Dubois and Prade's fuzzy elements: a challenge for formal fuzzy logic

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Abstract. Although Dubois and Prade's fuzzy elements and gradual sets cannot be represented as objects or fuzzy predicates in first-order versions of known systems of propositional fuzzy logic, they can be represented in higher-order fuzzy logic. However, this logical representation does not simplify handling these notions, compared to informal semantic methods of traditional fuzzy mathematics. Finding a more direct logical representation of Dubois and Prade's notions presents a challenge for formal fuzzy logic, as the notions do not respect certain presuppositions that are in the core of all well-developed systems of formal fuzzy logic (esp. the principle of persistence). Rendering fuzzy elements and gradual sets as logical primitives would therefore require deep design changes of the underlying logic. A preliminary analysis of why this is so is offered.

1 Dubois and Prade's fuzzy elements

In [2], Dubois and Prade introduce the notion of fuzzy element by the following definition:

Definition 1. *Let S be a set and L a complete lattice with top 1 and bottom 0. A* fuzzy *(or* gradual*)* element *e in S is identified with a (partial) assignment function $a_e \colon L \setminus \{0\} \to S$. The partial assignment function can always be made total by defining $a^*_e(\lambda) = a_e(\inf\{\alpha \in \mathrm{Dom}(a_e) \mid \alpha \geq \lambda\})$.*

A prototypical example is the fuzzy middle-point of a fuzzy interval A, defined as the assignment of the middle point of the α-level of A to each $\alpha \in L \setminus \{0\}$. The assignment function need not be monotone nor injective (cf. the middle point of certain asymmetric intervals). Such fuzzy elements are met in many real-life situations (e.g., the average salary of older people).

The proclaimed motivation for introducing fuzzy elements is distinguishing *impreciseness* (i.e., intervals) from *fuzziness* (i.e., gradual change from 0 to 1). A general defining condition for a fuzzy notion is that its cuts be the corresponding crisp notion.

The authors proceed by defining gradual subsets, induced fuzzy sets, and the membership of fuzzy elements to fuzzy sets as follows:[1]

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[1] I generalize their definition to cover infinite cases as well and correct some typos.

Definition 2. *A* gradual subset G *in* S *is identified with its assignment function* $a_G\colon L \setminus \{0\} \to 2^S$.

Definition 3. *The* gradual set G induced by a family $\{e_i\}_{i\in I}$ of fuzzy elements *is defined by* $a_G(\lambda) = \{a^*(\lambda) \mid (\exists i \in I)(a = a_{e_i})\}$ *for all* $\lambda \in \bigcup_{i\in I} \mathrm{Dom}(a_{e_i})$.

Definition 4. *The membership function of the* fuzzy set induced by the gradual set G *is* $\mu_G(s) = \sup\{\lambda \mid s \in a_G(\lambda)\}$.

Definition 5. *A fuzzy element* e belongs *to a fuzzy set* F *iff* $(\forall s \in S)(\forall \lambda \in L \setminus \{0\})[(a_e(\lambda) = s) \Rightarrow (\mu_G(s) \geq \lambda)]$.

Definition 6. *The* degree of membership *of a fuzzy element* e *in a fuzzy set* F *is a fuzzy element of* L *defined by its assignment function* $a_{e\in F} = \mu_F(a_e(\lambda))$ *for all* $\lambda \in \mathrm{Dom}(a_e)$

2 A representation of fuzzy elements in higher-order fuzzy logic

Fuzzy elements and gradual sets represent the horizontal (cut-wise) view of fuzzy sets (systems of cuts), while traditional fuzzy set theory represents fuzzy sets vertically (by membership degrees of elements). First-order fuzzy logic formalizes fuzzy sets by representing them as fuzzy predicates; although the latter can also be represented by cuts, all usual systems of first-order fuzzy logic require that the cuts are nested. This requirement is built already in the propositional core of common fuzzy logics, which all presuppose the *principle of persistence:* if φ is guaranteed to be at least α-true, then it is also guaranteed to be at least β-true for all $\beta \leq \alpha$. Since Dubois and Prade's gradual sets do not meet this requirement (the α-cuts need not be nested), the known systems of first-order fuzzy logic cannot represent them as fuzzy predicates. The aim of this section is to show that gradual sets can nevertheless be represented in *higher-order* versions of known fuzzy logics.

Higher-order fuzzy logic ŁΠ has been introduced in [1] (called *fuzzy class theory* FCT there, since it in fact axiomatizes Zadeh's idea of fuzzy sets of arbitrary orders). It can easily be generalized to a wider class of fuzzy logics; here we shall define it over any logic $\mathcal{F}$ containing MTLΔ [3]. The system seems to be equivalent to fuzzy type theory of [5]; the constructions below can be carried out in the latter as well (in some respects even more directly, as its type hierarchy explicitly contains the type of inner truth values as a basic type). We sketch the basic definitions of higher-order fuzzy logic here for reference; for details, see [1].

Definition 7. *Let* $\mathcal{F}$ *be a logic containing* MTLΔ. Henkin-style higher-order fuzzy logic $\mathcal{F}$ *(denoted by* $\mathcal{F}_\omega$*) is a theory over multi-sorted first-order* $\mathcal{F}$ *with crisp equality and sorts for objects (lowercase variables), classes (uppercase variables), classes of classes (calligraphic variables), etc., with subsorts for* n*-tuples (for all* $n \geq 0$*) in each order. Apart from the obvious necessary function symbols* *and axioms for tuples (tuples equal iff their respective constituents equal etc.), the only primitive symbols are the membership predicates* $\in$ *between successive sorts and the comprehension terms* $\{x \mid \varphi(x)\}$ *for all variables* x *and formulae* φ*. The axioms for* $\in$ *are the following at each order:*

(i) The comprehension axioms: $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$*, where* φ *may contain any parameters and comprehension terms.*

(ii) The extensionality axiom: $(\forall x)\Delta(x \in X \leftrightarrow x \in Y) \to X = Y$.

Definition 8. *All usual abbreviations and definitions known from classical or fuzzy mathematics are assumed (e.g., writing* Rxy *instead of* $\langle x, y\rangle \in R$ *etc.). In particular we define in* $\mathcal{F}_\omega$ *the following operations and properties (at each order of the hierarchy of classes):*

$\bigcup \mathcal{A}$	$=_{\mathrm{df}}$	$\{x \mid (\exists A \in \mathcal{A})(x \in A)\}$	*class union*
$\bigcap \mathcal{A}$	$=_{\mathrm{df}}$	$\{x \mid (\forall A \in \mathcal{A})(x \in A)\}$	*class intersection*
$X \subseteq Y$	$\equiv_{\mathrm{df}}$	$(\forall x)(x \in X \to x \in Y)$	*inclusion*
$A \sqsubseteq B$	$\equiv_{\mathrm{df}}$	$\Delta(A \subseteq B)$	*strict inclusion*
$\mathrm{Crisp}\, A$	$\equiv_{\mathrm{df}}$	$(\forall x)\Delta(x \in A \vee x \notin A)$	*crispness*
$\mathrm{Pow}\, A$	$=_{\mathrm{df}}$	$\{X \mid X \subseteq A\}$	*power class*
$\mathrm{Dom}\, R$	$=_{\mathrm{df}}$	$\{x \mid (\exists y)Rxy\}$	*domain*
$\mathrm{Rng}\, R$	$=_{\mathrm{df}}$	$\{y \mid (\exists x)Rxy\}$	*range*
$\mathrm{Fnc}\, F$	$\equiv_{\mathrm{df}}$	$(\forall xyy')(Fxy \,\&\, Fxy' \to y = y')$	*functionality*

If $\Delta\,\mathrm{Fnc}\, F$*, we can use the functional notation* $y = Fx$ *besides* Fxy*.*

Furthermore for any (definable) k-ary connective $\mathbf{c}$ *we define the corresponding class operation* $\mathrm{Op}_{\mathbf{c}}(A_1, \dots, A_k) =_{\mathrm{df}} \{x \mid \mathbf{c}(x \in A_1, \dots, x \in A_k)\}$*. For binary connectives, we use the usual notation, e.g.,* $A \cap B = \mathrm{Op}_{\&}(A, B)$*,* $\backslash A = \mathrm{Op}_{\neg} A$*,* $\mathrm{Ker}\, A = \mathrm{Op}_{\Delta} A$*,* $\emptyset = \mathrm{Op}_{\bar{0}}$*, etc.*

Dubois and Prade's fuzzy elements and gradual sets can be represented in $\mathcal{F}_\omega$ by means of the (rather standard, cf. [6]) construction of *inner truth values.* Let $\underline{0}$ be an element and $\underline{1} =_{\mathrm{df}} \{\underline{0}\}$. The class $\mathrm{L} =_{\mathrm{df}} \mathrm{Ker}\,\mathrm{Pow}\,\underline{1}$ represents the system of truth values within the theory:

- The truth value of a formula φ is represented by the class $\overline{\varphi} =_{\mathrm{df}} \{\underline{0} \mid \varphi\}$, as by definition, $\overline{\varphi} \sqsubseteq \underline{1}$ and $\varphi \leftrightarrow (\underline{0} \in \overline{\varphi})$.
- Vice versa, every $\alpha \sqsubseteq \underline{1}$ represents the truth value of a formula (e.g., of $\underline{0} \in \alpha$, since $\mathcal{F}_\omega \vdash (\forall \alpha \sqsubseteq \underline{1})(\overline{\underline{0} \in \alpha} = \alpha)$).

The correspondence is an order-isomorphism in the sense that $(\varphi \to \psi) \leftrightarrow (\overline{\varphi} \subseteq \overline{\psi})$ for any formulae φ and ψ.

Furthermore, there is a correspondence between propositional connectives and class operations on L: $\overline{\varphi \,\&\, \psi} = \overline{\varphi} \cap \overline{\psi}$, $\overline{\neg\varphi} = \underline{1} \setminus \overline{\varphi}$, $\overline{\bar{0}} = \emptyset$, etc.; in general, $\overline{\mathbf{c}(\varphi_1, \dots, \varphi_k)} = \underline{1} \cap \mathrm{Op}_{\mathbf{c}}(\overline{\varphi_1}, \dots, \overline{\varphi_k})$ for any definable connective $\mathbf{c}$. Because of this correspondence, the elements of L are called *inner truth values* and class operations on L *formal connectives.*

The correspondence extends to the suprema and infima of truth values, as $\mathcal{F}_\omega$ proves that $\overline{\bigvee_{\alpha \in \mathcal{A}} (\underline{0} \in \alpha)} = \bigcup_{\alpha \in \mathcal{A}} \alpha$ and $\overline{\bigwedge_{\alpha \in \mathcal{A}} (\underline{0} \in \alpha)} = \bigcap_{\alpha \in \mathcal{A}} \alpha$ for any $\mathcal{A} \sqsubseteq \mathrm{L}$.

Notice, however, that here $\mathcal{A}$ is a *class* of inner truth values; by the axiom of comprehension, the union (intersection) exists for any class $\mathcal{A}$, even though the system of *semantic* truth values need not be a complete lattice.[2] Thus from the point of view of the theory, inner truth values always form a complete lattice, even though there may be undefined suprema or infima of some sets of semantic truth values in a particular semantical model.[3] This feature is caused by $\mathcal{F}_\omega$ being a first-order theory and is already well-known from classical metamathematics.

With truth values represented in $\mathcal{F}_\omega$ as elements of L, fuzzy elements can now be defined in $\mathcal{F}_\omega$ as functions from $\mathrm{L} \setminus \{\emptyset\}$ to a basic class S:

Definition 9. *A fuzzy element of S (in higher-order fuzzy logic $\mathcal{F}_\omega$) is any (second-order) class $\mathcal{E}$ such that*

$$\mathrm{Crisp}\,\mathcal{E} \;\&\; \mathrm{Dom}\,\mathcal{E} \sqsubseteq \mathrm{L} \setminus \{\emptyset\} \;\&\; \mathrm{Rng}\,\mathcal{E} \sqsubseteq S \;\&\; \mathrm{Fnc}\,\mathcal{E}$$

It can be made total by defining

$$\mathcal{E}^* = \{\langle \lambda, s\rangle \mid \langle \bigcup\{\alpha \mid (\alpha \geq \lambda) \;\&\; (\alpha \in \mathrm{Dom}\,\mathcal{E})\}, s\rangle \in \mathcal{E}\}$$

Definition 10. *A gradual subset of S in $\mathcal{F}_\omega$ is any (second-order) class $\mathcal{G}$ such that* $\mathrm{Crisp}\,\mathcal{G} \;\&\; \mathrm{Dom}\,\mathcal{E} \sqsubseteq \mathrm{L} \setminus \{\emptyset\} \;\&\; \mathrm{Rng}\,\mathcal{E} \sqsubseteq \mathrm{Ker}\,\mathrm{Pow}\,S \;\&\; \mathrm{Fnc}\,\mathcal{G}$.

Definition 11. *In $\mathcal{F}_\omega$, the gradual set induced by a family $\mathrm{E} = \{\mathcal{E}_i\}_{i\in I}$ of fuzzy elements is defined as $\mathcal{G}\mathrm{r}(\mathrm{E}) = \{s \mid \bigcup\{\lambda \mid (\exists i \in I)(\langle \lambda, s\rangle \in \mathcal{E}_i)\}\}$.*

Definition 12. *In $\mathcal{F}_\omega$, the fuzzy class induced by a gradual set $\mathcal{G}$ is defined as* $\mathrm{Fuz}(\mathcal{G}) = \{s \mid \underline{0} \in \bigcup\{\lambda \mid \langle \lambda, s\rangle \in \mathcal{G}\}\}$.

Definition 13. *In $\mathcal{F}_\omega$, the (Dubois–Prade) degree of membership of a fuzzy element $\mathcal{E}$ in a fuzzy set F is a fuzzy element of* L *defined as* $\mathcal{M}\mathrm{emb}(\mathcal{E}, F) = \{\langle \lambda, \kappa\rangle \mid \lambda \in \mathrm{Dom}\,\mathcal{E} \;\&\; \kappa = \overline{\mathcal{E}(\lambda) \in F} = \{\underline{0} \mid \mathcal{E}(\lambda) \in F\}\}$.

Remark 1. Notice that unlike semantic Definitions 1–6, the apparatus of $\mathcal{F}_\omega$ facilitates further generalizations obtained by dropping the condition $\mathrm{Crisp}\,\mathcal{F}$ resp. $\mathrm{Crisp}\,\mathcal{G}$ in Definitions 9 and 10 (with further adjustments in the definition of Fnc to avoid the crispness of $=$).

3 A challenge for formal fuzzy logic

It can be observed that the formal representatives in $\mathcal{F}_\omega$ of Dubois and Prade's notions are rather complex objects (namely 2nd-order classes). Although this

[2] Only the *safeness* of the structure is required in first-order fuzzy logic, (i.e., the existence of all suprema and infima that are truth values of formulae, see [4]). The theory of all complete structures differs from that of safe structures and need not be axiomatizable in first-order fuzzy logics.

[3] Namely, some of such sets that are not definable by a formula of $\mathcal{F}_\omega$ and thus need not correspond to a class.

does not complicate handling them in the formal framework of higher-order logic (cf., e.g., fuzzy or crisp Dedekind reals which also form a third-order class), their relation to traditional fuzzy sets (i.e., first-order classes), however, is far from perspicuous. Furthermore, the apparatus of formal fuzzy logic does not simplify handling fuzzy elements and gradual sets (unlike traditional fuzzy sets), since they are represented by crisp functions like in their informal semantic treatment.[4] Considering the fundamental role fuzzy elements are to play in Dubois and Prade's recasting of fuzzy set theory, it would certainly be preferable to have fuzzy elements and gradual sets rendered more directly in formal fuzzy logic—as primitive notions rather than complex defined entities. However, this encounters the problems described in the beginning of Section 2, esp. the violation of the principle of persistence by gradual sets.

The reason why the new notions depart so radically from the presuppositions of formal fuzzy logics may reside in the conceptual difference between the approaches to fuzziness in formal fuzzy logic and in traditional fuzzy mathematics (employed by Dubois and Prade). In the latter, fuzzy sets may represent *imprecision* and membership degrees *gradual change.* In formal fuzzy logic, on the contrary, membership degrees are the degrees of *truth;* and fuzzy sets represent rather a continuous decrease of the satisfaction of truth conditions rather than interval-like imprecision. In traditional fuzzy mathematics, membership degrees are just indices parameterizing the membership into a fuzzy set; therefore the operations can be defined regardless of the rules of gradual inference (e.g., by cuts). In formal fuzzy logic, on the other hand, truth degrees are what is preserved under graded inference (i.e., preserved w.r.t. the ordering of truth values—hence the persistence principle). This determines a different treatment of truth degrees in formal fuzzy logic (e.g., most cutworthy definitions are disqualified from the point of view of graded inference, as cuts are generally not preserved under strong conjunction).[5]

Since a direct logical rendering of gradual sets would need to drop the principle of persistence, it would have to adopt an entirely different concept of the truth preservation under inference; such a radical change would consequently affect virtually all logical notions. Unfortunately, many straightforward approaches are not viable, as they would trivialize the theory.[6]

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[4] Recall, however, that by Remark 1, the formal fuzzy setting of $\mathcal{F}_\omega$ allows fuzzifying these notions, which is harder to carry out in their classical models of Definitions 1–6.

[5] It is worth considering if this difference should not result in adopting a new name for "fuzziness" in one of these traditions, thus clarifying the terminology.

[6] E.g., a truth preservation based on the identity (rather than $\leq$) of truth degrees would reduce truth degrees to mere indices exactly in the way traditional mathematics does; however, it would trivialize the logic to classical. From the opposite point of view, this could be an indication that by treating membership degrees as mere indices (rather than truth degrees that should be $\leq$-preserved under graded inference), traditional fuzzy mathematics does not in fact step out of the classical framework; it is gradual inference what makes things genuinely fuzzy, rather than mere employing some set of indices like $[0, 1]$.

No doubt fuzzy elements are a natural notion, abundant in many real-life situations; therefore the above difficulties should not stop us from investigating them. There are no obstacles to investigating them in the traditional framework of fuzzy mathematics (in semantical models, as a formal fuzzy logician would say). Nevertheless, they present a challenge to current formal fuzzy logic, which can render them only indirectly in a higher-order setting: a further analysis is needed whether they can or cannot be treated propositionally or as a primitive first-order notion in a radically new system of fuzzy logic. Until such a logic is developed, we have, unfortunately, to treat fuzzy elements and gradual sets from the formal point of view as just what they are in the higher-order model of Section 2 (and, in fact, in Definition 1 as well)—namely as *functions* from the set of (formal) truth values, rather than a new primitive notion of formal fuzzy logic.

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