starting point

(2) Contraction:

Is then Int a good logic for vagueness?

—Some people do use it, but consider the following:

Partially true = imperfectly true

By using imperfect premises many times, imperfection increases

If the imperfectly true inductive premise of the Sorites is used several times, the conclusion is still guaranteed to be fairly true

Yet after using it 100.000 times, the imperfection of the argument accumulates and the conclusion is completely false

$\Rightarrow$ Repeated usage of an imperfectly true premise

makes the argument *weaker*

(even the same premise: cf. the Sorites inductive premise)

$\Rightarrow$ $\varphi \,\&\, \varphi \not\leftrightarrow \varphi$ for vague φ

(3) Prelinearity:

Is then Int − C = FL_{ew} a good logic for vagueness?

—Quite possibly, but:

In fuzzy logic we also aim at *comparing* the truth of propositions

(fuzzy logic studies a *comparative notion of truth*)

The following principle of prelinearity

- Has proved useful in fuzzy logic
- Seems faithful to certain features of "measuring" truth
- Can be maintained while solving the Sorites
- Therefore we shall adopt it

(recall the strategy: drop from Bool only the necessary)

Prelinearity: "either φ is weaker than ψ or vice versa"

= in sufficiently strong logics: $(\varphi \to \psi) \vee (\psi \to \varphi)$

$\Rightarrow$ The "default" fuzzy logic =

= Int − Contraction + Prelinearity

= FL_{ew} + Prelinearity

= MTL, indeed one of the most important fuzzy logics

Further axioms can be added in special situations

($\neg\neg$) if $\neg$ is involutive . . . IMTL

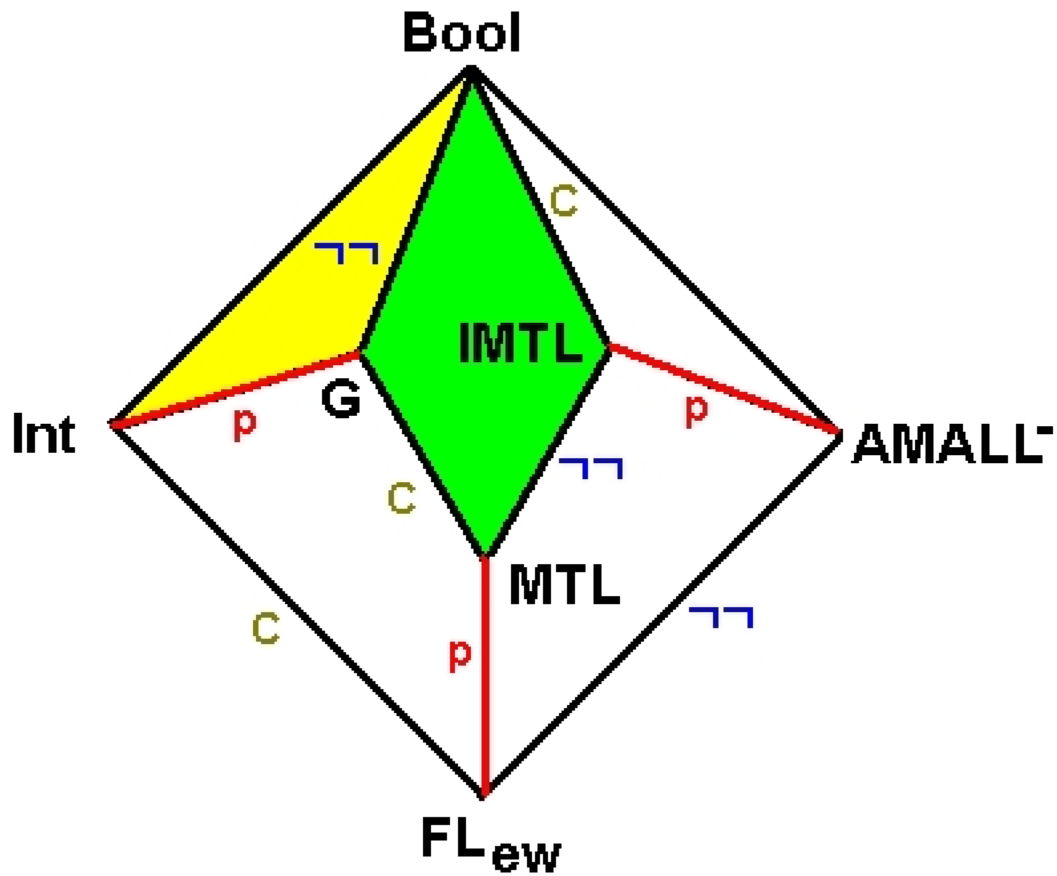
C if conjunction is idempotent . . . Gödel logic

LEM if all propositions involved are crisp . . . Bool, etc

The most important fuzzy logics are expansions of MTL

Fuzzy logics = intuitionistic non-contractive prelinear logics

Fuzzy logics among substructural logics:



T-norm based fuzzy logics

(the $[0,1]$-based account)

Hájek's (1998) approach

Goal: Generalize bivalent classical logic to $[0, 1]$

Strategy: Impose some reasonable constraints on the truth functions of propositional connectives to get a well-behaved logic

Implementation:

- As a design choice, we assume the truth-functionality of all connectives w.r.t. $[0, 1]$
- We require some natural conditions of &
- A truth function of & satisfying these constraints will determine the rest of propositional calculus

The requirements of the truth function of conjunction, $*\colon [0,1]^2 \to [0,1]$

Commutativity: $x * y = y * x$

- When asserting two propositions, it does not matter in which order we put them down

- The commutativity of classical conjunction, which holds for crisp propositions, seems to be unharmed by taking into account also fuzzy propositions

- Thus, by using a non-commutative conjunction we would generalize to fuzzy-tolerance, not the Boolean logic, but rather some other logic that models order-dependent assertions of propositions (e.g., some kind of temporal logic)

Associativity: $(x * y) * z = x * (y * z)$

- When asserting three propositions, it is irrelevant which two of them we put down first (be they fuzzy or not)

Monotony: *if* $x \leq x'$, *then* $x * y \leq x' * y$

- Increasing the truth value of the conjuncts should not decrease the truth value of their conjunction

Classicality: $x * 1 = x$ *(thus also* $x * 0 = 0$*)*

- $0, 1$ represent the classical truth values for crisp propositions
- Conjunction with full truth should not change the truth value

Continuity: $*$ *is continuous*

- An infinitesimal change of the truth value of a conjunct should not radically change the truth value of the conjunction

We could add further conditions on & (e.g., idempotence), but it has proved suitable to stop here, as it already yields a rich and interesting theory and further conditions would be too limiting

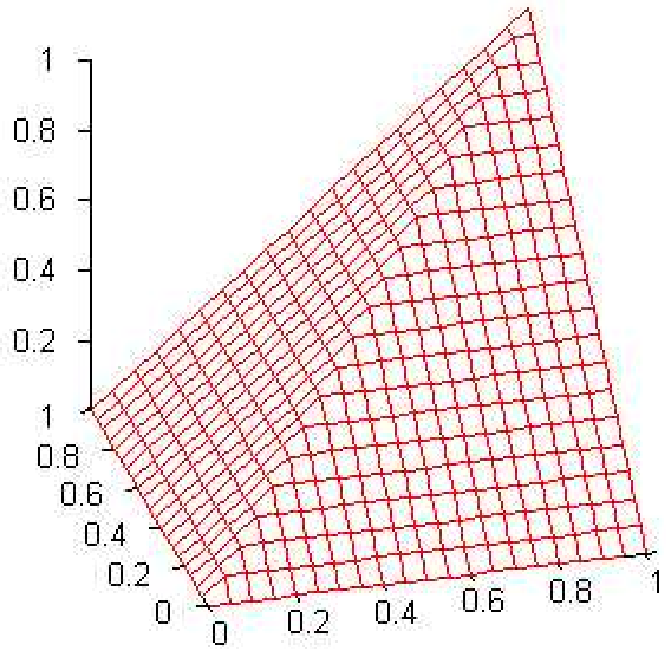
Such functions have previously been studied in the theory of probabilistic metric spaces and called *triangular norms* or shortly *t-norms* (continuous, as we require continuity):

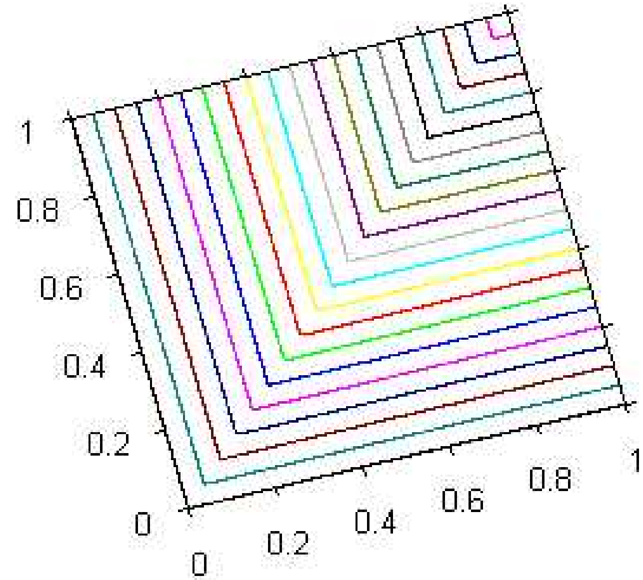
Definition: A binary function $*\colon [0,1] \to [0,1]$ is a t-norm iff it is commutative, associative, monotone, and 1 is a neutral element

Fact: A t-norm $*$ is continuous iff it is continuous in one variable, ie, iff $f_x(y) = x * y$ is continuous for all $x \in [0,1]$ (analogously for left- and right-continuity)

Prominent examples of continuous t-norms (1):

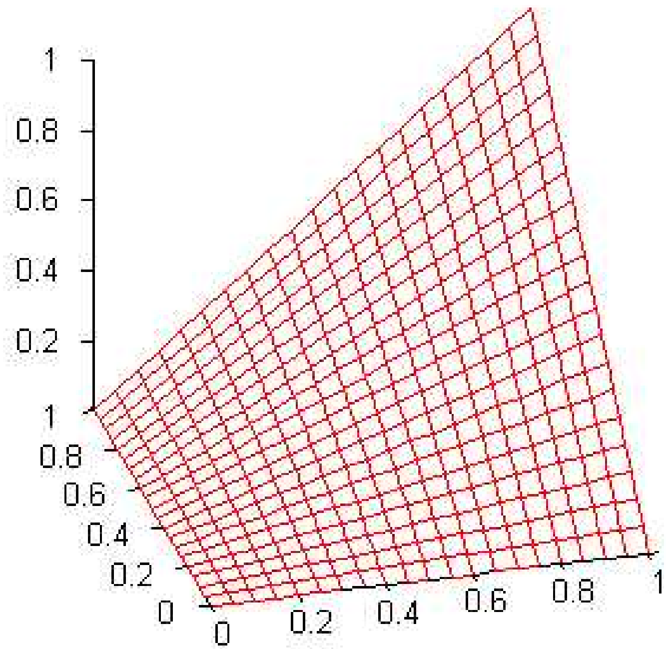
The minimum t-norm: $x *_{\mathsf{G}} y = \min(x, y)$

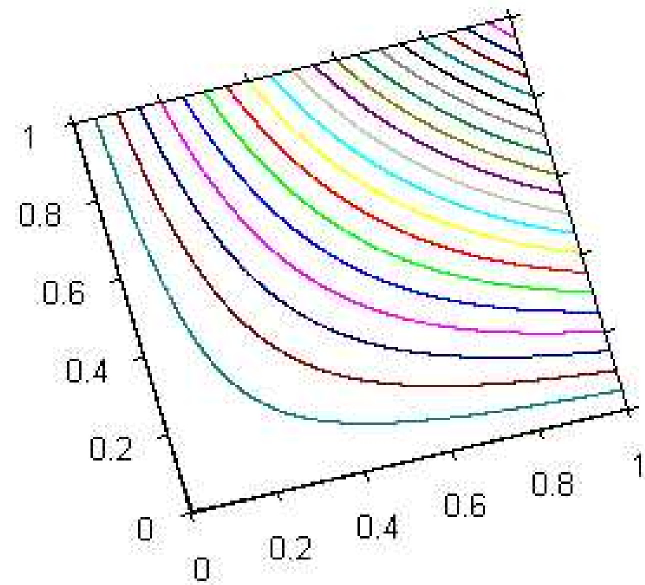




Prominent examples of continuous t-norms (2):

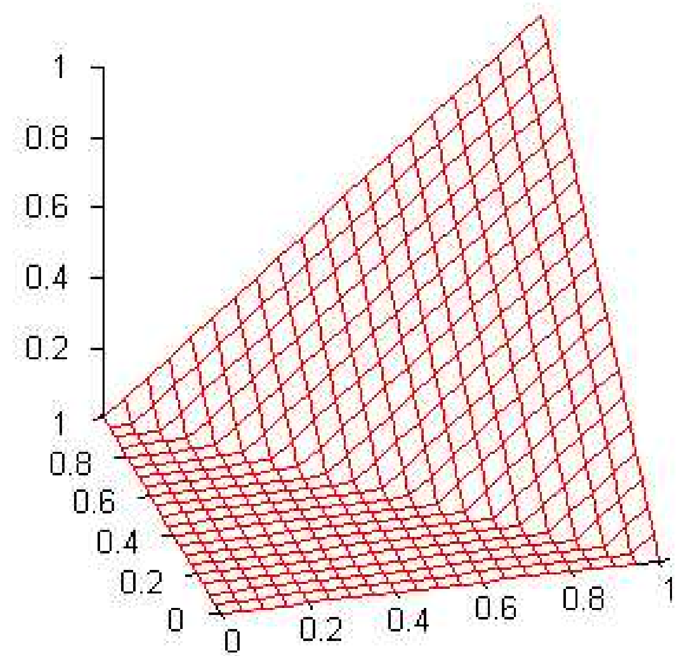
The product t-norm: $x *_{\prod} y = x \cdot y$

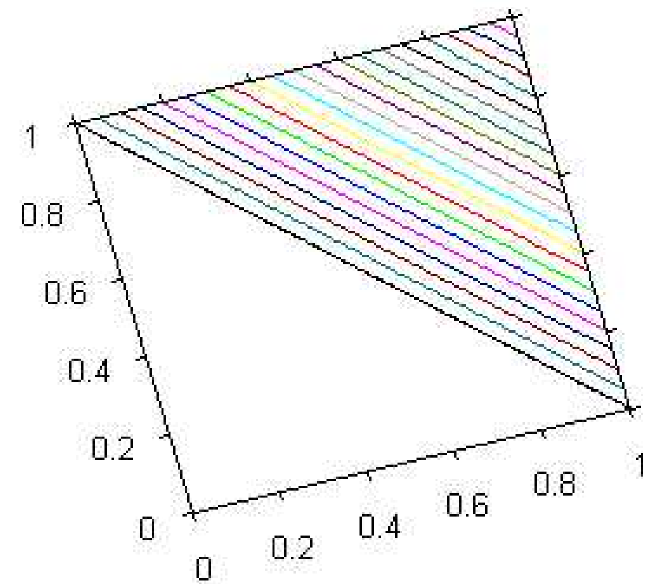




Prominent examples of continuous t-norms (3):

The Łukasiewicz t-norm: $x *_{Ł} y = \max(0, x + y - 1)$





Mostert–Shield's characterization of continuous t-norms

The idempotents (ie, such x that $x * x = x$)
of any continuous t-norm form a closed subset of $[0, 1]$

Its complement is an (at most countable) union of open intervals

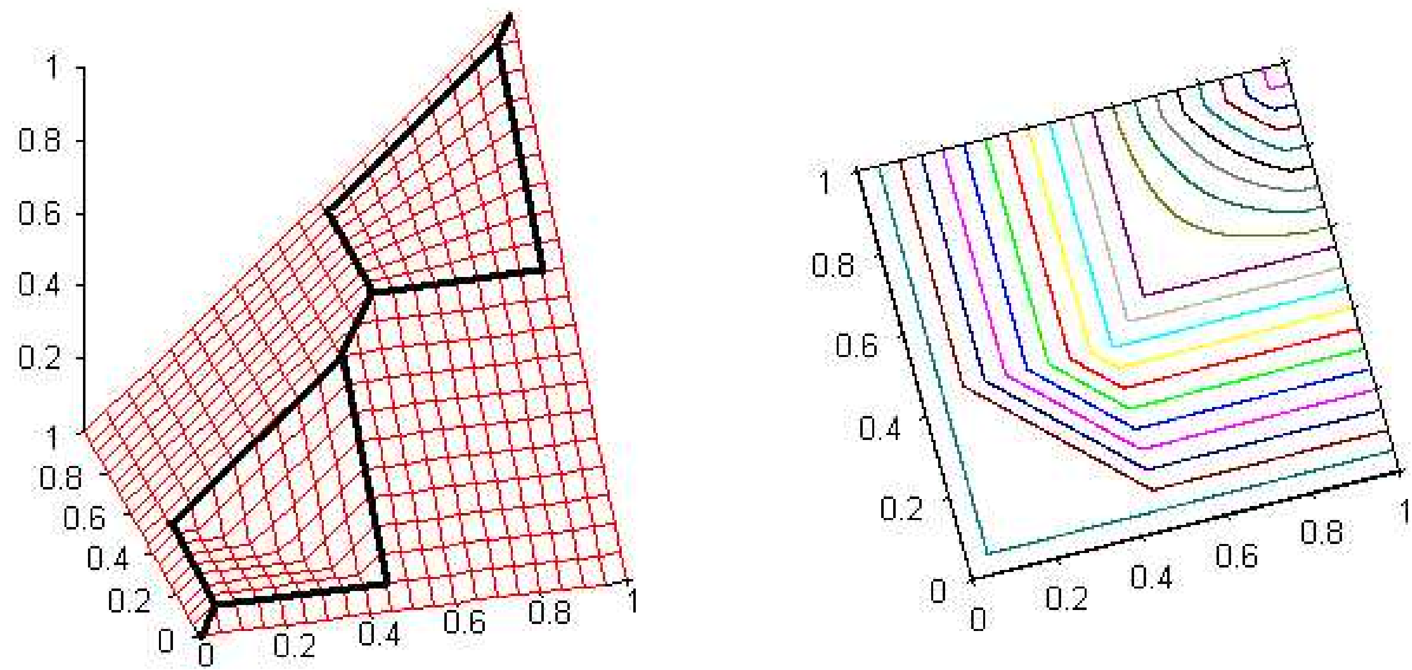
The restriction of $*$ to each of these intervals is isomorphic
to $*_{Ł}$ or $*_{\Pi}$

On the rest of $[0, 1]$ it coincides with $*_{G}$

All continuous t-norms can be obtained this way

= All continuous t-norms are ordinal sums of
isomorphic copies of $*_{Ł}, *_{\Pi}, *_{G}$

Example: Ordinal sum of $*_{Ł}$ on $[0.05, 0.45]$, $*_{\Pi}$ on $[0.55, 0.95]$, and the default $*_{G}$ elsewhere



Residua of continuous t-norms

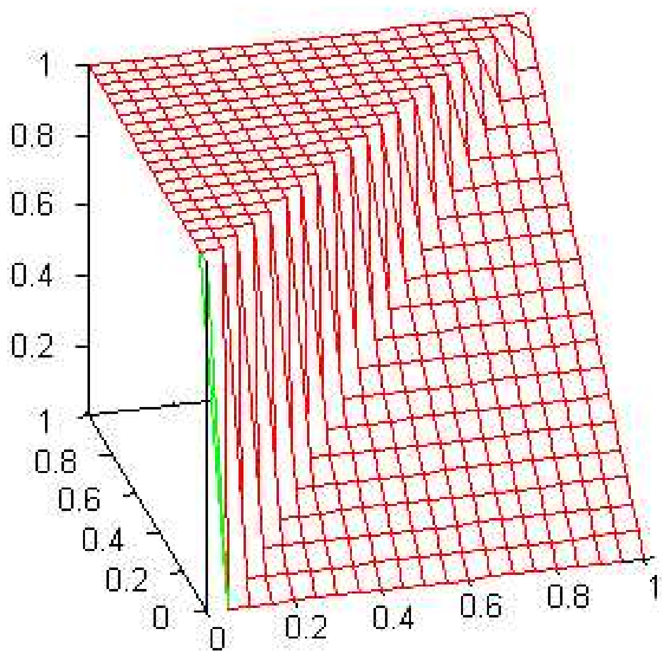
Each continuous t-norm $*$ uniquely determines its residuum $\Rightarrow$

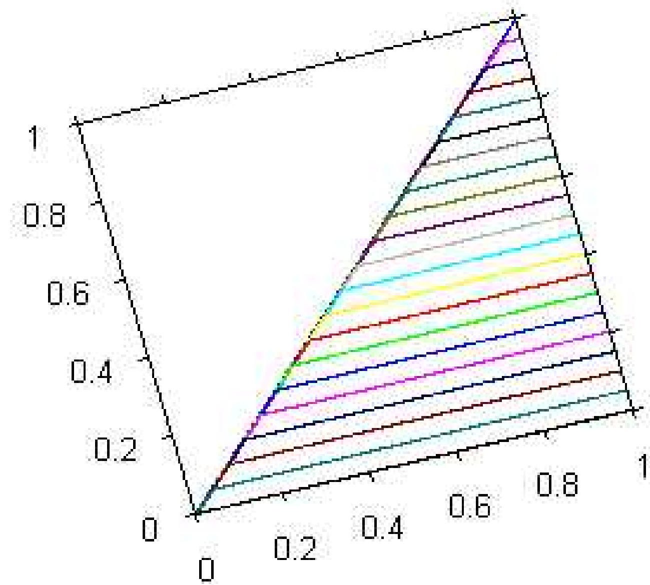
ie, such operation $\Rightarrow$ that there holds: $z * x \leq y$ iff $z \leq x \Rightarrow y$

The residuum $\Rightarrow$ of a continuous t-norm $*$

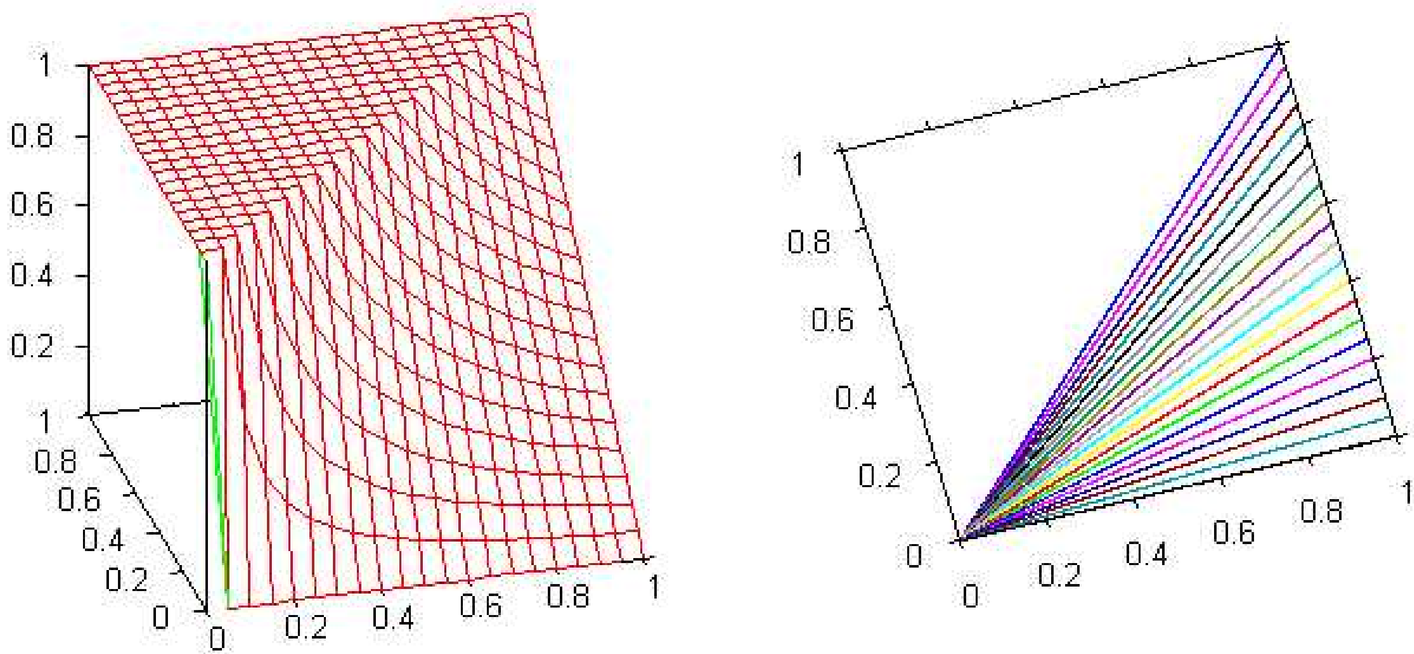
- Can explicitly be defined as $(x \Rightarrow y) = \sup\{z \mid z * x \leq y\}$ (in fact, max)
- $x \Rightarrow -$ is, for each $x \in [0,1]$, the right adjoint to the functor $- * x$ on $[0,1]$ taken as a poset category
- Makes the t-algebra $[0,1]_* = ([0,1], \min, \max, *, \Rightarrow, 1)$ a (bounded integral commutative) residuated lattice
- Is the maximal (ie, weakest) function that makes internalized graded modus ponens $x * (x \Rightarrow y) \leq y$ valid (which makes it a natural truth function for implication)

The residuum of $*_G$: Gödel implication $x \Rightarrow_G y = \begin{cases} y & \text{if } x > y \\ 1 & \text{otherwise} \end{cases}$



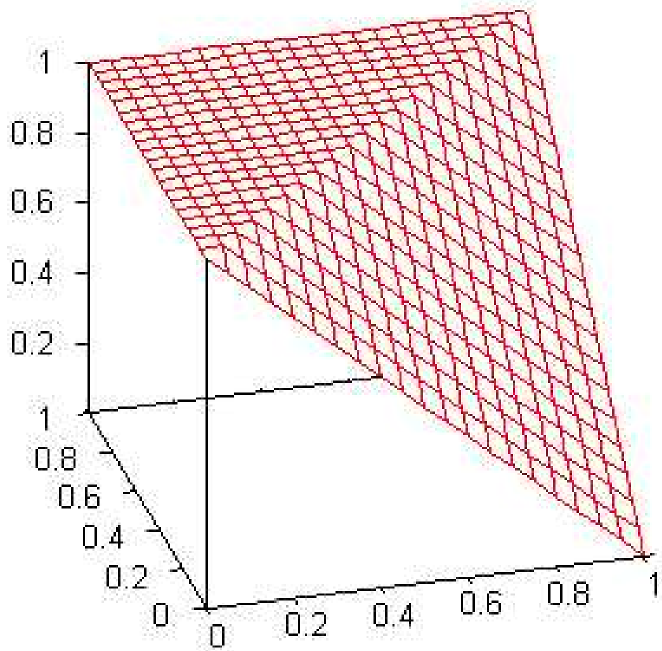


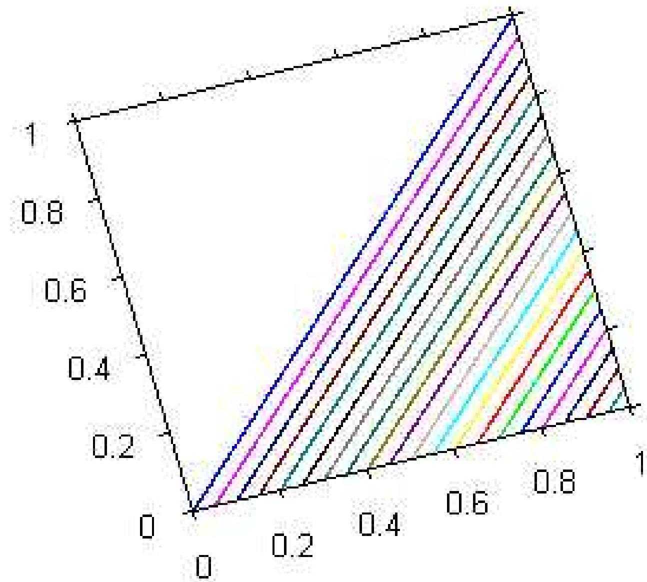
The residuum of $*_\Pi$: Goguen implication $x \Rightarrow_{\mathsf{G}} y = \begin{cases} \frac{y}{x} & \text{if } x > y \\ 1 & \text{if } x \leq y \end{cases}$



The residuum of $*_{Ł}$: Łukasiewicz implication

$$x \Rightarrow_{Ł} y = \min(1, 1 - x + y)$$





Basic properties of the residua of continuous t-norms:

- $(x \Rightarrow y) = 1$ iff $x \leq y$
- $(1 \Rightarrow y) = y$
- $\min(x, y) = x * (x \Rightarrow y)$
- $\max(x, y) = \min((x \Rightarrow y) \Rightarrow y, (y \Rightarrow x) \Rightarrow x)$

$\Rightarrow$ Lattice connectives min and max are definable from $*$ and $\Rightarrow$

Further we define:

- Negation as reductio ad absurdum: $\neg x = (x \Rightarrow 0)$
- Equivalence as bi-implication:

$$(x \Leftrightarrow y) = (x \Rightarrow y) * (y \Rightarrow x) = \min(x \Rightarrow y, y \Rightarrow x)$$

$\Rightarrow$ The choice of a continuous t-norm determines
the truth functions of all usual propositional connectives

Propositional calculi of continuous t-norms

Evaluation in $[0,1]$ . . . a mapping $e\colon \mathbf{Var} \to [0,1]$

Given $*$ and the truth functions of other connectives as above,
e extends inductively to $e_*\colon \mathbf{Form} \to [0,1]$

φ is a $*$-tautology . . . $e_*(\varphi) = 1$ for all evaluations e

Propositional calculus PC($*$) = the set of all $*$-tautologies

Some formulae (eg, $p \to p$) are tautologies of *all* PC($*$)
(call them t-tautologies)

The set of all t-tautologies =
Hájek's basic fuzzy logic BL of all continuous t-norms

Axioms of BL

BL turns out to be axiomatizable by the following axiom schemes:

$$((\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)))$$
$$(\varphi \,\&\, \psi) \to \varphi$$
$$(\varphi \,\&\, (\varphi \to \psi)) \to (\psi \,\&\, (\psi \to \varphi))$$
$$(\varphi \to (\psi \to \chi)) \to ((\varphi \,\&\, \psi) \to \chi)$$
$$((\varphi \,\&\, \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$$
$$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$$
$$0 \to \varphi$$

and the rule of modus ponens $(\varphi, \varphi \to \psi \;/\; \psi)$, with definitions:

$$\begin{aligned} \varphi \wedge \psi &\equiv_{\mathrm{df}} \varphi \,\&\, (\varphi \to \psi) \\ \varphi \vee \psi &\equiv_{\mathrm{df}} ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi) \\ \varphi \leftrightarrow \psi &\equiv_{\mathrm{df}} (\varphi \to \psi) \wedge (\psi \to \varphi) \\ \neg\varphi &\equiv_{\mathrm{df}} \varphi \to 0 \end{aligned}$$

General semantics of BL

The class of all $[0,1]_*$ is the standard semantics of BL

General algebraic semantics is the class of all BL-algebras
= divisible prelinear bounded integral commutative residuated lattices

where (in $\mathsf{FL_{ew}}$-algebras):

divisible . . . $x \wedge y = x * (x \Rightarrow y)$ holds
(then always y "divides" x by $x \Rightarrow y$, for $y < x$)

prelinear . . . $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$ holds
(true, ia, in all linear $\mathsf{FL_{ew}}$-algebras)

Fact: BL-algebras form a variety

Completeness theorems

Theorem: The following conditions are equivalent:

- φ is provable in BL
- φ is an $\mathbf{L}$-tautology for all BL-algebras $\mathbf{L}$
  (general completeness)
- φ is an $\mathbf{L}$-tautology for all linear BL-algebras $\mathbf{L}$
  (linear completeness)
- φ is an $\mathbf{L}$-tautology for all standard BL-algebras $\mathbf{L}$
  (standard completeness)

Linear completeness is proved as usual, but instead of complete theories take linear ones, ie, $T \vdash \varphi \to \psi$ or $T \vdash \psi \to \varphi$ for all pairs φ, ψ. For standard completeness, a partial embedding of a linear counterexample in some t-algebra needs to be constructed.

Local deduction theorem

Theorem: $T, \varphi \vdash \psi$ iff $T \vdash \varphi^n \to \psi$ for some n

where $\varphi^n \equiv_{\text{df}} \underbrace{\varphi \,\&\, \ldots \,\&\, \varphi}_{n\times}$

Hint: The induction step for MP needs the premise twice

Complexity

Theorem: The set of all BL tautologies is coNP-complete

(positive or 1-satisfiability NP-complete)

Łukasiewicz propositional logic Ł = PC($*_{Ł}$)

Axiomatized by BL + $\neg\neg\varphi \to \varphi$

Equivalent to Łukasiewicz original axioms

- $\varphi \to (\psi \to \varphi)$
- $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
- $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
- $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

with $\varphi \,\&\, \psi \equiv_{\text{df}} \neg(\varphi \to \neg\psi)$ and $0 \equiv_{\text{df}} \neg(\varphi \to \varphi)$

(∞-valued; n-valued Ł$_n$ . . . with the excluded $(n+1)$-st)

Algebras for Ł . . . MV-algebras

$= $ BL-algebras with $((x \Rightarrow 0) \Rightarrow 0) = x$

$[0,1]_{*_{Ł}}$ = the standard MV-algebra ($\neg x = 1 - x$ in $[0,1]_{*_{Ł}}$)

Completeness, deduction, and complexity as in BL

Gödel–Dummett logic G $= \mathsf{PC}(*_\mathsf{G})$

Axiomatized by

- BL + $\varphi \to \varphi \mathbin{\&} \varphi$ (contraction), or
- Int + $(\varphi \to \psi) \vee (\psi \to \varphi)$ (Dummett's axiom of prelinearity)

G-algebras = BL-algebras with $* = \wedge$

$\neg x = 1 - \operatorname{sgn} x$ in the standard G-algebra $[0,1]_{*_\mathsf{G}}$

Completeness and complexity as in BL, classical deduction thm

$\Rightarrow$ complete w.r.t. G-chains = linear Heyting algebras

The set of $\mathbf{L}$-tautologies depends only on $\mathsf{Card}(\mathbf{L})$

$\Rightarrow$ G_n are the only extensions, as $\bigcap \mathsf{G}_n = \mathsf{G}$

Further axiomatic extensions of BL

- SBL = BL + $\varphi \wedge \neg\varphi \to 0$

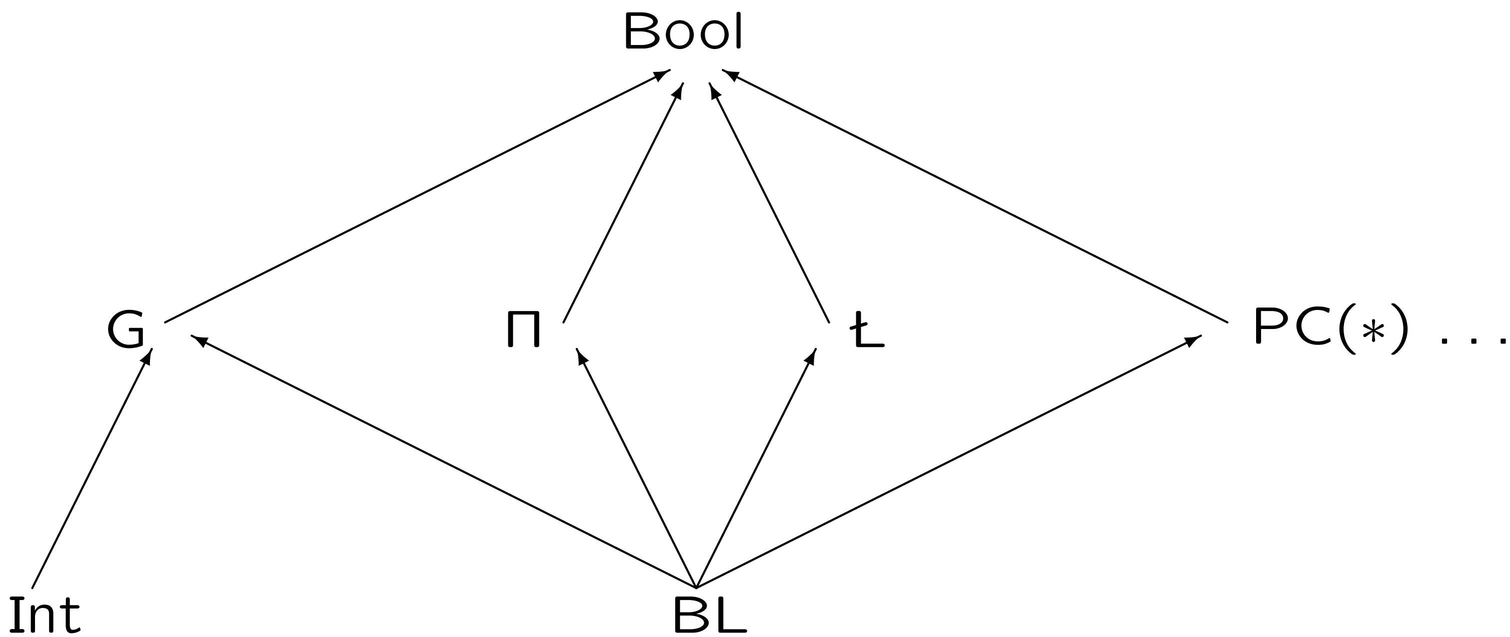
- Product logic Π = PC($*_{\Pi}$)
  $= \text{SBL} + \neg\neg\chi \,\&\, (\varphi \,\&\, \chi \to \psi \,\&\, \chi) \to (\varphi \to \psi)$
  $= \text{BL} + \neg\neg\chi \,\&\, (\chi \to \varphi \,\&\, \chi) \to (\varphi \,\&\, \neg\neg\varphi)$

- PC($*$) are axiomatic extensions of BL for all $*$
  Notation if $*$ is a finite ordinal sum: $Ł \oplus Ł$, $Ł \oplus \Pi \oplus Ł$, ...
  Fact: $\text{BL} = Ł \oplus Ł \oplus Ł \oplus \cdots$

- WCBL = BL + $\neg(\chi \,\&\, \varphi) \vee ((\chi \to \varphi \,\&\, \chi) \to \varphi)$
  (both Ł and Π extend WCBL)

- Classical logic Bool = BL + $\varphi \vee \neg\varphi$ (LEM)

Main t-norm fuzzy logics (as of 1998)



Monoidal t-norm logic MTL

= the most prominent example of post-1998 fuzzy logics

It can be observed that left-continuity of $*$ is sufficient
for the residuum (ie, $\Rightarrow$ such that $z * x \leq y$ iff $z \leq x \Rightarrow y$ holds)
to be defined as $(x \Rightarrow y) = \sup\{z \mid z * x = y\}$

$\Rightarrow$ We can weaken the condition of the continuity of $*$

. . . MTL = the logic of left-continuous t-norms

(turns out to be even more important than BL)

Differences from BL:

- The minimum is no longer definable from $*, \Rightarrow, 0$

  ($\wedge$ has to be added as a primitive connective)

- The BL axiom $(\varphi \,\&\, (\varphi \to \psi)) \to (\psi \,\&\, (\psi \to \varphi))$ fails in MTL

  (it has to be replaced by three weaker axioms
  ensuring the lattice behavior of $\wedge$)

Axioms of MTL (besides MP):

$$((\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)))$$
$$(\varphi \,\&\, \psi) \to \varphi$$
$$(\varphi \wedge \psi) \to \varphi$$
$$(\varphi \wedge \psi) \to (\psi \wedge \varphi)$$
$$(\varphi \,\&\, (\varphi \to \psi)) \to (\varphi \wedge \psi)$$
$$(\varphi \to (\psi \to \chi)) \to ((\varphi \,\&\, \psi) \to \chi)$$
$$((\varphi \,\&\, \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$$
$$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$$
$$0 \to \varphi$$

$\mathsf{MTL} = \mathsf{FL_{ew}} + (\varphi \to \psi) \vee (\psi \to \varphi)$

$\mathsf{BL} \quad = \mathsf{MTL} + (\varphi \wedge \psi) \to (\varphi \,\&\, (\varphi \to \psi))$

Completeness, deduction: like in BL

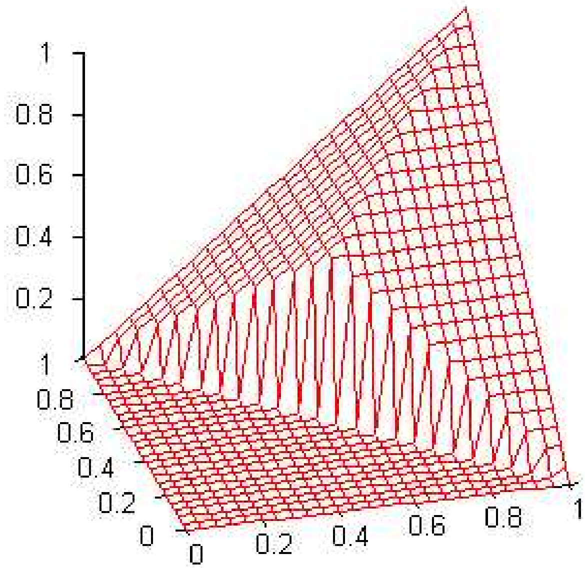
Complexity: an open problem

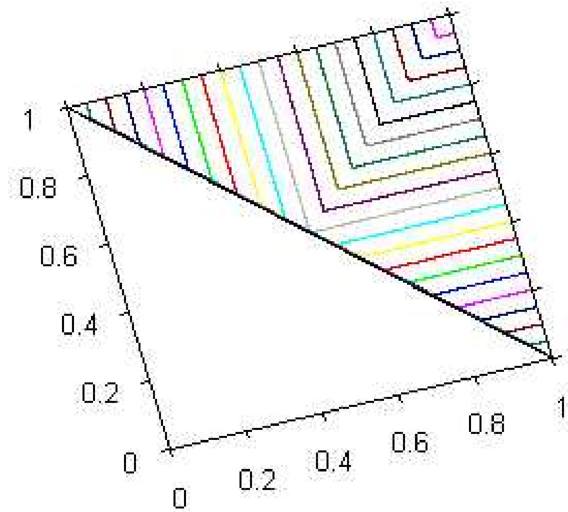
Example: $*_{\mathsf{NM}}$ nilpotent minimum (left-, not right-continuous)

$x *_{\mathsf{NM}} y = \min(x, y)$ if $x + y > 1$, otherwise 0 (Fodor 1995)

Its logic NM = IMTL + $\neg(\varphi \mathbin{\&} \psi) \vee ((\varphi \wedge \psi) \to (\varphi \mathbin{\&} \psi))$

(Wang 1997; Esteva, Godo 2001)





New systems of fuzzy logic

. . . papers since 1998

Drop some requirements, axioms, or connectives:

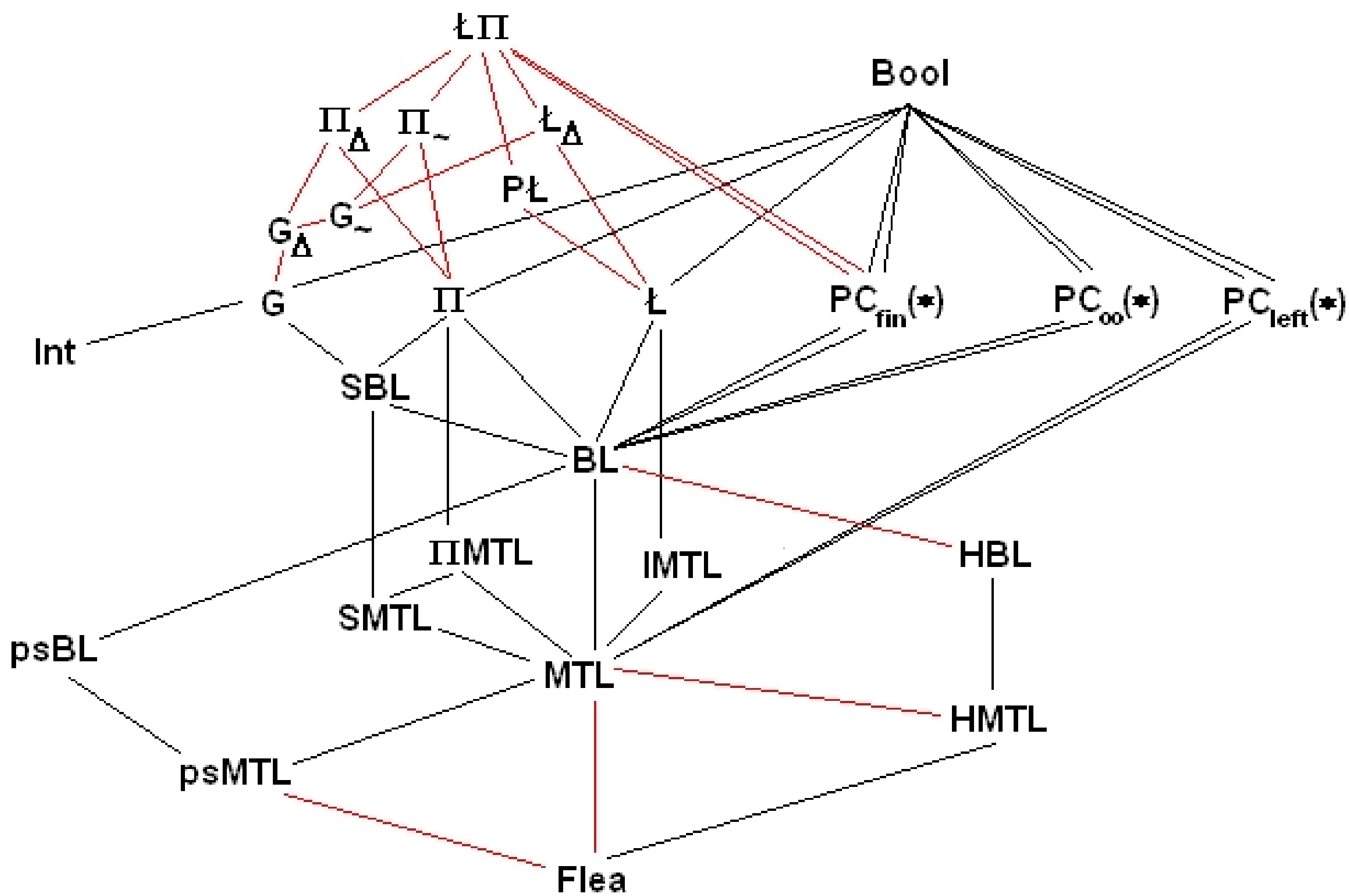
- Only left-continuity needed for residuation = MTL, NM
- Dropping commutativity of & = psBL, psMTL
- Discarding $\bot$ = hoop logics
- Combining the three above deletions = flea logic
- Not requiring the unit of & to equal $\top$ = uninorm logic UL

Add new axioms to these logics:

- MTL + involutiveness/cancellativity = IMTL/ΠMTL
- MTL + strictness of $\neg$ = SMTL

Add new connectives:

- Baaz Δ: BL_Δ, G_Δ, IMTL_Δ, . . . $\Delta x = 1 - \mathrm{sgn}(1 - x)$
- Involutive negation: $\mathrm{SBL}_\sim$, $\mathrm{G}_\sim$, $\Pi_\sim$ $\sim x = 1 - x$
- Combine conjunctions: PŁ, ŁΠ have both $\&_{Ł}$ and $\&_\Pi$
- Truth constants: RPL, $Ł\Pi\frac{1}{2}$, . . . fixed truth values


ŁΠ
Bool
Π∆
Π~
Ł∆
PŁ
G~
G∆
G
Π
Ł
PCfin(*)
PC∞(*)
PCleft(*)
Int
SBL
BL
ΠMTL
IMTL
HBL
SMTL
psBL
MTL
HMTL
psMTL
Flea

Implicative fuzzy logics

(a general theory of fuzzy logics)

A universal approach to fuzzy logics

The multitude of fuzzy logics calls for a general unifying theory
The theory of implicative fuzzy logics is such a generalization

Cintula: *Weakly implicative (fuzzy) logics I*, Arch. Math. Logic 2006

Chagrov (*K voprosu ob obratnoi matematike modal'noi logiki,*
Online Journal Logical Studies, 2001)
distinguishes three stages in the development of a field in logic.
In fuzzy logic, they manifest as follows:

First stage: Emerging of fuzzy logic (since 1965)

- 1965: Zadeh's fuzzy sets, 1968: 'fuzzy logic' (Goguen)
- 1970s: systems of fuzzy 'logic' lacking good metatheory
- 1970s–1980s: first 'real' logics (Pavelka, Takeuti–Titani, . . .), discussion of many-valued logics in the fuzzy context

Second stage: Particular fuzzy logics (since the 1990s)

- Hájek's monograph (1998): BL, (G, Ł,) Π
- New logics: MTL, HBL, SBL, $\Pi_{\sim}$, ŁΠ, ...
- Algebraic semantics, hyper-sequent proof theory,
  Kripke-style and game-theoretic semantics, ...
- First-order, higher-order, and modal fuzzy logics

Systematic treatment of particular fuzzy logics

The theory of weakly implicative fuzzy logics marked
the beginning of the third stage:

Third stage: Universal fuzzy logic (since ∼2004)

- General methods to prove metamathematical properties
- Systematization of existing fuzzy logics
- The position of fuzzy logics in the logical landscape

Systematic treatment of classes of fuzzy logics

Definitions and design choices for the general framework

$\mathcal{L} = \langle \mathbf{Var}, \mathbf{Conn} \rangle$

Propositional language = variables + connectives (with arity)

$\mathbf{Form}(\mathcal{L})$ = the smallest set containing $\mathbf{Var}$ closed under $\mathbf{Conn}$

Logic . . . $\mathsf{L} \subseteq \mathcal{P}(\mathbf{Form}(\mathcal{L})) \times \mathbf{Form}(\mathcal{L})$,
a substitution-closed Tarski consequence relation
(write $X \vdash_{\mathsf{L}} \varphi$ for $\langle X, \varphi \rangle \in \mathsf{L}$)

(If a logic is defined by a Hilbert-style calculus,
we understand it as its deducibility relation)

L is finitary . . . if $X \vdash_{\mathsf{L}} \varphi$ then $\exists$ finite $X' \subseteq X$ s.t. $X' \vdash_{\mathsf{L}} \varphi$

Matrix semantics

$\mathcal{L}$-matrix $\mathbf{M} = \langle A, D \rangle$, where

A is an algebra with the signature of $\mathcal{L}$

D is its subset of designated values

$\mathbf{M}$-evaluation . . . any morphism $e\colon \mathbf{Form}(\mathcal{L}) \to A$

φ is valid under an $\mathbf{M}$-evaluation e iff $e(\varphi) \in D$

$X \models_{\mathcal{K}} \varphi$ iff any e validating all $\psi{\in}X$ validates also φ, in any $\mathbf{M}{\in}\mathcal{K}$
(semantic consequence w.r.t. a class $\mathcal{K}$ of matrices)

$\mathbf{M}$ is an L-matrix iff $X \vdash_{\mathsf{L}} \varphi$ implies $X \models_{\{\mathbf{M}\}} \varphi$

Weakly implicative logics

Requirements on L:

- $\vdash_{\mathsf{L}} \varphi \to \varphi$
- $\varphi \to \psi, \psi \to \chi \vdash_{\mathsf{L}} \varphi \to \chi$
- $\varphi, \varphi \to \psi \vdash_{\mathsf{L}} \psi$
- $\varphi \to \psi, \psi \to \varphi \vdash_{\mathsf{L}} c(\chi_1, \dots, \varphi, \dots, \chi_n) \to c(\chi_1, \dots, \psi, \dots, \chi_n)$
  for all connectives c in the language

Generalization of Rasiowa's *implicative logics*

Rasiowa: An Algebraic Approach to Non-Classical Logics, 1974

Implicative logics = Weakly implicative logics + Weakening

Weakly implicative logics ⊆ Finitely equivalential logics

$E(x, y) = \{x \to y, y \to x\}$

Examples:

- Intermediary logics (incl. Int, Bool)
- Classical modal logics (ie, with $\varphi \leftrightarrow \psi$ / $\Box\varphi \leftrightarrow \Box\psi$,
  incl. normal modal logics K, T, S4, S5, . . .)
- Substructural logics (FL_X, CFL_X, BCK, BCI, . . .)
- Usual fuzzy logics (MTL, BL, Ł, G, Π, . . .)

Non-examples:

- Non-monotonic logics (default, adaptive, . . .),
- Logics with non-standard syntax
  (labeled deduction, evaluated syntax, . . .)
- First- and higher-order logics (propositional logics only)
- Kleene three-valued logic (as $\nvdash_{K_3} \varphi \to \varphi$)

Matrices for weakly implicative logics

Can be pre-ordered:

$$x \leq_{\mathbf{M}} y \quad \equiv_{\mathsf{df}} \quad x \to_{\mathbf{M}} y \in D$$

(The requirements on weakly implicative logics express that $\leq_{\mathbf{M}}$ is reflexive, transitive, congruent, and D is upper.)

(Strong) completeness wrt the class of all L-matrices:

$$X \vdash_{\mathsf{L}} \varphi \quad \text{iff} \quad X \models_{\mathbf{M}} \varphi \text{ holds for each L-matrix } \mathbf{M}$$

Weakly implicative logics = logics of (pre-)ordered matrices

Weakly implicative fuzzy logics

A weakly implicative logic L will be called fuzzy iff
L is complete w.r.t. the class of all linearly ordered L-matrices

Fuzzy logics = logics of linearly ordered matrices

Examples:

- MTL, BL, and their schematic extensions (G, Ł, Π, . . .)
- Hoop logic and its extensions
- Expressively rich fuzzy logics (ŁΠ, PŁ, . . .)
- Intermediary logics extending G
- Relevance logics extending RM

Non-examples:

- BCK, Int, intermediary logics not extending G
- Usual modal logics (K, S4, S5, . . .)
- (C)FL_X (except for Bool), linear logic

Why linearly ordered matrices

1. Fuzzy logic investigates the comparative notion of truth

   If the truth-values are to be understood as *degrees* of truth, they should be comparable

2. The methods commonly used in metamathematics of fuzzy logics (which actually distinguish them among other logics)

   - Subdirect representation
   - Proof by cases
   - Construction of linear theories

   work exactly over *linear matrices*

3. The class approximates the interests of the fuzzy community

Equivalent conditions for L weakly implicative:

- L is complete w.r.t. linearly ordered L-matrices

  (linear semantics)

- For each T, φ s.t. $T \nvdash_{\mathsf{L}} \varphi$ there is a linear theory $T' \supseteq T$ s.t. $T' \nvdash_{\mathsf{L}} \varphi$ (where T' is linear iff $T' \vdash_{\mathsf{L}} \varphi \to \psi$ or $T' \vdash_{\mathsf{L}} \psi \to \varphi$)

  (linear extension property)

Equivalent conditions for L weakly implicative finitary:

- L is complete w.r.t. linearly ordered L-matrices

  (linear semantics)

- For each T, φ s.t. $T \nvdash_{\mathsf{L}} \varphi$ there is a linear theory $T' \supseteq T$ s.t. $T' \nvdash_{\mathsf{L}} \varphi$ (where T' is linear iff $T' \vdash_{\mathsf{L}} \varphi \to \psi$ or $T' \vdash_{\mathsf{L}} \psi \to \varphi$)

  (linear extension property)

- The following meta-rule is valid:

$$\frac{X, \varphi \to \psi \vdash_{\mathsf{L}} \chi \qquad X, \psi \to \varphi \vdash_{\mathsf{L}} \chi}{X \vdash_{\mathsf{L}} \chi}$$

  (prelinearity property)

- Each L-matrix is a subdirect product of linear ones

  (linear subdirect decomposition property)

Direct and subdirect products

The direct product $\prod \mathcal{I}$ of a set of L-matrices $\mathcal{I}$ is the matrix whose:

- Domain is the Cartesian product of the domains of all $\mathbf{M} \in \mathcal{I}$
- Set of designated values is the Cartesian product of the sets of designated values of all $\mathbf{M} \in \mathcal{I}$
- Operations are defined pointwise

$\mathbf{M}$ is a subdirect product of $\mathcal{I}$ if there is an embedding $f\colon \mathbf{M} \to \prod \mathcal{I}$ such that $\pi_A(f(X)) = A$ for each $(A, D) \in \mathcal{I}$

(a submatrix of the direct product with all projections total)

Matrices for fuzzy logics are subdirect products of linear ones

(ie, only chains are subdirectly irreducible)

$=$ "measuring truth along linear scales"

Some properties of weakly implicative fuzzy logics

- All fuzzy logics are distributive (ie, & over $\wedge$, $\vee$)
- The intersection of an arbitrary system of fuzzy logics is fuzzy
- Any axiomatic extension of a fuzzy logic is fuzzy
  $\Rightarrow$ every logic L has the weakest fuzzy extension $\mathcal{F}(\mathrm{L})$
- MTL is the weakest fuzzy logic extending $\mathrm{FL_{ew}}$
  (ie, $\mathrm{MTL} = \mathcal{F}(\mathrm{FL_{ew}})$)
- Similarly $\mathrm{G} = \mathcal{F}(\mathrm{Int})$, $\mathrm{IMTL} = \mathcal{F}(\mathrm{AMALL})$, $\mathrm{UL} = \mathcal{F}(\mathrm{FL_e})$
  In all of these cases, $\mathcal{F}(\mathrm{L}) = \mathrm{L} + (\varphi \to \psi) \vee (\psi \to \varphi)$

Moral: To make your logic L fuzzy, add prelinearity
(often equivalent to adding the axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$)
The result is often the weakest fuzzy logic extending L

Deductive fuzzy logics

(fuzzy substructural logics)

Partial transmission of partial truth

- The Hilbert-style rules of fuzzy logics are intended to transmit the full truth of fuzzy propositions
  (from fully true premises to fully true conclusions)
- The conditions of weakly implicative fuzzy logic furthermore ensure transmission of partial truth by $\to$
  (the consequent at least as true as the antecedent)
- But only fully true transmission of partial truth
  Requirements only on $\to$ as the principal connective
  Recall: $e(\varphi \to \psi) = 1$ iff $\varphi \leq \psi$
  This is not ensured in weakly implicative fuzzy logics:
  $e(\varphi \to \psi)$ close to 1 only if $e(\psi)$ not *much* less than $e(\varphi)$
  (will be precisified)
- To ensure also partially true transmission of partial truth, the logics need to satisfy additional requirements
  $\Rightarrow$ the class of deductive fuzzy logics

Deductive fuzzy logics

Partial truth transmission internalized by $\to$
Implications can be nested

$\Rightarrow$ we need rules for partially true implications

Deductive fuzzy logics = Weakly implicative fuzzy logics +

- $\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$ (antecedent antitony)
- $\vdash (\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$ (consequent monotony)
- $\varphi \to (\psi \to \chi) \dashv\vdash (\varphi \,\&\, \psi) \to \chi$ (residuation)

Deductive fuzzy logics
= those weakly implicative fuzzy logics which internalize

- Local consequence by $\to$ and
- Comma between premises by &

$\varphi_1, \ldots, \varphi_k \approx\!\!\!\!|\; \psi$ iff $\models \varphi_1 \,\&\, \ldots \,\&\, \varphi_k \to \psi$

Local and global consequence relation

A distinction analogous to that in modal or first-order logics

Global consequence relation = full truth preservation

$\varphi \models \psi$

iff $(\forall e)(e(\varphi) = 1 \Longrightarrow e(\psi) = 1)$

iff $(\forall e)((\forall \alpha)(e(\varphi) \geq \alpha) \Longrightarrow (\forall \alpha)(e(\psi) \geq \alpha))$

iff $(\forall e)((\forall \alpha)(\alpha \Vdash_e \varphi) \Longrightarrow (\forall \alpha)(\alpha \Vdash_e \psi))$

Local consequence relation = partial truth preservation

$\varphi \models\!\!\approx \psi$

iff $(\forall e)(e(\varphi) \geq e(\psi))$

iff $(\forall e)(\forall \alpha)(e(\varphi) \geq \alpha \Longrightarrow e(\psi) \geq \alpha)$

iff $(\forall e)(\forall \alpha)(\alpha \Vdash_e \varphi \Longrightarrow \alpha \Vdash_e \psi)$

Observe:

- Hilbert-style rules capture $\models$
- But it is $\mid\!\approx$ that is substructural

  $\varphi, \varphi \vdash \psi$ iff $\varphi \vdash \psi$

  yet generally not: $\varphi, \varphi \mid\!\approx \psi$ (ie, $\vdash \varphi \,\&\, \varphi \to \psi$) iff $\vdash \varphi \to \psi$

$\mid\!\approx$ is more important (even though often neglected)

as it allows inference even with imperfectly true premises

Fine-tuning the internalization

Additional axioms can internalize further properties of $\models\!\!\approx$

Usually adopted:

- The rule of exchange: $\varphi \to (\psi \to \chi) \vdash \psi \to (\varphi \to \chi)$
  or equivalently $\vdash \varphi \to ((\varphi \to \psi) \to \psi)$
  (internalizes the commutativity of the comma)
- The rule of weakening: $\varphi \vdash \psi \to \varphi$
  (excludes multiple degrees of full truth)
- The rule of contraction? $\varphi \to (\varphi \to \psi) \vdash \varphi \to \psi$
  BUT: multiple "imperfect" premises increase imperfection
  $\Rightarrow$ contraction better omitted

Adding connectives to deductive fuzzy logics

Lattice connectives:

Contraction-free logics have two meaningful conjunctions

(as known from substructural logics)

- Strong conjunction = cumulation of premises = our &
- Weak conjunction = a choice of one premise = $\wedge$

Requirements that express the lattice behavior of $\wedge$:

$$\begin{aligned} &\vdash \varphi \wedge \psi \to \varphi \\ &\vdash \varphi \wedge \psi \to \psi \\ \chi \to \varphi, \chi \to \psi &\vdash \chi \to \varphi \wedge \psi \end{aligned}$$

Dually for weak disjunction (change $\to$ to $\leftarrow$ and $\wedge$ to $\vee$)

Strong disjunction usually not definable (due to *asymmetric* $\vdash$)

unless involutive negation present

Baaz Δ = internalization of full truth

$\Rightarrow$ internalization of $\vdash$ by the Δ-deduction theorem:

$\varphi \vdash \psi$ iff $\vdash \Delta\varphi \to \Delta\psi$

Thm: In weakly implicative fuzzy logics, Δ is axiomatized by

- the modal rules of S4 (K, T, 4, Nec) plus
- the 3 structural rules (W, E, C) for Δ-prefixed formulae

L_Δ = L + the above axioms for Δ

Thm: If L is weakly implicative fuzzy,

then L_Δ is weakly implicative fuzzy

$\bot$ = the least truth (“ex $\bot$ quodlibet”): $\bot \to \varphi$

0 = a chosen element bounding falsity (below 0 = “absurdum”)

negation = reductio ad absurdum: $\neg\varphi$ is $\varphi \to 0$

Optional requirements on the derived connectives fine-tune the resulting logic (involutiveness of $\neg$, . . .)

Characterization of deductive fuzzy logics

Residuation makes them part of Ono's substructural logics
(logics of residuated lattices)

Deductive fuzzy logics
= Cintula's weakly implicative fuzzy logics
with residuation and $\rightarrow$-monotony conditions
= Ono's substructural logics (of residuated lattices)
$\cap$ Cintula's weakly implicative fuzzy logics
(of linearly ordered matrices)

Deductive fuzzy logics with exchange
= congruent expansions of Metcalfe's uninorm logic UL
(with weakening: of MTL)

The class contains all of the most important fuzzy logics
(MTL, BL, Ł, G, Π, NM, ŁΠ, . . .)

Logic-based justification of fuzzy logic

Observe: The axioms of fuzzy logic can be justified without a reference to real numbers:

1. (Full) preservation of full truth
   $\Rightarrow$ Substitution-invariant Tarski consequence relation
2. (Full) preservation of partial truth
   $\Rightarrow$ Weakly implicative fuzzy logics
3. Partial preservation of partial truth
   $\Rightarrow$ Deductive fuzzy logics

Standard completeness theorems = if we are measuring truth, we can as well measure it by real numbers

First-order fuzzy logics

Language of first-order fuzzy logics

For simplicity, we shall only work with languages without sorts of variables (but they can easily be added)

Predicate language $\mathcal{L} = (\mathbf{P}, \mathbf{F}, \mathbf{A})$, where

$\mathbf{P}$ = a non-empty set of predicate symbols

$\mathbf{F}$ = a set of function symbols

$\mathbf{A}$ = the arity function $\mathbf{P} \cup \mathbf{F} \to \omega$

(functions of arity 0 = individual constants)

Logical symbols:

Individual variables $x, y, z, \ldots$ (denote the set by $\mathbf{Var}$)

Connectives of a given propositional logic L

Quantifiers $\forall, \exists$

Terms and formulae, as well as free variables and substitutability, are defined as usual

Axioms of first-order fuzzy logics

Let L be a propositional substructural logic in Ono's sense
(for simplicity, we assume exchange)

First-order logic $\mathrm{L}\forall^-$ = L + Rasiowa's axioms for quantifiers:

$(\forall 1)$	$(\forall x)\varphi(x) \to \varphi(t)$	if t is free for x in φ
$(\exists 1)$	$\varphi(t) \to (\exists x)\varphi(x)$	if t is free for x in φ
$(\forall 2)$	$(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$	if x is not free in χ
$(\exists 2)$	$(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$	if x is not free in χ

and the rule of generalization: from φ infer $(\forall x)\varphi$

If L is fuzzy and has lattice disjunction $\vee$, then the first-order *fuzzy* logic $\mathrm{L}\forall$ is defined as $\mathrm{L}\forall^-$ plus the following axiom:

$(\forall 3)$ $\quad (\forall x)(\varphi \vee \chi) \to (\forall x)\varphi \vee \chi \quad$ if x is not free in χ
(needed for completeness w.r.t. chains)

Semantics of first-order fuzzy logics

Let $\mathcal{L} = (\mathbf{P}, \mathbf{F}, \mathbf{A})$ be a predicate language and $\mathbf{L}$ an L-algebra.

An $\mathbf{L}$-structure for $\mathcal{L}$ is $\mathbf{M} = (M, \{P_\mathbf{M}\}_{P \in \mathbf{P}}, \{F_\mathbf{M}\}_{F \in \mathbf{F}})$, where

- M is a non-empty set (the universe of discourse)
- $P_\mathbf{M} \colon M^{\mathbf{A}(P)} \to \mathbf{L}$ (fuzzy relation of appropriate arity)
- $F_\mathbf{M} \colon M^{\mathbf{A}(P)} \to M$ (individual function of appropriate arity)

An $\mathbf{M}$-evaluation is a mapping that assigns each variable an element of M

Denote $v[x \mapsto a](y) = \begin{cases} a & \text{if } y = x \\ v(y) & \text{otherwise} \end{cases}$

The values of terms and the truth values of formulae for an $\mathbf{M}$-evaluation v are defined inductively as:

$$\begin{array}{rcl}
\|x\|^{\mathbf{L}}_{\mathbf{M},v} & = & v(x) \\
\|F(t_1,\dots,t_n)\|^{\mathbf{L}}_{\mathbf{M},v} & = & F_{\mathbf{M}}(\|t_1\|^{\mathbf{L}}_{\mathbf{M},v},\dots,\|t_n\|^{\mathbf{L}}_{\mathbf{M},v}) \\
\|P(t_1,\dots,t_n)\|^{\mathbf{L}}_{\mathbf{M},v} & = & P_{\mathbf{M}}(\|t_1\|^{\mathbf{L}}_{\mathbf{M},v},\dots,\|t_n\|^{\mathbf{L}}_{\mathbf{M},v}) \\
\|c(\varphi_1,\dots,\varphi_n)\|^{\mathbf{L}}_{\mathbf{M},v} & = & c_{\mathbf{L}}(\|\varphi_1\|^{\mathbf{L}}_{\mathbf{M},v},\dots,\|\varphi_n\|^{\mathbf{L}}_{\mathbf{M},v}) \\
\|(\forall x)\varphi\|^{\mathbf{L}}_{\mathbf{M},v} & = & \inf_{a\in M}\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v[x\to a]} \\
\|(\exists x)\varphi\|^{\mathbf{L}}_{\mathbf{M},v} & = & \sup_{a\in M}\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v[x\to a]}
\end{array}$$

for each $F \in \mathbf{F}$, $P \in \mathbf{P}$, and each connective c

(generalized Tarski conditions)

Subtlety: The inf and sup need not exist in $\mathbf{L}$. Options:

- Use only complete lattices $\mathbf{L}$
  (but then $\mathbf{L}$-structures often not axiomatizable: BL, Ł, ...)
- Use Rasiowa's *interpretations* = Hájek's *safe structures*

The $\mathbf{L}$-structure $\mathbf{M}$ is safe . . . $\|(\exists x)\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ is defined for all φ, v
(ie, all needed suprema and infima exist)

φ is valid in an $\mathbf{L}$-structure $\mathbf{M}$ ($\mathbf{M} \models \varphi$)
. . . $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = 1_{\mathbf{L}}$ for all $\mathbf{M}$-evaluations v

φ is an $\mathbf{L}$-tautology ($\models \varphi$)
. . . $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = 1_{\mathbf{L}}$ for each valuation v in a *safe* $\mathbf{L}$-structure $\mathbf{M}$

An $\mathbf{L}$-structure $\mathbf{M}$ is a model of a theory T ($\mathbf{M} \models T$)
. . . $\mathbf{M} \models \varphi$ for all $\varphi \in T$

The completeness theorem

Theorem (general completeness): $T \vdash_{\text{L}\forall^-} \varphi$ iff
for each L-algebra $\mathbf{L}$ and each $\mathbf{L}$-model M of T, $M \models \varphi$

Theorem (linear completeness): $T \vdash_{\text{L}\forall} \varphi$ iff
for each L-chain $\mathbf{L}$ and each $\mathbf{L}$-model M of T, $M \models \varphi$

Standard completeness:

- Only a few first-order fuzzy logics (G∀, MTL∀) do have it
  $\Rightarrow$ Their sets of standard tautologies are Σ_1
- Standard tautologies of others are not axiomatizable
  Arithmetical complexity of standard Ł∀ . . . Π_2-complete
  Standard Π∀ is not arithmetical

Formal fuzzy mathematics

Formal fuzzy mathematics

First-order fuzzy logic is strong enough to support non-trivial formal mathematical theories

Mathematical concepts in such theories show gradual rather than bivalent structure

Examples:

- Skolem, White (1960, 1979): naive set theory over Ł
- Takeuti–Titani (1994): ZF-style fuzzy set theory in a system close to Gödel logic (⇒ contractive)
- Hájek–Haniková (2003): ZF-style set theory over $BL_\triangle$
- Novák (2004): Church-style fuzzy type theory over $IMTL_\triangle$
- Běhounek–Cintula (2005): higher-order fuzzy logic

Theory of identity

Logic: Any first-order fuzzy logic

Axioms:

- $x = x$ (reflexivity—all things are identical to themselves)
- $x = y \mathbin{\&} \varphi(x) \to \varphi(y)$
  (Leibniz identity law—indiscernibility of identicals)

In sufficiently strong logics (eg, with Δ), $=$ comes out crisp:

$$\vdash x = y \vee \neg(x = y)$$

Models can then be factorized so that $=_{\mathbf{M}}$ is realized as the identity of individuals

Hájek–Haniková fuzzy set theory

Logic: First-order BL_Δ with identity
Language: $\in$
Axioms:

- $\Delta(\forall u)(u \in x \leftrightarrow u \in y) \to x = y$ (extensionality)
- $(\exists z)\,\Delta(\forall y)\neg(y \in z)$ (empty set $\emptyset$)
- $(\exists z)\,\Delta(\forall u)(u \in z \leftrightarrow (u = x \vee u = y))$ (pair $\{x, y\}$)
- $(\exists z)\,\Delta(\forall u)(u \in z \leftrightarrow (\exists y)(u \in y \,\&\, y \in x))$ (union $\bigcup$)
- $(\exists z)\,\Delta(\forall u)(u \in z \leftrightarrow \Delta(\forall u \in x)(u \in y))$ (weak power)
- $(\exists z)\,\Delta(\emptyset \in z \,\&\, (\forall x \in z)(x \cup \{x\} \in z))$ (infinity)
- $(\exists z)\,\Delta(\forall u)(u \in z \leftrightarrow (u \in x \,\&\, \varphi(u, x))$, z not free in φ (separation)
- $(\exists z)\,\Delta[(\forall u \in x)(\exists v)\varphi(u, v) \to (\forall u \in x)(\exists v \in z)\varphi(u, v)]$, z not free in φ (collection)
- $\Delta(\forall x)((\forall y \in x)\varphi(y) \to \varphi(x)) \to \Delta(\forall x)\varphi(x)$ ($\in$-induction)
- $(\exists z)\,\Delta((\forall u)(u \in z \vee \neg(u \in z)) \,\&\, (\forall u \in x)(u \in y))$ (support)

Semantics:

A cumulative hierarchy of BL-valued fuzzy sets

Features:

- Contains an inner model of classical ZF:
  (as the subuniverse of hereditarily crisp sets)
- Conservatively extends classical ZF with fuzzy sets
- Generalizes Takeuti–Titani's construction
  in a non-contractive fuzzy logic

Fuzzy class theory = (Henkin-style) higher-order fuzzy logic

Logic: Any first-order deductive fuzzy logic with Δ and $=$

Originally: ŁΠ for its expressive power

Language:

- Sorts of variables for atoms, classes, classes of classes, etc.
- Subsorts for k-tuples of objects at each level
- $\in$ between successive sorts
- At all levels: $\{x \mid \ldots\}$ for classes, $\langle \ldots \rangle$ for tuples

Axioms (for all sorts):

- $\langle x_1, \ldots, x_k \rangle = \langle y_1, \ldots, y_k \rangle \to x_1 = y_1 \mathbin{\&} \ldots \mathbin{\&} x_k = y_k$ (tuple identity)
- $(\forall x)\, \Delta(x \in A \to x \in B) \to A = B$ (extensionality)
- $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$ (class comprehension)

Semantics:

Zadeh-style fuzzy sets of all orders over a crisp ground set
(Henkin-style $\Rightarrow$ non-standard models exist,
full higher-order fuzzy logic non-axiomatizable)

Features:

- Suitable for the reconstruction and graded generalization of large parts of traditional fuzzy mathematics
- Several mathematical disciplines have been developed within its framework, using it as a foundational theory, eg:
  Fuzzy relations (Běhounek, Bodenhofer, Cintula, Daňková)
  Fuzzy numbers (Běhounek, Horčík)
  Fuzzy topology (Běhounek, Kroupa)
  Fuzzy quantifiers (Cintula, Horčík)
- The results obtained trivialize initial parts of traditional fuzzy set theory

Cantor-Łukasiewicz set theory

Logic: First-order Łukasiewicz logic Ł∀

Language: $\in$, set comprehension terms $\{x \mid \varphi\}$

Axioms:

- $y \in \{x \mid \varphi\} \leftrightarrow \varphi(y)$ (unrestricted comprehension)

Features:

- Non-contractivity of Ł blocks Russell's paradox
- Consistency conjectured by Skolem (1960), proved by White (1979)
- Adding extensionality is contradictory
- Open problem: define a reasonable arithmetic in CŁ (some negative results by Hájek, 2005)

Some logic-oriented applications

(epistemic, deontic, . . .)

The sorites paradox—fuzzy logic solution

Take a non-contractive fuzzy logic (eg, standard Łukasiewicz)

(1) The premise Px_0 is fully true: $\|Px_0\| = 1$
(2) The inductive premise is *almost* true $\Rightarrow$ looks quite plausible

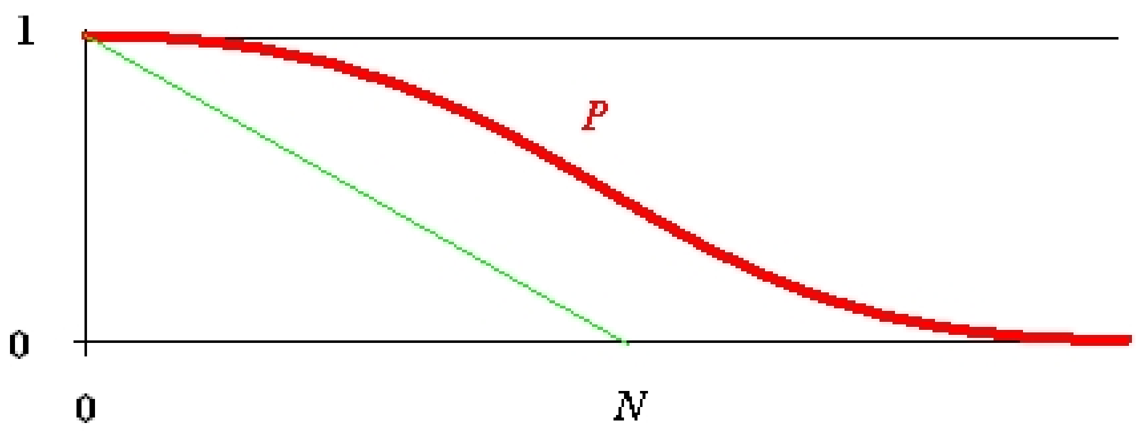
$$\left\|(\forall n)(Px_n \to Px_{n+1})\right\| = 1 - \varepsilon$$

To infer Px_i one has to use it i times
$(\forall x)\varphi(x) \to \varphi(a)$, but not $(\forall x)(\varphi(x) \to \varphi(a) \,\&\, \varphi(b))$
$\Rightarrow$ we have to specify the inductive premise for all $n = 1, \ldots, i$ to get the chain of implications

$\Rightarrow$ only $\|Px_i\| \geq 1 - i\varepsilon$ is guaranteed in $[0,1]_{*_Ł}$,

For $n \geq \frac{1}{\varepsilon}$, $\|Px_n\|$ can be 0
(eg, $\|Px_n\| = 1 - n\varepsilon$ models (1)+(2) in $[0,1]_{*_Ł}$)

$[0,1]_{*_Ł}$-model of the sorites



Higher-order vagueness objection: $\exists$ first n s.t. $\|Px_n\| < 1$

Solution: values close to 1 behave like 1 $\Rightarrow$ represent *True*

Define *True* as a fuzzy set of truth values close to 1

(can be done in higher-order fuzzy logic)

Then the series *gradually* diverges from *True*

Cost-awareness

Non-contractive fuzzy logic can be used for modeling costs of premises (similarly as linear logic models resources)

Prelinearity ... comparability of prices:
Either φ is cheaper than ψ or vice versa

Subdirect products ... several components of price,
each of them scalar

1 = for free
& combines prices ($*_Ł$ sums them, $*_\Pi$ sums logarithms, ...)
$\rightarrow$ transmits affordability and measures the difference in prices

$[0,1]_{*_Ł}$ ... limited resources: $(\forall x)(\exists n)(x^n = 0)$
$[0,1]_{*_\Pi}$ ... unlimited resources: $(\forall x)(\forall n)(x^n > 0)$

Logical omniscience paradox

Standard epistemic logic validates K: $K\varphi \,\&\, K(\varphi \to \psi) \to K\psi$

(the agent can make logical inference)

which leads to $\vdash K\chi$ for all propositional tautologies χ

Fuzzy logic solution:

Distinguish three kinds of knowledge:

- Implicit knowledge: omniscience is unproblematic
- Explicit knowledge: the axiom K is not valid
- Feasible knowledge: omniscience undesirable

Feasible knowledge is a *fuzzy modality:*

Agent can make inference, but at some costs

$\Rightarrow$ The axiom K is true only to a degree close to 1

The truth value of $K\varphi$ decreases by each step of inference

Logical omniscience is an instance of the Sorites!

Deontic paradoxes

Some of the paradoxes of deontic logic are caused
by the bivalence of the classical logic

Classically, all obligations have the same weight
$\Rightarrow$ counter-intuitive consequences

Fuzzy logic can measure the strength of obligations:

Truth value is proportional, eg, to the punishment threat

Conjunction combines the punishments

Subdirect products ... several components of the punishment

Prelinearity = comparability of punishments in all components
(eg, fines, years in prison, ...)

$[0,1]_{*_{Ł}}$ ... limited punishment scale

$[0,1]_{*_{\Pi}}$ ... unlimited punishment scale

In modeling vagueness, fuzzy truth values can also be
interpreted as 'punishments for inaccurateness'

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