IT4I

Formal Calculi of Fuzzy Relations

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Outline

Outline

Fuzzy sets

Recall:

Fuzzy set = a set with weighted membership, $A\colon V \to \mathbf{L}$

Conventions:

- $\mathbf{L}$ usually a (residuated) lattice, often $[0,1] \subseteq \mathbb{R}$
- Crisp subsets of V identified with fuzzy sets $C\colon V \to \{0,1\} \subseteq \mathbf{L}$

Fuzzy sets

Notation:

- Membership degrees: $A(x)$ or just Ax
- Write $A = \{x \parallel f(x)\}$ if $Ax = f(x)$ for all $x \in V$

Examples:

$$A \cap B = \{x \parallel Ax \wedge Bx\} \qquad (A \cap B)x \equiv \min(Ax, Bx)$$

$$V = \{x \parallel 1\} \qquad \text{analogously } \emptyset = \{x \parallel 0\}$$

$$\ker A = \{x \parallel Ax = 1\} \qquad \text{analogously } \operatorname{supp} A$$

Fuzzy relations

Recall:

- (Binary) fuzzy relation = a fuzzy set of pairs, $R\colon V^2 \to \mathbf{L}$
- n-Ary fuzzy relation = a fuzzy set of n-tuples, $S\colon V^n \to \mathbf{L}$
- Fuzzy relation from X to Y (for $X, Y \subseteq V$) . . . $T\colon X \times Y \to \mathbf{L}$
  (treat as $V^2 \to \mathbf{L}$: add 0's elsewhere, ignore $V^2 - (X \times Y)$)

Fuzzy relations

Notation:

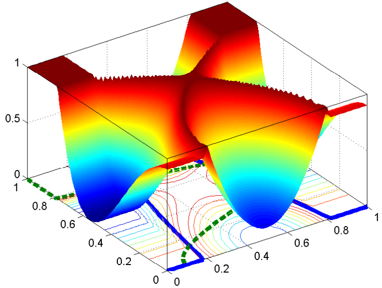
- Membership degrees: Rxy, $Sxyz$, ...
- Write $R = \{xy \parallel f(x,y)\}$ if $Rxy = f(x,y)$ for all $x, y \in V$

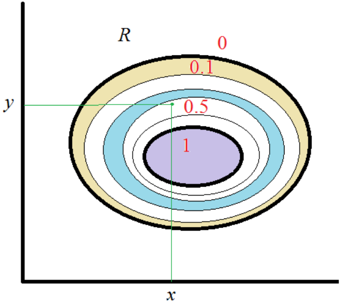
Examples:

$$\operatorname{supp} R = \{xy \parallel Rxy > 0\}$$
$$\mathsf{Id} = \{xy \parallel x = y\}$$

Representation of fuzzy relations

- Graphs (3D or contours = a system of horizontal cuts)





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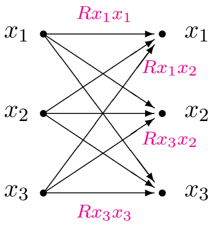
Representation of fuzzy relations

- Matrices (esp. for finite fuzzy relations):

$$R = \begin{pmatrix} Rx_1x_1 & Rx_1x_2 & \cdots & Rx_1x_n \\ Rx_2x_1 & Rx_2x_2 & \cdots & Rx_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ Rx_nx_1 & Rx_nx_2 & \cdots & Rx_nx_n \end{pmatrix}, \quad \text{e.g.,} \quad R = \begin{pmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.9 & 0.3 \\ 0.1 & 0 & 1 \end{pmatrix}$$

Representation of fuzzy relations

- Co-graphs (weighted co-graphs of classical relations):

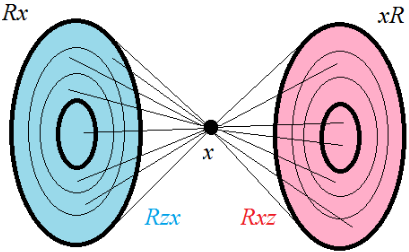


Representation of fuzzy relations

For a binary fuzzy relation R define:

$Ry = \{x \parallel Rxy\}$ foreset (of $x \in V$)

$xR = \{y \parallel Rxy\}$ afterset (of $x \in V$)



Formalization of the theory of fuzzy relations

Formalization = specification of:

- A formal language (formulae for statements about fuzzy relations)
- An interpretation of the formal language
  (translation of formulae into statements about fuzzy relations)
- Derivation rules (to derive true statements about fuzzy relations)

Formalization of the theory of fuzzy relations

Why formalize?

- We restrict the language in order to enable formal (symbolic) manipulation with statements about fuzzy relations
- Derivation of schematic theorems
- Automatic (machine) deduction
- Metamathematical results (on expressivity, complexity, ...)

Predicate fuzzy logic: language

Example of formalization: predicate fuzzy logic

We restrict the mathematical language of fuzzy set theory to:

- Algebraic operations in $\mathbf{L}$: $\wedge$, $\vee$, $*$, $\Rightarrow_*$, $\triangle$, ... (logical connectives)
- Infima and suprema $\bigwedge, \bigvee$ (written as the quantifiers $\forall, \exists$)

Optionally:

- Crisp equality $=$ of elements and fuzzy sets ($=, \leq$ on $\mathbf{L}$ definable)
- (Eliminable) set-terms $\{\vec{x} \parallel \varphi\}$
- Variables for fuzzy sets of fuzzy relations ($\mathcal{A}, \mathcal{B}, \dots$) etc.
  (then add the axioms of equality, comprehension, and extensionality)

Predicate fuzzy logic: interpretation

The meaning of symbols in predicate fuzzy logic:

Atomic formulae:

- Unary predicate symbols P: fuzzy sets
- Unary atomic formulae Px: membership degree of x in P
- n-Ary predicate symbols R: fuzzy relations
- n-Ary atomic formulae Rxy, $Sxyz$, ...:
  membership degree of the n-tuple in the fuzzy relation

Predicate fuzzy logic: interpretation

Connectives:

- Min-conjunction $\wedge$: minimum (in the lattice $\mathbf{L}$ of degrees)
- Max-disjunction $\vee$: maximum
- Strong conjunction $\&$: a (left-)continuous t-norm $*$
  - T-norm $= *$: $[0,1]^2 \to [0,1]$ commutative associative monotone, unit 1
    - E.g.: product, min, the Łukasiewicz t-norm $\max(\alpha + \beta - 1, 0)$
  - More generally, the monoidal operation $*$ of the residuated lattice $\mathbf{L}$
- Implication $\to$: the residuum $\Rightarrow_*$ of $*$
  - $(\alpha \Rightarrow_* \beta) = \sup_{\gamma * \alpha \leq \beta} \gamma$
  - $(\alpha \Rightarrow_* \beta) = 1$ iff $\alpha \leq \beta$
- Negation $\neg$: the function $\alpha \Rightarrow_* 0$ (e.g., $1 - \alpha$ for $*$ Łukasiewicz)
- Equivalence $\leftrightarrow$: the biresiduum $\min(\alpha \Rightarrow_* \beta, \beta \Rightarrow_* \alpha)$
- Delta $\triangle$: indicator of $\alpha = 1$ (in linear $\mathbf{L}$)

Predicate fuzzy logic: interpretation

Quantifiers:

- Universal quantifier $\forall$: infimum (in the complete lattice $\mathbf{L}$)
- Existential quantifier $\exists$: supremum

Example:

$(\forall x)(\exists y)(Px \wedge Qy \to Rxy)$

$\bigwedge_x \bigvee_y (\min(Px, Qy) \Rightarrow_* Rxy)$

Predicate fuzzy logic: axiomatic system

Aim: generate only (and all?) formulae φ s.t. always $\varphi = 1$
= soundness (and, sometimes, completeness)

(a) For a given (left-)continuous t-norm $*$:
- For the Łukasiewicz $*$: Łukasiewicz logic
- For the minimum: Gödel logic
- For the product: product logic, etc.

(b) For a given class of continuous t-norms:
- All continuous t-norms: Hájek's logic BL
- All left-continuous t-norms: MTL, etc.

(c) Variations: adding connectives, relaxing conditions, ...

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Predicate fuzzy logic: axiomatic system

Example: Łukasiewicz fuzzy logic

Axioms:

$\varphi \to (\psi \to \varphi)$	$\alpha \leq (\beta \Rightarrow_* \alpha)$
$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	$\alpha \Rightarrow_* \beta \leq (\beta \Rightarrow_* \gamma) \Rightarrow_* (\alpha \Rightarrow_* \gamma)$
$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$	$\max(\alpha, \beta) \leq \max(\beta, \alpha)$
$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$	$(\alpha \Rightarrow_* 0) \Rightarrow_* (\beta \Rightarrow_* 0) \leq (\beta \Rightarrow_* \alpha)$
$(\forall x)Px \to Py$	$\bigwedge_x Px \leq Py$
$(\forall x)(\varphi \to Px) \to (\varphi \to (\forall x)Px)$	$\bigwedge_x(\alpha \Rightarrow_* Px) \leq \alpha \Rightarrow_* \bigwedge_x Px$

Rules: Uniform substitution and

$\varphi, \varphi \to \psi \vdash \psi$	if $\alpha = 1$ and $\alpha \leq \beta$ then $\beta = 1$
$Px \vdash (\forall x)Px$	if $Px = 1$ for all x, then $\bigwedge_x Px = 1$

Outline

Fuzzy relational operations (formalized)

Recall: In formal fuzzy logic, predicates are interpreted by fuzzy relations (and unary predicates by fuzzy sets)

$\Rightarrow$ Predicate fuzzy logic is in fact a formal calculus for fuzzy relations!

Recall: Restricting the language enables formal manipulation, automatic deduction, schematic theorems, metamathematical results

Large expressive power of predicate fuzzy logic $\Rightarrow$
can be used as the language of the theory of fuzzy relations

Fuzzy relational operations (formalized)

Notice: Fuzzy relations are fuzzy sets

$\Rightarrow$ Fuzzy set-theoretical operations apply to fuzzy relations, too

$R \cap S =_{\text{df}} \{xy \parallel (Rxy \wedge Sxy)\}$ — min-intersection

$(R \cap S)xy = Rxy \wedge Sxy$ — analogously: $\Cap, \cup$

$-R =_{\text{df}} \{xy \parallel \neg Rxy)\}$ — complement

$(-R)xy = (Rxy \Rightarrow_* 0)$ — also written $\overline{R}, R^c, \backslash R, \ldots$

$\ker R =_{\text{df}} \{xy \parallel \triangle Rxy)\}$ — kernel

$xy \in \ker R$ iff $Rxy = 1$ — analogously: supp $(\neg\triangle\neg)$

$\bigcap \mathcal{A} =_{\text{df}} \{xy \parallel (\forall R)(\mathcal{A}R \to Rxy)\}$ — fuzzy set–intersection

$(\bigcap \mathcal{A})xy = \bigwedge_R (\mathcal{A}(R) \Rightarrow_* Rxy)$ — analogously: $\bigcup$

Properties of fuzzy relations

Fuzzy set–theoretical properties of fuzzy relations:

$\mathsf{Hgt}\, R \equiv_{\mathsf{df}} (\exists xy) Rxy$ — height

$\mathsf{Hgt}\, R = \bigvee_{x,y} Rxy$ — analogously: Plt $(\forall / \bigwedge)$

$R \subseteq S \equiv_{\mathsf{df}} (\forall xy)(Rxy \to Sxy)$ — graded inclusion

$(R \subseteq S) = \bigwedge_{x,y} (Rxy \Rightarrow_* Sxy)$

$R \between S \equiv_{\mathsf{df}} (\exists xy)(Rxy \,\&\, Sxy)$ — graded compatibility

$(R \between S) = \bigvee_{x,y} (Rxy * Sxy)$ — $= \mathsf{Hgt}(R \Cap S)$

Notice: $\subseteq$ is *graded*

$(R \subseteq S) = 1$ iff $Rxy \leq Sxy$ for all xy … $\triangle(R \subseteq S)$

Properties of fuzzy relations

Example (for $*$ Łukasiewicz):

$$\begin{pmatrix} 0.7 & 0.1 & 0.3 \\ 0.2 & 0.8 & 0.3 \\ 0.1 & 0.2 & 0.3 \end{pmatrix} \subseteq \begin{pmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.9 & 0.3 \\ 0.1 & 0 & 1 \end{pmatrix} \quad = 0.8$$

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Properties of fuzzy relations

Fuzzy set–theoretical laws apply to fuzzy relations, too. Examples:

$$(R \cap S) \cup T = (R \cup T) \cap (S \cup T)$$
$$R \cap S \subseteq R \cup S$$
$$\mathrm{Hgt}(R \cap S) \leq \mathrm{Hgt}\, R \wedge \mathrm{Hgt}\, S, \quad \text{etc.}$$

Derivation in fuzzy logic (MTL, using Lemma: $(\exists x)(\varphi \wedge \psi) \to (\exists x)\varphi \wedge (\exists x)\psi$):

$$\mathrm{Hgt}(R \cap S) \longleftrightarrow (\exists xy)((R \cap S)xy) \longleftrightarrow (\exists xy)(Rxy \wedge Sxy) \longrightarrow$$
$$(\exists xy)Rxy \wedge (\exists xy)Sxy \longleftrightarrow \mathrm{Hgt}\, R \wedge \mathrm{Hgt}\, S$$

$$\mathrm{Hgt}(R \cap S) = \bigvee_{xy}((R \cap S)xy) = \bigvee xy(Rxy \wedge Sxy) \leq$$
$$\bigvee_{xy} Rxy \wedge \bigvee xySxy = \mathrm{Hgt}\, R \wedge \mathrm{Hgt}\, S$$

Basic fuzzy relational operations

$\emptyset^2 =_{\mathsf{df}} \{xy \parallel 0\}$ — empty fuzzy relation

$\emptyset^2 xy = 0$ for all xy — analogously: $V^2 = \{xy \parallel 1\}$

$\mathsf{Id} =_{\mathsf{df}} \{xy \parallel x = y\}$ — crisp identity relation

$\mathsf{Id}\, xy = 1$ if $x = y$

$\mathsf{Id}\, xy = 0$ otherwise

$R^{-1} =_{\mathsf{df}} \{xy \parallel Ryx\}$ — converse relation

$R^{-1} xy = Ryx$

Basic fuzzy relational operations

$\mathrm{dom}\, R =_{\mathrm{df}} \{x \parallel (\exists y) Rxy\}$ — domain

$(\mathrm{dom}\, R)x = \bigvee_y Rxy$ — analogously: rng

$R^{\rightarrow} A =_{\mathrm{df}} \{y \parallel (\exists x)(Ax \mathbin{\&} Rxy)\}$ — image

$(R^{\rightarrow} A)y = \bigvee_x (Ax * Rxy)$ — analogously: preimage $^{\leftarrow}$

$A \times B =_{\mathrm{df}} \{xy \parallel Ax \mathbin{\&} By\}$ — Cartesian product

$(A \times B)xy = Ax * By$ — analogously: n-ary, powers

Example:

$$\mathrm{dom} \begin{pmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.9 & 0.3 \\ 0.1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.9 \\ 1 \end{pmatrix}$$

Defined properties of fuzzy relations

Refl $R \equiv_{\mathrm{df}} (\forall x) Rxx$ — reflexivity

Refl $R = \bigwedge_x Rxx$

Sym $R \equiv_{\mathrm{df}} (\forall xy)(Rxy \to Ryx)$ — symmetry

Sym $R = \bigwedge_{x,y}(Rxy \Rightarrow_* Ryx)$

Trans $R \equiv_{\mathrm{df}} (\forall xyz)(Rxy \mathbin{\&} Ryz \to Rxz)$ — transitivity

Trans $R = \bigwedge_{x,y,z}(Rxy * Ryz \Rightarrow_* Rxz)$

Notice *graded* properties again:

Trans $R = 1$ iff $Rxy * Ryz \leq Rxz$ for all x, y, z (crisp ‘$*$-transitivity’)

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Defined properties of fuzzy relations

Compound properties:

$\triangle\,\mathrm{Sim}\,R \equiv_{\mathrm{df}} \triangle(\mathrm{Refl}\,R \wedge \mathrm{Sym}\,R \wedge \mathrm{Trans}\,R)$ . . . similarity relation

$\mathrm{Sim}\,R = 1$ iff $\mathrm{Refl}\,R = \mathrm{Sym}\,R = \mathrm{Trans}\,R = 1$

(a.k.a. indistinguishability, fuzzy equivalence relation)

Similarly: fuzzy preorders, fuzzy orders, . . .

Modifications (Höhle, Bodenhofer): e.g.,

$\mathrm{Refl}_E\,R \equiv_{\mathrm{df}} (\forall xy)(Exy \to Rxy)$ for an indistinguishability E

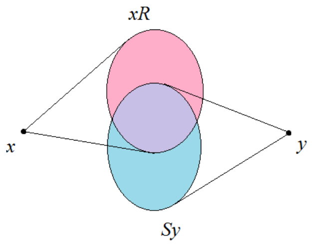
(Sup-T) composition of fuzzy relations

Fuzzy relational composition

$$R \circ S =_{\mathsf{df}} \{xy \parallel (\exists z)(Rxz \mathbin{\&} Szy)\}$$ sup-T composition

$$(R \circ S)xy = \bigvee_z (Rxz * Szy)$$

Fore/afterset characterization: $(R \circ S)xy \equiv xR \between Sy \equiv xR \cap Sy \neq \emptyset$ (crisp)



$(R \circ S)xy$

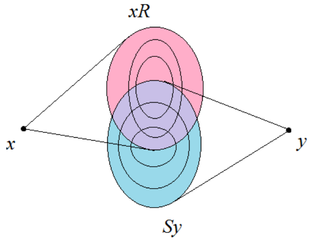
(Sup-T) composition of fuzzy relations

Fuzzy relational composition

$$R \circ S =_{\mathsf{df}} \{xy \parallel (\exists z)(Rxz \,\&\, Szy)\}$$ sup-T composition

$$(R \circ S)xy = \bigvee_z (Rxz * Szy)$$

Fore/afterset characterization: $(R \circ S)xy = xR \between Sy = \mathrm{Hgt}(xR \cap Sy)$ (fuzzy)



$(R \circ S)xy$

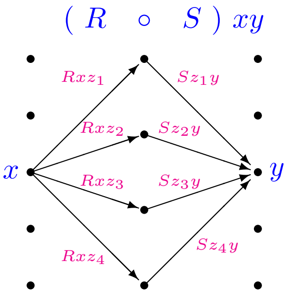
(Sup-T) composition of fuzzy relations

Fuzzy relational composition

$$R \circ S =_{\mathsf{df}} \{xy \parallel (\exists z)(Rxz \,\&\, Szy)\}$$ sup-T composition

$$(R \circ S)xy = \bigvee_z (Rxz * Szy)$$

Co-graph visualization:



(Sup-T) composition of fuzzy relations

Fuzzy relational composition

$$R \circ S =_{\text{df}} \{xy \parallel (\exists z)(Rxz \mathbin{\&} Szy)\}$$

sup-T composition

$$(R \circ S)xy = \bigvee_z (Rxz * Szy)$$

Matrix 'multiplication':

$$\begin{array}{c|c}
\circ & \begin{pmatrix} Sx_1x_1 & \cdots & Sx_1x_n \\ \vdots & \ddots & \vdots \\ Sx_nx_1 & \cdots & Sx_nx_n \end{pmatrix} \\ \hline
\begin{pmatrix} Rx_1x_1 & \cdots & Rx_1x_n \\ \vdots & \ddots & \vdots \\ Rx_nx_1 & \cdots & Rx_nx_n \end{pmatrix} & \begin{pmatrix} (R \circ S)x_1x_1 & \cdots & (R \circ S)x_1x_n \\ \vdots & \ddots & \vdots \\ (R \circ S)x_nx_1 & \cdots & (R \circ S)x_nx_n \end{pmatrix}
\end{array}$$

Properties of sup-T composition

Theorem

- $(R \circ S) \circ T = R \circ (S \circ T)$ (associativity)
- $R \circ \mathsf{Id} = \mathsf{Id} \circ R = R$ (identity)
- $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$ (transposition)
- $R_1 \subseteq R_2 \to R_1 \circ S \subseteq R_2 \circ S$ (monotony)
- $\left(\bigcup_{R \in \mathcal{A}} R\right) \circ S = \bigcup_{R \in \mathcal{A}} (R \circ S)$ (union distributivity)
- $\left(\bigcap_{R \in \mathcal{A}} R\right) \circ S \subseteq \bigcap_{R \in \mathcal{A}} (R \circ S)$ (intersection subdistributivity)

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Derivation in predicate fuzzy logic

$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$ (transposition)

Proof:

$$(R \circ S)^{-1}xy \longleftrightarrow (R \circ S)yx \longleftrightarrow (\exists z)(Ryz \mathbin{\&} Szx) \longleftrightarrow (\exists z)(S^{-1}xz \mathbin{\&} R^{-1}zy) \longleftrightarrow (S^{-1} \circ R^{-1})xy$$

$$(R \circ S)^{-1}xy = (R \circ S)yx = \bigvee_z (Ryz * Szx) = \bigvee_z (S^{-1}xz * R^{-1}zy) = (S^{-1} \circ R^{-1})xy$$

Derivation in predicate fuzzy logic

$(\bigcap_{R\in\mathcal{A}} R) \circ S \subseteq \bigcap_{R\in\mathcal{A}}(R \circ S)$ (intersection subdistributivity)

Proof:

$Q_L xy \equiv_{\mathrm{df}} ((\bigcap_{R\in\mathcal{A}} R) \circ S)xy \longleftrightarrow$	$Q_L xy = ((\bigcap_{R\in\mathcal{A}} R) \circ S)xy =$
$(\exists z)((\forall R)(\mathcal{A}(R) \to Rxy)\ \&\ Szy) \longrightarrow$	$\bigvee_z (\bigwedge_R(\mathcal{A}(R) \Rightarrow_* Rxy) * Szy) \leq$
$(\exists z)(\forall R)(\mathcal{A}(R) \to Rxy\ \&\ Szy) \longrightarrow$	$\bigvee_z \bigwedge_R(\mathcal{A}(R) \Rightarrow_* (Rxy * Szy)) \leq$
$(\forall R)(\exists z)(\mathcal{A}(R) \to Rxy\ \&\ Szy) \longrightarrow$	$\bigwedge_R \bigvee_z(\mathcal{A}(R) \Rightarrow_* (Rxy * Szy)) \leq$
$(\forall R)(\mathcal{A}(R) \to (\exists z)(Rxy\ \&\ Szy)) \longleftrightarrow$	$\bigwedge_R(\mathcal{A}(R) \Rightarrow_* \bigvee_z(Rxy * Szy)) =$
$(\forall R)(\mathcal{A}(R) \to (R \circ S)xy) \longleftrightarrow$	$\bigwedge_R(\mathcal{A}(R) \Rightarrow_* (R \circ S)xy) =$
$(\bigcap_{R\in\mathcal{A}}(R \circ S))xy \equiv_{\mathrm{df}} Q_R xy$	$(\bigcap_{R\in\mathcal{A}}(R \circ S))xy = Q_R xy$
Thus provably,	$Q_L xy \leq Q_R xy$
$Q_L xy \to Q_R xy$	$(Q_L xy \Rightarrow_* Q_R xy) = 1$
$(\forall xy)(Q_L xy \to Q_R xy)$	$\bigwedge_{x,y}(Q_L xy \Rightarrow_* Q_R xy) = 1$
$Q_L \subseteq Q_R$	$(Q_L \subseteq Q_R) = 1$

Inf-R compositions of fuzzy relations

Compositions of fuzzy relations (a.k.a. fuzzy relational products):

$R \circ S =_{\mathrm{df}} \{xy \parallel (\exists z)(Rxz \,\&\, Szy)\}$ — sup-T composition

$(R \circ S)xy = \bigvee_z (Rxz * Szy)$

$R \lhd S =_{\mathrm{df}} \{xy \parallel (\forall z)(Rxz \to Szy)\}$ — inf-R composition

$(R \lhd S)xy = \bigwedge_z (Rxz \Rightarrow_* Szy)$ — a.k.a. BK-(sub)product

$R \rhd S =_{\mathrm{df}} \{xy \parallel (\forall z)(Szy \to Rxz)\}$ — BK-superproduct

$(R \rhd S)xy = \bigwedge_z (Szy \Rightarrow_* Rxz)$

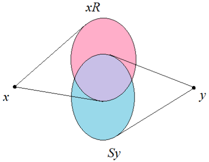
$R \,\square\, S =_{\mathrm{df}} \{xy \parallel (\forall z)(Rxz \leftrightarrow Szy)\}$ — BK-squareproduct

$(R \,\square\, S)xy = \bigwedge_z (Rxz \Leftrightarrow_* Szy)$

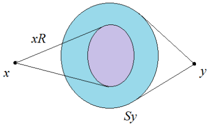
Inf-R compositions of fuzzy relations

Characterization in terms of foresets and aftersets:

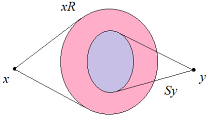
$$(R \circ S)xy = xR \between Sy$$
$$(R \lhd S)xy = xR \subseteq Sy$$
$$(R \rhd S)xy = Sy \subseteq xR$$



$(R \circ S)xy$



$(R \lhd S)xy$



$(R \rhd S)xy$

Inf-R compositions of fuzzy relations

Example (crisp/graded):

Rps = patient p shows symptom s
Ssd = s is a symptom of disease d

$(R \circ S)pd$ = some symptoms shown by patient p are symptoms of disease d
$(R \lhd S)pd$ = all symptoms shown by patient p are symptoms of disease d
$(R \rhd S)pd$ = all symptoms of disease d are shown by patient p
$(R \Box S)pd$ = patient p shows exactly the symptoms of disease d

Expressivity of fuzzy relational products

Many relational properties can be characterized by means of fuzzy relational products

Examples:

$$\mathrm{Trans}\,R \leftrightarrow R \subseteq R \circ R \qquad \mathrm{Trans}\,R \leftrightarrow R \subseteq R \triangleright R^{-1}$$

$$\mathrm{Refl}\,R \rightarrow R \circ R \subseteq R \qquad \mathrm{Refl}\,R \leftrightarrow R \triangleright R^{-1} \subseteq R$$

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Properties of BK-products

Theorem

- $R \triangleright S = (S^{-1} \triangleleft R^{-1})^{-1}$
  $R \square S = (R \triangleleft S) \cap (R \triangleright S)$ (interdefinability)
- $(R \triangleleft S)^{-1} = S^{-1} \triangleright R^{-1}$
  $(R \triangleright S)^{-1} = S^{-1} \triangleleft R^{-1}$ (transposition)
- $R \triangleleft (S \triangleleft T) = (R \circ S) \triangleleft T$
  $(R \triangleright S) \triangleright T = R \triangleright (S \circ T)$ (residuation)
- $(R \triangleleft S) \triangleright T = R \triangleleft (S \triangleright T)$ (quasi-associativity)

Properties of BK-products

Theorem

- $R_1 \subseteq R_2 \to R_2 \lhd S \subseteq R_1 \lhd S$
  $S_1 \subseteq S_2 \to R \lhd S_1 \subseteq R \lhd S_2$ (monotony)
- $\bigcap_{R \in \mathcal{A}} (R \lhd S) = \left(\bigcup_{R \in \mathcal{A}} R\right) \lhd S$
  $\bigcap_{S \in \mathcal{A}} (R \lhd S) = R \lhd \bigcap_{S \in \mathcal{A}} S$ (union subdistributivity)
- $\bigcup_{R \in \mathcal{A}} (R \lhd S) \subseteq \left(\bigcap_{R \in \mathcal{A}} R\right) \lhd S$
  $\bigcup_{S \in \mathcal{A}} (R \lhd S) \subseteq R \lhd \bigcup_{S \in \mathcal{A}} S$ (intersection distributivity)

De Baets–Kerre modification of BK-products

De Baets–Kerre modification of BK-products

$$R \lhd' S =_{\mathsf{df}} \{xy \parallel (\forall z)(Rxz \to Szy) \wedge (\exists z)Rxz\}$$

$$(R \lhd' S)xy = \bigwedge_z (Rxz \Rightarrow_* Szy) \wedge \bigvee_z Rxz$$

(analogously for $\rhd', \Box'$)

Motivation: to avoid the ‘useless pairs’

(good for some applications, many BK-properties preserved)

Modifications of BK-products

Example:

Rps = patient p shows symptom s
Ssd = s is a symptom of disease d

$(R \lhd S)pd$ = all symptoms shown by patient p are symptoms of disease d
(trivially true if p shows no symptoms)

$(R \lhd' S)pd$ = patient p shows some symptoms, all of which are symptoms of d
(much more relevant for diagnosis)

Other side-condition modifications of fuzzy relational products possible—
e.g., to exclude *incompatible* pairs (Štěpnička, Holčapek, Cao)

Outline

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Tarski's relational calculus

Tarski's relational calculus for crisp relations =
an equational calculus using $\cup, -, \circ, {}^{-1}, \mathsf{Id}$

Strong expressive power—can express:

- Classical first-order logic with max 3 variables
- Peano arithmetic $\Rightarrow$ incomplete (Gödel's theorems apply)
- ZFC set theory $\Rightarrow$ math can be done w/o quantifiers

Tarski's relational calculus

Axioms of crisp relational calculus

$$R \cup S = S \cup R$$
$$R \cup (S \cup T) = (R \cup S) \cup T$$
$$-(-R \cup S) \cup -(-R \cup -S)) = R \qquad \text{= axioms of BA}$$

$$R \circ (S \circ T) = (R \circ S) \circ T$$
$$R \circ \mathsf{Id} = R$$
$$(R^{-1})^{-1} = R$$
$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$
$$(R \cup S)^{-1} = R^{-1} \cup S^{-1}$$
$$(R \cup S) \circ T = (R \circ T) \cup (S \circ T)$$
$$(R^{-1} \circ -(R \circ S)) \cup -S = -S$$

Rules: uniform replacement, substitution of equals for equals

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Fuzzy relational calculus

Fuzzy relational calculus = Tarski's calculus adapted for fuzzy relations

A simple way: replace the axioms of BA by the axioms of fuzzy logic

Example for Łukasiewicz logic: add $\uplus, \emptyset^2$, replace BA-axioms with MV-axioms:

$$R \uplus S = S \uplus R \qquad -(-R) = R$$

$$R \uplus (S \uplus T) = (R \uplus S) \uplus T \qquad R \cup S = -(-R \uplus S) \uplus S$$

$$R \uplus \emptyset^2 = R \qquad R \cup S = S \cup R$$

$$R \uplus -\emptyset^2 = -\emptyset^2$$

Theorem: The 'non-logical' relational axioms carry over to fuzzy relations based on any (left-)continuous t-norm (Kohout 2002)

Only sketched by Kohout, many open problems (expressive power, . . .)

Composition-based calculus for fuzzy sets and relations

Extension to fuzzy set-theoretical notions (Běhounek–Daňková 2009) by including BK-products and identification of fuzzy sets and truth degrees with certain fuzzy relations

Compare:

$$R \circ S =_{\mathsf{df}} \{xy \parallel (\exists z)(Rxz \mathbin{\&} Szy)\} \qquad (R \circ S)xy = \bigvee_z (Rxz * Szy)$$
$$R^{\leftarrow} A =_{\mathsf{df}} \{x \parallel (\exists z)(Rxz \mathbin{\&} Az)\} \qquad (R^{\rightarrow} A)x = \bigvee_z (Rxz * Az)$$

Idea: Add a dummy constant $\mathbf{o}$ and identify Ax with $S_A\, x\mathbf{o}$

This reduces $^{\leftarrow}$ to $\circ$:

$$R \circ S_A =_{\mathsf{df}} \{x\mathbf{o} \parallel (\exists z)(Rxz \mathbin{\&} S_A z\mathbf{o})\} \qquad (R \circ S_A)x\mathbf{o} = \bigvee_z (Rxz * S_A z\mathbf{o})$$

Composition-based calculus for fuzzy sets and relations

Consequently, properties of $\circ$ transfer to $^{\leftarrow}$, e.g.:

$$(R_1 \subseteq R_2) \to (R_1 \circ S \subseteq R_2 \circ S) \qquad \left(\bigcap_{R \in \mathcal{A}} R\right) \circ S \subseteq \bigcap_{R \in \mathcal{A}} (R \circ S)$$

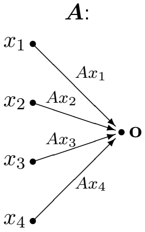
$$(R_1 \subseteq R_2) \to (R_1 {}^{\leftarrow} A \subseteq R_2 {}^{\leftarrow} A) \qquad \left(\bigcap_{R \in \mathcal{A}} R\right) {}^{\leftarrow} A \subseteq \bigcap_{R \in \mathcal{A}} (R {}^{\leftarrow} A)$$

Similarly for many other fuzzy set- and relation-theoretical notions

Fuzzy relational representation of fuzzy sets

In general, identify a fuzzy set A with the fuzzy relation $\boldsymbol{A} = A \times \{\mathbf{o}\}$:
(i.e., a $1 \times N$ fuzzy relation if finite)

$$\boldsymbol{A} = \begin{pmatrix} Ax_1 \\ \vdots \\ Ax_n \end{pmatrix}$$



Fuzzy relational representation of fuzzy sets

Then $R^{\leftarrow}A = R \circ \boldsymbol{A} = \begin{pmatrix} Rx_1x_1 & \cdots & Rx_1x_n \\ \vdots & \ddots & \vdots \\ Rx_nx_1 & \cdots & Rx_nx_n \end{pmatrix} \circ \begin{pmatrix} Ax_1 \\ \vdots \\ Ax_n \end{pmatrix} = \begin{pmatrix} (R^{\leftarrow}A)x_1 \\ \vdots \\ (R^{\leftarrow}A)x_n \end{pmatrix}$

Similarly: $R^{\rightarrow}A = R^{-1} \circ \boldsymbol{A}$, $\mathrm{dom}\, R = R \circ \boldsymbol{V}$, $A \times B = \boldsymbol{A} \circ \boldsymbol{B}^{-1}$, etc.

$\circ$	$\begin{pmatrix} Bx_1 & \cdots & Bx_n \end{pmatrix}$
$\begin{pmatrix} Ax_1 \\ \vdots \\ Ax_n \end{pmatrix}$	$\begin{pmatrix} (A \times B)x_1x_1 & \cdots & (A \times B)x_1x_n \\ \vdots & \ddots & \vdots \\ (A \times B)x_nx_1 & \cdots & (A \times B)x_nx_n \end{pmatrix}$

Fuzzy relational representation of truth degrees

Furthermore, identify a truth degree α with the fuzzy set $\boldsymbol{\alpha} = \{\mathbf{o} \parallel \alpha\}$
(i.e., a 1×1 fuzzy relation)

$$\boldsymbol{\alpha} = \left(\, \alpha \, \right)$$

$\boldsymbol{\alpha}$:

Fuzzy relational representation of truth degrees

Then, e.g., $A \subseteq B = (\forall x)(Ax \to Bx) = (\forall x)(\boldsymbol{A}x\mathbf{o} \to \boldsymbol{B}x\mathbf{o}) =$
$(\forall x)(\boldsymbol{A}^{-1}\mathbf{o}x \to \boldsymbol{B}x\mathbf{o}) = \boldsymbol{A}^{-1} \triangleleft \boldsymbol{B}$

$$\begin{array}{c|c} \triangleleft & \begin{pmatrix} Bx_1 \\ \vdots \\ Bx_n \end{pmatrix} \\ \hline \begin{pmatrix} Ax_1 & \cdots & Ax_n \end{pmatrix} & \begin{pmatrix} A \subseteq B \end{pmatrix} \end{array}$$

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Fuzzy relational representation of truth degrees

Similarly: Hgt $A = \boldsymbol{V}^{-1} \circ \boldsymbol{A}$, Plt $A = \boldsymbol{V}^{-1} \lhd \boldsymbol{A}$, $(A \between B) = \boldsymbol{A}^{-1} \circ \boldsymbol{B}$, etc.

Moreover, propositional connectives reduce to relational products, too:

$$(\alpha \mathbin{\&} \beta) = \boldsymbol{\alpha} \circ \boldsymbol{\beta}$$
$$(\alpha \rightarrow \beta) = \boldsymbol{\alpha} \lhd \boldsymbol{\beta}$$
$$(\alpha \leftrightarrow \beta) = \boldsymbol{\alpha} \,\square\, \boldsymbol{\beta}$$

$\circ$	$(\quad \alpha \quad)$
$(\,\beta\,)$	$(\alpha \mathbin{\&} \beta)$

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Transfer of properties of compositions

Properties of compositions apply to the notions reduced to compositions:

Examples:

$$
\begin{aligned}
(\textstyle\bigcup_{R\in\mathcal{A}} R)\circ S &= \textstyle\bigcup_{R\in\mathcal{A}}(R\circ S) & \quad R\circ\textstyle\bigcup_{S\in\mathcal{A}} S &= \textstyle\bigcup_{S\in\mathcal{A}}(R\circ S)\\
(\textstyle\bigcup_{R\in\mathcal{A}} R)^{\rightarrow} A &= \textstyle\bigcup_{R\in\mathcal{A}}(R^{\rightarrow} A) & \quad R^{\rightarrow}\textstyle\bigcup_{A\in\mathcal{A}} A &= \textstyle\bigcup_{A\in\mathcal{A}}(R^{\rightarrow} A)\\
(\textstyle\bigcup_{R\in\mathcal{A}} R)^{\leftarrow} A &= \textstyle\bigcup_{R\in\mathcal{A}}(R^{\leftarrow} A) & \quad R^{\leftarrow}\textstyle\bigcup_{A\in\mathcal{A}} A &= \textstyle\bigcup_{A\in\mathcal{A}}(R^{\leftarrow} A)\\
(\textstyle\bigcup_{A\in\mathcal{A}} A)\times B &= \textstyle\bigcup_{A\in\mathcal{A}}(A\times B) & \quad A\times\textstyle\bigcup_{B\in\mathcal{A}} B &= \textstyle\bigcup_{B\in\mathcal{A}}(A\times B)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Dom}(\textstyle\bigcup_{R\in\mathcal{A}} R) &= \textstyle\bigcup_{R\in\mathcal{A}} \mathrm{Dom}\, R\\
\mathrm{Rng}(\textstyle\bigcup_{R\in\mathcal{A}} R) &= \textstyle\bigcup_{R\in\mathcal{A}} \mathrm{Rng}\, R\\
\mathrm{Hgt}(\textstyle\bigcup_{A\in\mathcal{A}} A) &\leftrightarrow \textstyle\bigvee_{A\in\mathcal{A}}(\mathrm{Hgt}\, A)
\end{aligned}
$$

Equational derivations of fuzzy set-theoretic laws

Example:

$$R^{\rightarrow}(\operatorname{rng} S) = \operatorname{rng}(S \circ R)$$

Recall:

$$R^{\rightarrow} A = R^{-1} \circ \boldsymbol{A}$$
$$\operatorname{rng} R = R^{\rightarrow} V = R^{-1} \circ \boldsymbol{V}$$

Derivation:

$$R^{\rightarrow}(\operatorname{rng} S) = R^{-1} \circ (S^{-1} \circ \boldsymbol{V}) = (R^{-1} \circ S^{-1}) \circ \boldsymbol{V} = (S \circ R)^{-1} \circ \boldsymbol{V} = \operatorname{rng}(S \circ R)$$

Equational derivations of fuzzy set-theoretic laws

Example:

$\mathrm{Hgt}(\mathrm{dom}\, R) = \mathrm{Hgt}(\mathrm{rng}\, R)$

Recall:

$$\mathrm{Hgt}\, A = \boldsymbol{A}^{-1} \circ \boldsymbol{V} = \boldsymbol{V}^{-1} \circ \boldsymbol{A}$$
$$\mathrm{dom}\, R = R^{\leftarrow} V = R \circ \boldsymbol{V}$$
$$\mathrm{rng}\, R = R^{\rightarrow} V = R^{-1} \circ \boldsymbol{V}$$

Derivation:

$$\mathrm{Hgt}(\mathrm{dom}\, R) = \boldsymbol{V}^{-1} \circ (R \circ \boldsymbol{V}) = (\boldsymbol{V}^{-1} \circ R) \circ \boldsymbol{V} = (R^{-1} \circ \boldsymbol{V})^{-1} \circ \boldsymbol{V} = \mathrm{Hgt}(\mathrm{rng}\, R)$$

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