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The Apparatus of Fuzzy Class Theory

Libor Běhounek and Petr Cintula

Institute of Computer Science
Academy of Sciences of the Czech Republic

Outline

1. Need for Fuzzy Class Theory

2. Building Fuzzy Class Theory

3. Using Fuzzy Class Theory

4. Advancing Fuzzy Class Theory

1. The need for Fuzzy Class Theory

Foundations of fuzzy mathematics

Traditional fuzzy mathematics:

- Fuzzy sets represented by their membership functions
  = classical (crisp) model of fuzzy sets
- Uses classical Boolean logic
  (since it reasons about crisp membership functions)
- Approaches fuzziness only indirectly
  (through the crisp model of membership function)

A genuine fuzzy approach:

- Fuzzy sets are a primitive notion
  (like sets in classical mathematics)
- Uses fuzzy logic
  (a logic appropriate for reasoning about fuzzy sets)
- Speaks directly about fuzzy phenomena
  (not via a crisp model)

Formal fuzzy logic

Fuzzy logic describes *the laws of truth preservation* in reasoning under (a certain form of) vagueness

Its interpretation in terms of truth degrees is just a *model* (a classical rendering of vague phenomena)

Although such models have originally been employed for discovering the laws of approximate reasoning, they must be regarded as secondary (essentially classical, not genuinely vague)

Formal theories over fuzzy logic—assuming that fuzzy logic faithfully approximates the laws of truth preservation in reasoning fraught with vagueness—are genuinely fuzzy

Axiomatization of fuzzy mathematics

The need for axiomatization of further areas of fuzzy mathematics is beyond doubt (axiomatization has always aided the development of mathematical theories)

Previous attempts:

- Usually designed ad hoc
- Only some concepts turned fuzzy
- Based on non-systematic intuitions or intended applications
- Often semi-classical (truth degrees, membership functions)

Fragmentation of fuzzy mathematics:

- Completely different sets of primitive concepts
- Incompatible formalisms
- Virtually impossible to combine any two theories

Formal fuzzy mathematics

We propose a unified methodology for the axiomatization of fuzzy mathematics

It would

- Facilitate the exchange of results between its branches
- Result in a systematic development of their concepts

Work deductively in a formal axiomatic theory over fuzzy logic, not in a particular model!

("Formalistic imperative")

$\Rightarrow$ Formal theories over Hájek-style fuzzy logics

Běhounek-Cintula: *From fuzzy logic to fuzzy mathematics: a methodological manifesto.* FSS 2006 (to appear)

Foundations of fuzzy mathematics

Architecture of classical mathematics:

- Logic: (first-order) Boolean logic
  governs reasoning in mathematical theories
- Foundations: set theory (type theory, . . .)
  a formal theory giving a general framework
- Particular disciplines: graph theory, topology, . . .
  formalized within the foundational theory

Proposed architecture of fuzzy mathematics:

- Logic: (first-order) fuzzy logic
  developed enough for building formal theories
- Foundations: a kind of formal fuzzy set theory
  proposed here
- Particular disciplines: fuzzy graph theory, fuzzy topology, . . .
  formalized within the foundational theory

Proposed foundations: Fuzzy Class Theory

Great flexibility and generality ⇒ "arbitrary" fuzzy logic $\mathcal{F}$
Analogy with classical foundations ⇒ higher-order
Axiomatizability ⇒ Henkin-style

Henkin-style higher-order fuzzy logic $\mathcal{F}$ = FCT over $\mathcal{F}$

Formal theory ⇒ axiomatic method
Intended models = Zadeh's fuzzy sets of any order over a fixed domain
Soundness ⇒ results are valid of real fuzzy sets

Unified formalism for various branches of fuzzy mathematics

Běhounek-Cintula: *Fuzzy class theory.* FSS 2005

2. Building Fuzzy Class Theory

Second-order logic $\mathcal{F}_2$—axiomatic system

Propositional axioms: axioms of the logic $\mathcal{F}$

First-order axioms: usual axioms of quantifiers

Second-order axioms:

- *Comprehension axioms:* $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$
  (witness the existence of all definable fuzzy sets)

- *Extensionality:* from $(\forall x)(x \in A \leftrightarrow x \in B)$ infer $A = B$
  (fuzzy sets are determined by their membership functions)

- *Axioms for tuples:* tuples equal iff all components equal, etc.
  (usual axioms, tuples usually come out crisp)

Second-order logic $\mathcal{F}_2$—semantics

Intended models of $\mathcal{F}_2$:

U a set (universe)

$\mathbf{L}$ an $\mathcal{F}$-algebra (of truth-values)

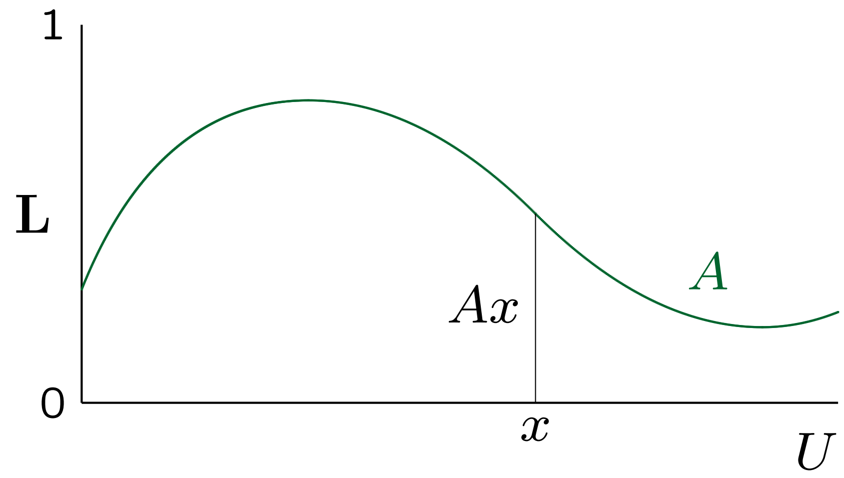
Object variables range over U

Class variables range over $\mathbf{L}^U$

$\|x \in A\| = \|A\|\,(\|x\|)$

$\|\varphi \circ \psi\| = \|\varphi\| \circ_{\mathbf{L}} \|\psi\|$

$\forall, \exists$ . . . $\inf, \sup$ resp.



Generally, second-order fuzzy logic is *not axiomatizable*

$\Rightarrow$ General (Henkin) models:

- Predicate variables range over a subset of $\mathbf{L}^U$
- Axioms insure that all definable fuzzy classes are present

Higher-order logic $\mathcal{F}_n$

Generalization to fuzzy sets of higher order
(fuzzy sets of fuzzy sets, of fuzzy sets of fuzzy sets, etc.):

Third-order fuzzy logic $\mathcal{F}_3$:

- New sort of variables for classes of classes $\mathcal{X}, \mathcal{Y}, \ldots$
- New membership predicate $X \in \mathcal{X}$
- Extensionality and comprehension axioms for classes of classes
- All definitions and theorems translate a level higher

Higher-order fuzzy logic $\mathcal{F}_\omega$

- Iterate the construction for all levels $n \in \omega$
- If necessary, mark the type of variables: $x^{(0)}, X^{(1)}, Y^{(n)}, \ldots$
- All definitions and theorems translate to all higher levels

Atomic formulae of $\mathcal{F}_\omega$

Variables

- $x, y, \ldots$ for atomic objects
- $X, Y, \ldots$ for (fuzzy) sets of atomic objects
- $\mathcal{X}, \mathcal{Y}, \ldots$ for (fuzzy) sets of (fuzzy) sets of atomic objects
- Generally $X^{(n)}$ for (fuzzy) sets of the n-th order
- Tuples $\langle X_1, \ldots, X_k \rangle$ of objects or sets of any order
- *No* variables for truth degrees: expressed by atomic formulae

Atomic formulae

- $x \in A$ (or Ax) expresses the membership degree of x in A
  Type match: $X^{(n)} \in Y^{(m)}$ wff iff $n \leq m$
- Crisp identity of objects $x = y$ or sets $A = B$ (or $X^{(n)} = Y^{(n)}$)
  (identical membership functions, intersubstitutable)

Logical connectives of ŁΠ$_\omega$ (standard semantics)

- T-norms $*$ or $\&_*$ (finite ordinal sums of G, Ł, Π) $Ax \wedge Bx$
- R-implications $\Rightarrow_*$ (their residua) $Ax * Bx \Rightarrow_* Ax$
- T-conorms, S-implications for $*$ (definable from $*$, $\Rightarrow_*$)
- Truth constants $\frac{m}{n}$ (added by an axiom) $Ax \vee \frac{m}{n}$
- Arithmetical operations $+$, $-$, $\cdot$, $:$ (bounded) $1 - Ax \cdot Bx$
- Comparisons $=$, $<$, $\leq$, $\neq$ of truth degrees $Ax \leq \frac{1}{2}$
- and any definable from these, e.g. negations $Ax \Rightarrow_* 0$
  (strict $Ax \neq 0$, involutive $1 - Ax$)

Notice the difference:

$Ax = Bx$ . . . *logical connective* (compares truth degrees)

$x = y$, $A = B$ . . . *atomic predicate* (identity of objects)

Esteva-Godo-Montagna: *The* ŁΠ *and* ŁΠ$\frac{1}{2}$ *logics.* AML 2001

Logical symbols of $\mathcal{F}_\omega$

Quantifiers:

$(\forall x)\varphi(x)$ ... $\inf_x \varphi(x)$ $\qquad$ $\inf_x Ax$ ... $(\forall x)Ax$

$(\exists x)\varphi(x)$ ... $\sup_x \varphi(x)$

φ crisp: $(\forall x)\varphi$ ... φ holds for all x

φ fuzzy: $(\forall x)\varphi$ ... the infimum of the truth values of φ

Set comprehension terms:

$A = \{x \mid \varphi(x)\}$ iff $Ax = \varphi(x)$ $\qquad$ $\{x \mid Ax \wedge Bx\}$

φ crisp: $\{x \mid \varphi\}$ ... the set of all x such that φ

φ fuzzy: $\{x \mid \varphi\}$ ... the set to which x belongs in the degree φ

FCT and FTT

Fuzzy class theory

- Russell style syntax
- Based of sets (classes)
- Comprehension terms
- Foundations oriented
- Stratified type hierarchy
- Simple axiomatization
- Set theoretical concepts are more natural
- Direct formalization of Zadeh's fuzzy sets
- Easy generalization to other fuzzy logics

B&C.: *Fuzzy class theory* FSS 2005

Fuzzy type theory

- Church style syntax
- Based on functions
- Lambda terms
- Natural language oriented
- Multi-dimensional type hierarchy
- Simple set of primitive concepts
- Truth values and functional concepts are more natural
- Direct formalization of certain linguistic concepts
- Formalization of natural human reasoning

Novák: *On fuzzy type theory* FSS 2004

These two theories seem to be equivalent

Defined symbols of FCT

Set constants:

- $\emptyset =_{\text{df}} \{x \mid 0\}$
- $\mathsf{V} =_{\text{df}} \{x \mid 1\}$
- $\mathsf{Id} =_{\text{df}} \{\langle x, y\rangle \mid x = y\}$

$\emptyset x = 0$ for all x

$\mathsf{V} x = 1$ for all x

Elementary set operations:

- $A \cap B =_{\text{df}} \{x \mid Ax \& Bx\}$
- $A \cup B =_{\text{df}} \{x \mid Ax \veebar Bx\}$
- $A \sqcap B =_{\text{df}} \{x \mid Ax \wedge Bx\}$
- $A \sqcup B =_{\text{df}} \{x \mid Ax \vee Bx\}$
- $\mathsf{Ker}\, A =_{\text{df}} \{x \mid Ax = 1\}$
- $\mathsf{Supp}\, A =_{\text{df}} \{x \mid Ax > 0\}$
- $\backslash A =_{\text{df}} \{x \mid \neg_G(Ax\}$
- $-A =_{\text{df}} \{x \mid \neg_{\text{Ł}}(Ax)\}$

$A \cap B = B \cap A$

$(\forall A)(A \sqcup A = A)$

strict complement

involutive complement

Properties of fuzzy sets

Crisp properties:

- Norm(A) $\equiv_{\text{df}}$ $(\exists x)(Ax = 1)$
- Crisp(R) $\equiv_{\text{df}}$ $(\forall x)[(Ax = 0) \vee (Ax = 1)]$
- Fuzzy(A) $\equiv_{\text{df}}$ $\neg$ Crisp(A)

Graded properties:

- Hgt(A) $\equiv_{\text{df}}$ $(\exists x)Ax$
  (the *truth degree* $\sup_x Ax$ is expressed by a *formula* of FCT)

Notice: Fuzzy properties of sets are objects of FCT:

$\mathcal{H}gt =_{\text{df}} \{A \mid \text{Hgt}(A)\}$ . . . a 2nd-order fuzzy set
to which A belongs in the degree of its height

Even *principles* can be formalized as objects of FCT
(Zadeh's extension principle is a certain 3rd order class)

Relations between fuzzy sets

Crisp relations:

- $A \sqsubseteq B \equiv_{\mathsf{df}} (\forall x)(Ax \leq Bx)$ (traditional) fuzzy set inclusion

Graded relations:

- $A \subseteq B \equiv_{\mathsf{df}} (\forall x)(Ax \rightarrow Bx)$ (graded) fuzzy set inclusion
- $A \approx B \equiv_{\mathsf{df}} (\forall x)(Ax \leftrightarrow Bx)$ fuzzy set similarity
- $A \parallel B \equiv_{\mathsf{df}} (\exists x)(Ax \& Bx)$ fuzzy set compatibility

Not only true or false, but *graded*

More general, but *easy to handle* in FCT:

the same form and similar proofs as in classical mathematics

Theory of fuzzy relations

In FCT, we define the following operations:

$A \times B$	$=_{\mathrm{df}}$	$\{\langle x, y\rangle \mid Ax \;\&\; By\}$	Cartesian product
$\mathrm{Dom}(R)$	$=_{\mathrm{df}}$	$\{x \mid Rxy\}$	Domain
$\mathrm{Rng}(R)$	$=_{\mathrm{df}}$	$\{y \mid Ryx\}$	Range
$R \circ S$	$=_{\mathrm{df}}$	$\{\langle x, y\rangle \mid (\exists z)(Rxz \;\&\; Szy)\}$	Composition
R^{-1}	$=_{\mathrm{df}}$	$\{\langle x, y\rangle \mid Ryx\}$	Inverse
Id	$=_{\mathrm{df}}$	$\{\langle x, y\rangle \mid x = y\}$	Identity

Properties of fuzzy relations

Crisp properties:

- Crisp(R) $\equiv_{\text{df}}$ $(\forall x, y)[(Rxy = 0) \vee (Rxy = 1)]$
- Fuzzy(R) $\equiv_{\text{df}}$ $\neg\,\text{Crisp}(R)$

Graded properties:

• Refl(R)	$\equiv_{\text{df}}$	$(\forall x)(Rxx)$	reflexivity
• Sym(R)	$\equiv_{\text{df}}$	$(\forall x, y)(Rxy \to Ryx)$	symmetry
• Trans(R)	$\equiv_{\text{df}}$	$(\forall x, y, z)(Rxy\ \&\ Ryz \to Rxz)$	transitivity
• ASym$_E$(R)	$\equiv_{\text{df}}$	$(\forall x, y)(Rxy\ \&\ Ryx \to Exy)$	E-asymmetry
• QOrd(R)	$\equiv_{\text{df}}$	Refl(R) & Trans(R)	quasi-ordering
• Ord$_E$(R)	$\equiv_{\text{df}}$	QOrd(R) & ASym$_E$(R)	E-ordering
• Sim(R)	$\equiv_{\text{df}}$	QOrd(R) & Sym(R)	similarity
• Equ(R)	$\equiv_{\text{df}}$	Sim(R) & $(\forall x, y)(\Delta Rxy \to x = y)$	equality

Graded properties of fuzzy relations

In classical mathematics: crisp R is reflexive $\equiv_{\text{df}}$ $(\forall x) Rxx$

Traditionally: fuzzy R is reflexive $\equiv_{\text{df}}$ for all x, $Rxx = 1$

In FCT, this is expressed by the formula $(\forall x)(Rxx = 1)$

In our approach: fuzzy R is reflexive $\equiv_{\text{df}}$ $(\forall x) Rxx$

Not only true or false, but *graded* (R more or less reflexive)

cf. *Gottwald: Fuzzy Sets and Fuzzy Logic: Foundations of Application—from a Mathematical Point of View, 1993*

More general, but *easy to handle* in FCT:

the same form and similar proofs as in classical mathematics

3. Using Fuzzy Class Theory

Exploiting the syntax

Formal syntactic manipulation can trivialize some areas of fuzzy mathematics

Example:

A large part of elementary theory of fuzzy sets can effectively be reduced to fuzzy propositional calculus

$\Rightarrow$ proofs of basic facts about fuzzy sets are trivialized
$\Rightarrow$ machine computable

Notation:

$$\mathsf{Op}_{\varphi}(X_1,\ldots,X_n) \;=_{\mathsf{df}}\; \{x \mid \varphi(x\in X_1,\ldots,x\in X_n)\}$$

$$\mathsf{Rel}^{\forall}_{\varphi}(X_1,\ldots,X_n) \;\equiv_{\mathsf{df}}\; (\forall x)\varphi(x\in X_1,\ldots,x\in X_n)$$

$$\mathsf{Rel}^{\exists}_{\varphi}(X_1,\ldots,X_n) \;\equiv_{\mathsf{df}}\; (\exists x)\varphi(x\in X_1,\ldots,x\in X_n)$$

E.g.:

$$\mathsf{Op}_{p\&q}(A,B) \;=_{\mathsf{df}}\; \{x \mid Ax \;\&\; Bx\} \;=\; A\cap B$$

$$\mathsf{Rel}^{\forall}_{p\rightarrow q}(A,B) \;\equiv_{\mathsf{df}}\; (\forall x)(Ax\rightarrow Bx) \;=\; A\subseteq B$$

$$\mathsf{Rel}^{\exists}_{\Delta p}(A,B) \;\equiv_{\mathsf{df}}\; (\exists x)\Delta Ax \;=\; \mathsf{Norm}\, A$$

Metatheorem 1:

$$\vdash \varphi(\psi_1, \ldots, \psi_n) \quad \text{(propositionally!)}$$

$$\text{iff} \;\; \vdash \mathsf{Rel}^{\forall}_{\varphi}(\mathsf{Op}_{\psi_1}(X_{11}, \ldots, X_{1k_1}), \ldots, \mathsf{Op}_{\psi_n}(X_{n1}, \ldots, X_{nk_n}))$$

$$\text{iff} \;\; \vdash \mathsf{Rel}^{\exists}_{\varphi}(\mathsf{Op}_{\psi_1}(X_{11}, \ldots, X_{1k_1}), \ldots, \mathsf{Op}_{\psi_n}(X_{n1}, \ldots, X_{nk_n}))$$

Metatheorem 2:

$$\vdash \mathop{\&}_{i=1}^{k} \varphi_i(\vec{\psi}_i) \longrightarrow \varphi'(\vec{\psi}') \quad \text{(propositionally)}$$

$$\text{iff} \;\; \vdash \mathop{\&}_{i=1}^{k} \mathsf{Rel}^{\forall}_{\varphi_i}\left(\overrightarrow{\mathsf{Op}_{\psi_i}}(\vec{X})\right) \longrightarrow \mathsf{Rel}^{\forall}_{\varphi'}\left(\overrightarrow{\mathsf{Op}_{\psi'}}(\vec{X})\right)$$

Metatheorem 3:

$$\vdash \mathop{\&}_{i=1}^{k} \varphi_i(\vec{\psi}_i) \longrightarrow \varphi'(\vec{\psi}') \quad \text{(propositionally)}$$

$$\text{iff} \;\; \vdash \mathop{\&}_{i=1}^{k-1} \mathsf{Rel}^{\forall}{}_{\varphi_i}\left(\overrightarrow{\mathsf{Op}_{\psi_i}}(\vec{X})\right) \;\&\; \mathsf{Rel}^{\exists}_{\psi_k}\left(\mathsf{Op}_{\psi_k}(\vec{X})\right) \longrightarrow$$

$$\longrightarrow \mathsf{Rel}^{\exists}_{\varphi'}\left(\overrightarrow{\mathsf{Op}_{\psi'}}(\vec{X})\right)$$

Examples of corollaries:

$\vdash \Delta p \to p$	proves	$\vdash \mathsf{Ker}\, A \subseteq A$
$\vdash (p \,\&\, q) \to p$	"	$\vdash A \cap B \subseteq A$
$\vdash (p \to q) \to (p \,\&\, r \to q \,\&\, r)$	"	$\vdash A \subseteq B \to A \cap C \subseteq B \cap C$
$\vdash (p \to q) \to (p \to q)$	"	$\vdash A \subseteq B \to (\mathsf{Hgt}\, A \to \mathsf{Hgt}\, B)$

...

Formal interpretations

Interpretation = syntactic translation of one theory into another theory, which preserves provability

Assign:

to each sort s of variables	...	a sort s^* and a function symbol F_s^*
function symbol F	...	a function symbol F^*
predicate symbol P	...	a predicate symbol P^*

Translate:

each variable x of sort s	...	the term $F_s^*(x^*)$ for x^* of sort s^*
term $F(t_1, \ldots, t_k)$	...	the term $F^*(t_1^*, \ldots, t_k^*)$
formula $P(t_1, \ldots, t_k)$	...	the formula $P^*(t_1^*, \ldots, t_k^*)$
formula $c(t_1, \ldots, t_k)$	...	the formula $c(t_1^*, \ldots, t_k^*)$
formula $(\forall x)\varphi$	...	the formula $(\forall x^*)\varphi^*$, dtto $\exists$

$*$ interprets T in S ... $S \vdash \varphi^*$ for all axioms φ of T, incl. ax. $=$

Proposition: If $*$ interprets T in S, then $T \vdash \varphi$ implies $S \vdash \varphi^*$.

The definition of interpretation can be further generalized, the proposition then requires additional preconditions. E.g.:

Translate connectives c of T by formulae φ of S
Require $S \vdash \varphi^*$ for all logical axioms φ of T

Example (upward type shift $\sharp$):

variable $x^{(n)}$	...	variable $x^{(n+1)}$
term $\{x^{(n)} \mid \varphi\}$	...	term $\{x^{(n+1)} \mid \varphi\}$
term $\langle x_1^{(n)}, \ldots, x_k^{(n)} \rangle$	...	term $\langle x_1^{(n+1)}, \ldots, x_k^{(n+1)} \rangle$
predicate $x^{(n)} \in X^{(n+1)}$	...	predicate $x^{(n+1)} \in X^{(n+2)}$

$\sharp$ interprets FCT in FCT
$\Rightarrow$ all theorems are preserved under the upward shift of types
$\Rightarrow$ we prove theorems only for the lowest orders

Example (relativization to A):

Let A be *crisp.*

Leave all predicate and function symbols absolute, translate

$$
\begin{array}{lll}
[(\forall x^{(n)})\varphi]^A & \ldots & (\forall x^{(n)} \in A^{(n)})\varphi^A \\
[(\exists x^{(n)})\varphi]^A & \ldots & (\exists x^{(n)} \in A^{(n)})\varphi^A
\end{array}
$$

where

$$
\begin{array}{rcl}
A^{(1)} & = & A \\
A^{(n+1)} & = & \mathsf{Ker}\,\mathsf{Pow}\,A^{(n)}
\end{array}
$$

Relativization to A interprets FCT in FCT

$\Rightarrow$ any crisp class can serve as V

Example (interpretation of classical theories):

Let T be a classical theory formulated in higher-order logic
$\Rightarrow$ almost any mathematical theory

Translate $y = F(x_1, \ldots, x_k)$ as $\langle x_1, \ldots, x_k, y\rangle \in F^*$
for each function symbol F of T

Let S be FCT plus the following axioms:

Crisp P for all predicate symbols P of T
Crisp F^* and $\langle x_1, \ldots, x_k, y\rangle \in F^* \mathbin{\&} \langle x_1, \ldots, x_k, y'\rangle \in F^* \rightarrow y = y'$
for all function symbols F of T

The translation *faithfully* interprets T in FCT+S
$*$ is faithful ... $T \vdash \varphi$ iff $S \vdash \varphi^*$, for all φ in the language of T

$\Rightarrow$ Any crisp structure can be introduced to FCT
$\Rightarrow$ available: $\mathsf{N}, \mathsf{Q}, \mathsf{R}$, crisp metrics, measures, topologies,...

Natural fuzzification of classical theories

Uniform methodology towards fuzzy mathematics:

Let T be a classical axiomatic theory

(e.g., the axioms of topology)

Remove all axioms Crisp P and Crisp F^*

from the above interpretation of T in FCT

Get a fuzzified version of the classical theory

!!! Not too mechanically:

- choose between classically equivalent definitions
- follow intuitive motivations, prefer good properties, ...

Example: fuzzified topology = fuzzy set of fuzzy neigborhoods

Optionally: keep some of the crispness assumptions

(controlled defuzzification)

Foreshadowed by U. Höhle (1987):

> “It is the opinion of the author that from a mathematical viewpoint the important feature of fuzzy set theory is the replacement of the two-valued logic by a multiple-valued logic. [. . . I]t is now clear how we can find for every mathematical notion its ‘fuzzy counterpart’. Since every mathematical notion can be written as a formula in a formal language, we have only to internalize, i.e. to interpret these expressions by the given multiple-valued logic.”

Fuzzy real numbers as Dedekind cuts w.r.t. a multiple-valued logic, FSS 1987

Graded theories

Consequence of the methodology and apparatus:

Not only $\in$, but all notions are naturally graded

In FCT we have:

- Graded inclusion $\subseteq$ . . . inclusion to a degree
- Graded reflexivity Refl . . . reflexivity to a degree
- Graded property "being a fuzzy topological space"
- . . .

Graded properties of fuzzy relations pursued to a certain extent already by Gottwald (1993, 2001), Bělohlávek (2002), . . .

The graded approach is important:

- Graded notions generalize the traditional (non-graded) ones

  traditional = Δ(graded)

- Graded notions allow to infer relevant information when the traditional conditions are *almost* fulfilled

  cf. 0.999-reflexivity

- Graded notions are easily be handled by FCT

  inferring by the rules of fuzzy logic

- Graded notions are more fuzzy

  properties of fuzzy sets crisp?

The advanced gradedness of the notions has consequences that have not been met in the traditional approaches:

Reading of graded theorems:

$$\text{Trans}\, R \,\&\, \text{Trans}\, S \rightarrow \text{Trans}(R \cap S)$$

"*The more* both R and S are transitive,
the more their intersection is transitive"

This is *stronger* than

Assume Trans R and Trans S *(to degree 1)*. Then Trans$(R \cap S)$.

$\Rightarrow$ Theorems should be formulated as implications,
proved by chains of provable implications in fuzzy logic

Implicational proofs

Proofs by chains of provable implications in fuzzy logic

= transmission of partial truth

Traditional proofs from 1-valid assumptions

= transmission of full truth only

Non-idempotent & in fuzzy logics (except for G)

⇒ multiple use of premises *matters*

$$\varphi \,\&\, \varphi \to \psi \quad \text{does not entail} \quad \varphi \to \psi$$

⇒ Premises in graded theorems have exponents of multiplicity:

$$\mathsf{Trans}^2 E \ \& \ \mathsf{Sym}\, E \ \to \ \mathsf{Ext}_E\, E$$

Multiple premises

The premises in theorems cumulate independently:

$$\mathsf{Refl}^2\, R \ \&\ \mathsf{Sym}^3\, R \ \&\ \mathsf{Trans}\, R \rightarrow \ \ldots$$

$\Rightarrow$ Meaningful properties:

- *Transitive similarity* . . . $\mathsf{Sim}\, R \ \&\ \mathsf{Trans}\, R$
  $\equiv \mathsf{Refl}\, R \ \&\ \mathsf{Sym}\, R \ \&\ \mathsf{Trans}^2\, R$
- *Reflexive preorder* . . . $\mathsf{Preord}\, R \ \&\ \mathsf{Refl}\, R$
  $\equiv \mathsf{Refl}^2\, R \ \&\ \mathsf{Trans}\, R$
- *Preorder similarity* . . . $\mathsf{Sim}\, R \ \&\ \mathsf{Preord}\, R$
  $\equiv \mathsf{Refl}^2\, R \ \&\ \mathsf{Sym}\, R \ \&\ \mathsf{Trans}^2\, R$
  $\equiv$ *Symmetric double preorder* . . . $\mathsf{Preord}^2\, R \ \&\ \mathsf{Sym}\, R$

$\Rightarrow$ Compound notions are just abbreviations (Sim, Preord, Ord, . . .)
rather than separate properties

The premises in theorems cumulate independently:
$\Rightarrow$ Definitions better without requiring particular properties
of their parameters

The properties of parameters appear only in theorems
in the required multiplicity

Example:

$$\begin{aligned} \text{Upper}\, A &\equiv_{\text{df}} (\forall xy)(Rxy \to (Ax \to Ay)) \\ A^{\uparrow} &\equiv_{\text{df}} \{x \mid (\forall y)(Ay \to Rxy)\} \end{aligned}$$

and dually Lower and $A^{\downarrow}$

Non-graded, these notions are most meaningful if R is a preorder
In the graded approach, we impose no restriction on R,
since theorems require properties with different multiplicities:

$A \subseteq B \to B^{\uparrow} \subseteq A^{\uparrow}$	no precondition on R
Trans $R \to$ Upper $A^{\uparrow}$	Trans R required
Trans^2 $R \to$ (Lower $A^{\downarrow}$ & Upper $A^{\uparrow}$)	Trans^2 R required

4. Advancing Fuzzy Class Theory

State of the art

General aspects:

- Natural language proofs, proof cookbook
- Metamethematical properties of higher-order fuzzy logics

Particular areas of mathematics:

fuzzy relations	-	Běhounek, Cintula, Bodenhofer, Daňková
fuzzy partitions	-	Cintula
fuzzy measures	-	Kroupa, Běhounek
fuzzy numbers	-	Horčík, Běhounek

Next steps:

- Fuzzy quantifiers
- Fuzzy functions
- Fuzzy cardinalities

Recent results

- Běhounek, Daňková: generalized compositions of fuzzy relations
  uniform treatment of fuzzy relational notions
- Cintula: fuzzy partitions via fuzzy quantifiers
  basic concepts generalized
- Horčík: fuzzy interval arithmetics
  dependent Zadeh extensions of arithmetical operations
- Kroupa, Běhounek: fuzzy filters and ultrafilters
  graded rendering of filter theory

⇒ Special session on Wednesday

Thank you for your attention

`www.cs.cas.cz/hp`

$\Rightarrow$ slides, preprints, . . .