Comments on *Fuzzy Logic and Higher-Order Vagueness* by Nicholas J.J. Smith

LIBOR BĚHOUNEK

Smith's article *Fuzzy logic and higher-order vagueness* [12] presents a solution to the problem of artificial precision, encountered in the degree-theoretical semantics of vagueness. The solution is based on fuzzy plurivaluationism, which has been discussed in more detail in Smith's book [11]. Within the degree-theoretical framework, fuzzy plurivaluationism is certainly the appropriate treatment of vague propositions: it has often been implicitly used by formal fuzzy logicians, too—namely, in their modeling of vague concepts by means of formal *theories* over fuzzy logic (see, e.g., [6, 7]). Such theories have (usually infinite) *classes* of models, which directly correspond to Smith's fuzzy plurivaluations. I therefore very much welcome that thanks to Smith, the multi-model fuzzy semantics of vague predicates has been explicitly spelt out in philosophical terms and discussed in the context of the philosophy of vagueness.

For the most part, Smith's article [12] and book [11] only deal with fuzzy (plurivaluationistic) *semantics* of vague predicates, putting aside its logical aspects.[1] Nevertheless, as I try to argue in this Comment, the logical facet of fuzzy plurivaluationism is quite relevant, and it supplements the fuzzy plurivaluationistic picture in important respects. In particular, it can play a rôle in the justification of fuzzy plurivaluationistic semantics, as well as in ascertaining an appropriate characterization of vagueness.

1. Fuzzy plurivaluation as the class of models of a theory over fuzzy logic

In the degree-theoretical semantics of such vague predicates that are based on real-valued quantities (e.g., the predicate *tall,* which is based on the quantity of *height*), the assignment of truth degrees (say, from the $[0,1]$ interval) to the values of the quantity (here, say, in feet) clearly cannot be uniquely determined in a way that the language users would agree upon. There are no facts connected with the use of language, nor any reasonable meaning postulates for such predicates, that would determine whether a $6'0''$ man should be tall to degree 0.7 or 0.8. In other words, as Smith notes, the meaning-determining facts do not narrow down the set of admissible models of vague predicates to a singleton set; so he rightly concludes that instead of a single "fuzzy" model (constituted by a $[0,1]$-valued function from feet to degrees), the degree-theoretical semantics of such predicates consists of a whole class of admissible fuzzy models.

Following Quine and 'Kripkenstein', Smith assumes the meaning-determining facts to be primarily based on the linguistic behavior and intentions of the speakers. However,

[1] The exception is §5.2 in [11], which puts forward a non-standard definition of logical consequence for fuzzy propositions. The aim of the definition is to maintain classical logic for vague predicates; however, it only works for a restricted set of logical connectives.

the speakers' behavior (including their intentions) is rather non-uniform, and sometimes even inconsistent, especially in the case of vague predicates. For instance, even the same speaker can on different occasions make contradictory statements about the tallness of the same person. The set of fuzzy models that such meaning-determining facts would delimit would therefore be not sharp, as assumed in Smith's fuzzy plurivaluationism, but rather vague and unsharp (in a statistical–probabilistic way).

Smith justifies the assumption of sharpness of the set of fuzzy models, de facto, by Ockham's razor (cf. [11, §6.2.2]): since sharp sets of fuzzy models suffice for the elimination of most paradoxes (including the paradox of artificial precision), there is no need to complicate the account by considering unsharp sets of fuzzy models. Such an explanation is, however, just meta-theoretical: it does not offer a deeper explanation within the theory itself as to why the semantics of vague predicates should be a sharp rather than unsharp set of fuzzy models.

Fuzzy plurivaluationistic semantics with sharp sets of fuzzy models in fact conforms better to a different conception of meaning determination, namely one which identifies the meaning of a word with the set of its *meaning postulates,* i.e., its semantic properties and relations that would be approved by competent speakers,[2] and which therefore have to be satisfied by the predicate's truth conditions (in our case, by the assignment of truth degrees to the values of the underlying quantity). These collectively accepted properties of the predicate can be understood as having been abstracted from the Quinean–Kripkean meaning-determining facts, which thus remain the ultimate factors determining the meaning of words. However, in contrast to the latter non-uniform, inconsistent, vague and variable facts, the derived meaning postulates are artificial extrapolations, thereby made consistent, unified, stable and precise. They are, in fact, the defining properties of the predicate, expressible in the rigorous language of logic, that link the values of the underlying quantities to their associated truth degrees. And since these defining properties are sharp, so is the set of membership functions they delimit.

Consider for example the vague predicate *tall.* In the last instance its meaning is certainly determined by the actual behavior and intentions of the speakers, changing over time and mood, and often contradictory. Nevertheless, from this chaotic evidence (and possibly also from the speakers' reflections on the meaning of the word) it is possible to extract the following condition that the intended usage of the term *tall* is presumed to satisfy:

(1) If X is tall, and Y is taller than X, then Y is tall as well.

This condition would be approved by a vast majority of competent speakers (even if they occasionally violate it themselves in some situations), and can be regarded as a meaning postulate for the predicate *tall:* those who would not recognize its validity do not understand the word *tall.* Further conditions on the predicate *tall* that would be

[2] Apparently, the requirement of the speakers' approval makes the set of such postulates vague again. Nevertheless, the number of generally accepted postulates is much smaller than the unmanageable set of the speakers' individual behaviors and intentions, and so a sharp specification can be defended more easily. Meaning postulates adhered to by only a proportion of the speakers can be seen, for example, as distinguishing between two or more alternative meanings of the vague word, which are still specified by a sharp set of defining properties.

approved by the speakers are, for instance, those related to prototypical cases, e.g.:

(2) Michael J. Fox is not tall, while

(3) Christopher Lee is tall.

In the degree-theoretical framework, these meaning postulates for the predicate *tall* can be reformulated as conditions on its membership function. Let us denote the height of an individual x by $h(x)$, and the truth degree of the atomic sentence "x is tall" by Hx. The meaning postulates (1)–(3) then correspond to the following conditions:

(4) $(h(x) > h(y)) \rightarrow (Hx \geq Hy)$

(5) $Ha = 0$

(6) $Hb = 1.$

Conditions of this kind have been mentioned by Smith in [11, §6.1.2]. However, it can be furthermore observed that the conditions (4)–(6) constitute the semantics of certain formulae in fuzzy logic.[3] In particular, they are the semantic conditions for (the full truth of) the following formulae:

(7) $(h(x) > h(y)) \rightarrow (Hx \rightarrow Hy)$, or equivalently, $Hx \,\&\, (h(x) > h(y)) \rightarrow Hy$

(8) $\neg Ha$

(9) $Hb.$

It can be noticed that the latter three formulae represent a straightforward formalization of the meaning postulates (1)–(3) in fuzzy logic. This fact is not accidental: it is a consequence of the manner in which fuzzy logic expresses relationships between the degrees of gradual predicates. Without going into details here, let us just briefly say that the meaning postulates (1)–(3) can be understood as expressing the axioms of a theory in fuzzy logic, straightforwardly formalized by (7)–(9). The class of (fuzzy) models of this theory then forms the fuzzy plurivaluation that represents the semantics of the predicate *tall;*[4] and this class is sharp, since the axioms (7)–(9) are required to be fully true.[5] A similar pattern can be found in the meaning postulates of other vague predicates.

Smith's plurivaluations are thus exactly the classes of models of formal fuzzy theories that straightforwardly formalize the meaning postulates of vague predicates. Since these meaning postulates do not speak directly about the degrees of truth (cf. (1)–(3)), the degrees are usually underdetermined by the theory, and the semantics of a vague predicate is indeed a sharp multi-element class of membership functions. The logical aspects thus elucidate the nature of Smith's plurivaluations.

[3] For detailed information on modern fuzzy logic see [2] or [5].

[4] The meaning postulates for the predicate *tall* are actually more complex than the simplified version (1)–(3) discussed in this Comment. A detailed analysis of the predicate would have to include, i.a., the meaning postulate that "imperceptible changes in height correspond to only negligible changes in the degree of tallness", formalized as the congruence of admissible membership functions w.r.t. a fuzzy indistinguishability relations on heights and degrees of truth (i.e., a certain generalization of Lipschitz continuity); cf. Smith's Closeness principle, [12, §4], discussed below in §2.

[5] Note that in the semantics of fuzzy logic, axioms required to be satisfied to the full degree have always sharp classes of models.

2. Vagueness as semantic indeterminacy, plus optional graduality

In his papers [10, 12] and book [11, ch. 3] Smith gives a definition of vagueness based on the principle of Closeness:

> If a and b are very close/similar in respects relevant to the application of F, then 'Fa' and 'Fb' are very close/similar in respect of truth.

Three objections to this definition have already been raised by Weatherson in [14]. Another problem of this definition can be seen in the fact that it is itself based on the vague terms *very close/similar,* and thus already its application requires an apparatus for handling vagueness. This aspect will be further discussed at the end of this Comment in §4.

As another serious problem with this definition I see its tautologicity: it can be argued that for *all* predicates, similarity in respects relevant to the application of F coincides with similarity in respect of truth—simply because whenever 'Fa' and 'Fb' are dissimilar in respect of truth, then precisely the respects in which a and b differ as regards the application of F are those relevant to its application (and since a and b differ in them, they are dissimilar, too). In other words, the *right* similarity relation that is relevant to the application of F is always the one given by the closeness of truth.[6]

Smith addresses the problem (related to the previous objection) that the Closeness principle is trivially satisfied even by predicates with sharp boundaries or just jump discontinuities by a modification of the Closeness principle [11, §3.3.4], requiring, in essence, the underlying quantity to change gradually at least on a part of the domain. This move, however, binds vagueness by definition to graduality, although their concomitance is not necessary (even if frequent): there are examples of totally bivalent predicates, the assumption of whose graduality would not even make sense, which are still vague: they are susceptible to the sorites paradox and have undecidable (i.e., borderline) cases, including higher-order vagueness. Examples of such predicates are, for instance, bivalent states (such as *pregnant* or *dead*) considered on a time scale several orders of magnitude finer than is the time in which they can change (in the latter cases, e.g., nanoseconds). Even though there is no exact nanosecond in which a woman becomes pregnant, still pregnancy is a bivalent predicate for which no degrees of truth would make sense (not even on the nanosecond scale).[7] Another class of examples are the

[6] Consequently, it is not always the one that straightforwardly suggests itself: for instance, the quantity relevant for the application of the predicate "weighs over 1 kg" [11, p. 142] is, in fact, not weight, but only a purely bivalent quantity derived from weight, which can be physically realized, e.g., by triggering a switch by the needle of a balance when setting the object against the Sèvres prototype of the kilogramme. (Similarly for other examples given by Smith, such as *tall* vs. *height* [11, p. 147], etc.) Even though Smith's intuitions on the quantities underlying the application of predicates are understandable, these quantities include, strictly speaking, many aspects that are *not* relevant for the given predicate. Such aspects then do not partake in the "respects relevant to the application of F", and the similarity in them is *irrelevant* for F. They are, nevertheless, relevant for formal fuzzy logic (which can be regarded as the logic of partial truth based on such gradual quantities), and in this looser sense they do underlie gradual vagueness.

[7] Smith would probably consider as relevant to the application of the predicate such aspects as the degree of penetration of a sperm cell into an egg cell, the progress of nidation, the vitality of the embryo, etc., all of which can change gradually. However, since the predicate "pregnant" (when understood as *stipulated* to be bivalent, for instance for legal purposes) cannot change with these parameters continuously, it does not conform to Smith's definition of vagueness.

predicates *determinately P,* which can be understood as bivalent (due to *determinately*), but arguably are still vague (with borderline cases and subject to the sorites).

Plurivaluationism based on fuzzy logic (as opposed to Smith's fuzzy plurivaluationism based simply on the $[0,1]$-valued semantics) offers a partly different perspective on the nature of vagueness, which avoids these difficulties. Since fuzzy logic admits finitely valued or even bivalent models besides the infinite-valued ones, graduality is not, from the fuzzy-logical perspective, an essential feature of fuzzy plurivaluations. It is contingent on the semantic postulates for a vague predicate (which constitute a theory over fuzzy logic, see §1 above) whether they admit infinite-valued gradual models only (as with the predicate *tall*),[8] or whether they also admit finitely valued or even bivalent models (perhaps even exclusively, as with the predicate *pregnant*). Rather than graduality, it is the semantic indeterminacy (caused by the character of meaning-determining facts, as discussed in §1) which is essential for vagueness.

From the perspective of fuzzy logic, the essence of vagueness is thus semantic indeterminacy, only optionally accompanied by graduality.[9] The semantics of vague predicates is therefore constituted by classes of models, in general fuzzy (because of the possibility that the predicate is also gradual), but as the case may be, also finitely-valued or even just two-valued. Thus although the degree-theoretical semantics of a vague predicate is *in general* a fuzzy plurivaluation, in the special case of bivalent vagueness it is a *classical* plurivaluation. From the viewpoint of fuzzy logic, classical plurivaluationism (or the essentially equivalent classical supervaluationism) is just a special case of fuzzy plurivaluationism: a case adequate for *some* of vague predicates (namely the bivalent ones), though certainly not all of them, as the meaning postulates of many vague predicates allow or even enforce their models to be fuzzy.[10]

As explained in §1, a (fuzzy or classical) plurivaluation is the class of models of a theory over a fuzzy logic (which in the case of classical plurivaluation becomes trivialized to classical logic). This theory expresses the constraints on admissible (fuzzy or classical) models, and so is the essence of what in the case of classical super- or plurivaluationism is called the *penumbral connections.* Thus in terms of fuzzy logic, penumbral connections are nothing else but the predicate's meaning postulates, which formalized as the axioms of a theory over fuzzy logic constrain its class of fuzzy models.

[8] That is, under the more complex semantic analysis of *tall,* taking into account its continuity with respect to height: see footnote 4 above.

[9] The difference between graduality on the one hand and vagueness as indeterminacy on the other hand is stressed also by Dubois and Prade [4, 3]; cf. also Zadeh quoted in [8, §2] in this volume.

[10] The reader may wonder how the jolt problem, i.e., the supertruth of the existence of a sharp transition point (such as the first nanosecond for the predicate *pregnant;* see [12, §6]) can be eliminated for bivalent vague predicates, considering that the fuzzy-plurivaluationistic solution is based on graduality. For bivalent predicates, however, the problem resides not in the neglect of graduality (which is not present), but rather must reside in a wrong choice of structures employed for modeling the extensions of predicates. Take for instance the predicate *pregnant* on the scale of nanoseconds. If time is modeled by a complete lattice (e.g., as usual, a bounded interval of either real numbers representing time instants or natural numbers representing successive time intervals) and standard sets are taken for extensions of predicates, then every predicate extension, including that of *pregnant,* necessarily has a first instant (i.e., "the first nanosecond") in every model. However, in non-standard models (formalized, e.g., in Vopěnka's Alternative Set Theory, [13]), subclasses of natural numbers, though bivalent, need not have first elements. Using such non-standard models of time might therefore avoid the jolt problem for *pregnant* and similar bivalent vague predicates (including *determinately P,* which is subject to the jolt problem even in fuzzy plurivaluationism).

3. Supertruth as deducibility in fuzzy logic

As correctly observed by Smith, fuzzy plurivaluationistic semantics for gradual vague predicates solves the problem of artificial precision. It also provide answers to other frequent objections to degree theories of vagueness and fuzzy logic.

For instance, the linear ordering of the system of truth degrees is frequently criticized, on the basis that for incommensurable pairs of gradual predicates (such as *green* and *big*) it makes no sense to compare their truth degrees (e.g., to say that a ball is more green than big, or vice versa). Even though fuzzy logic does employ linear systems of degrees (most often, the real unit interval), fuzzy plurivaluationism answers the objection in a convincing way (described in [11, §6.1.4], here slightly simplified): since the degrees of both properties vary across admissible fuzzy models, and since the incommensurable predicates are not tied by any meaning postulates (or "penumbral connections"), the sentence p: "a is more P than Q" is true in some models (in which the degree of Pa is larger than that of Qa), but false in others (in which the opposite is true). However, since the semantics of the predicates P and Q is the *whole class* of fuzzy models admitted by their meaning postulates, neither the sentence p nor its negation can be claimed—none of them is *supertrue.* Consequently, the truth status of the sentence p is not semantically determined, and so the properties are indeed incommensurable, even if a linear system of mutually comparable truth values is employed in each fuzzy model.

As seen from this example, for the assessment of (super)truth of sentences involving vague predicates one needs to know which of them are true in all models forming the plurivaluation. However, since the plurivaluation is the class of models of a fuzzy theory that formalizes meaning postulates, the latter question is equivalent to asking which sentences are true in all fuzzy models of this theory; in other words, which are its *consequences in fuzzy logic.* As a matter of fact, the consequence relation of fuzzy logic and the corresponding deduction rules have literally been *designed* to determine the supertruth of sentences of fuzzy plurivaluationism.[11]

4. Plurivaluations taken seriously

Smith's analysis of meaning-determining facts suggests that plurivaluations, be they fuzzy or classical, are to be taken seriously. The meaning of a vague predicate is the *whole* class of (fuzzy or classical) models, and there is nothing that would determine the meaning more narrowly. As already discussed in the previous paragraph, the only semantically grounded statements about such predicates are therefore the "supertrue" sentences—i.e., those valid in *all* fuzzy models admitted by the meaning postulates.

These facts are often overlooked, and *single* membership functions are frequently taken for the semantics of vague predicates (especially in the fuzzy literature, e.g., [15]). This not only contradicts the arguments pointing to fuzzy plurivaluationism, but also leads to inadequate models that retain graduality (which is just accidental to vagueness) while indeterminacy (which is substantial for vagueness) has been removed. Consequently, what is modeled are no longer vague, but artificially precisified gradual predicates. The artificial precisification can be an expedient simplification in technical practice. It should be, however, kept in mind that the precisification is in most cases com-

[11] From the perspective of fuzzy logic, however, exactly the converse is the case: fuzzy plurivaluationism is just the semantic representation of what derivations in fuzzy logic are about.

pletely arbitrary, and that taking a slightly different membership function can lead to radically different results.[12] Unless the choice of a particular membership function is justified by some aspect of the technical application, the results may be just artifacts of the choice and have little in common with the original vague predicates modeled.

In some cases, the neglect of the plurivaluationistic nature of vague predicates is subtler, but still casts a shadow of doubt on the meaningfulness of the model. At the very least, the use of specific membership degrees in these models calls for explanation.

Consider, for example, the use of a fixed (though arbitrary) threshold r in Cerami and Pardo's r-semantics for counterfactuals (see [1, §5] in this volume). Justification of its meaningfulness would require clarification of the notion of possible world used in the definition: if the meaning of a vague predicate in a possible world (e.g., the meaning of *tall* in a world in which I measure $6'4$ instead of my actual height) is a fuzzy plurivaluation, then a sentence (e.g., "I am tall") may exceed the fixed threshold r in only some of the models from the fuzzy plurivaluation; exceeding the threshold is then a semantically meaningless criterion. A fixed threshold could only be meaningful if possible worlds were comprised of precisified membership functions; however, it remains to be clarified whether such a conception of possible worlds is reasonable (this question is loosely connected with the problem whether vagueness is in the world or only in language).

Similarly it should be made clear what is the meaning of specific truth degrees assigned by the speakers to vague propositions in Sauerland's study ([9, §2.2] in this volume), considering that the semantics of a vague predicate is in fact a fuzzy plurivaluation, and so the truth degrees of such propositions are only determined up to certain limits. A more careful analysis would probably find out that rather than degrees of *truth* (which are not uniquely determined), Sauerland actually studies (something like) the degrees of the speakers' subjective agreement with statements (which, on the other hand, might be unique and ascertainable). It is then no wonder that the degrees do not follow the patterns of fuzzy logic, as they represent a completely different (most probably even non-truth-functional) modality than that studied by fuzzy logic.

Finally, the plurivaluationistic nature of vague predicates poses another problem for Smith's definition of vagueness, or rather for its coherence with fuzzy plurivaluationism. As noted in §2 of this Comment, the Closeness principle is based on the vague notion of "very close/similar". According to fuzzy plurivaluationism, the meaning of this term is a *class* of fuzzy models of the closeness or similarity relation, rather than a single such model. Across these models, there may be considerable differences in the truth value of the sentence "a and b are very close/similar in respects relevant to the application of F", as well as the sentence " 'Fa' and 'Fb' are very close/similar in respect of truth". Whether the condition of the Closeness principle for F is satisfied therefore can (and for many vague predicates F probably also will) vary across the models in the plurivaluation that represents the meaning of the principle; yet no fact determining the meaning of this

[12] For instance, the truth value of $Ax \rightarrow Bx$ in standard Gödel fuzzy logic is 1 if the truth values of Ax and Bx both equal 0.4, but drops to as low as 0.39 if the value of Bx is decreased by just 0.01. The truth value of the proposition can thus be radically different even for very close membership functions (here, differing just by 0.01). A similar effect can occur even in standard Łukasiewicz logic despite the continuity of truth functions of all its propositional connectives, as the truth functions of more complex formulae can grow very rapidly and yield large differences for negligible changes in degrees.

principle can decide between these models. Consequently, the proposition "F is vague" is itself vague, and its truth status is for many predicates F undetermined, being neither supertrue nor superfalse in the plurivaluation that represents its meaning.

Even if one does not adopt fuzzy plurivaluationism as the theory of vagueness that should apply to the Closeness principle, still the undeniable semantic underdeterminacy of the expression "very close/similar" makes any application of this definition problematic and calls for an explanation. The conception sketched above in §2, which identifies vagueness with the very indeterminacy of meaning (and only optional graduality), on the other hand, does not refer to the vague concept of closeness/similarity, and therefore does not suffer from this problem.

These arguments notwithstanding, it should be stressed that the Closeness-based definition of vagueness is not central to Smith's degree-theoretical approach, nor to the fuzzy-plurivaluationistic solution to the problem of artificial precision. In [11] and [12] it in fact only plays a rôle of a supporting argument, from which the rest of the theory is essentially independent. From the perspective of fuzzy logic, Smith's fuzzy plurivaluationism represents adequate degree-theoretical semantics for gradual vagueness. As I tried to hint here, fuzzy logic might contribute to its picture by elucidating the nature of fuzzy plurivaluations (as the models of meaning postulates formalized in fuzzy logic), proposing a different definition of vagueness, and supplementing the picture with classically plurivaluational, bivalent (or finite-valued) vagueness.

BIBLIOGRAPHY

[1] Cerami M., Pardo P.: Many-valued semantics for vague counterfactuals. This volume, 2011.
[2] Cintula P., Hájek P., Noguera C. (eds.): *Handbook of Mathematical Fuzzy Logic.* College Publications, to appear in 2011.
[3] Dubois D.: Have fuzzy sets anything to do with vagueness? This volume, 2011.
[4] Dubois D., Prade H.: Gradual elements in a fuzzy set. *Soft Computing* 12: 165–175, 2008.
[5] Hájek P.: *Metamathematics of Fuzzy Logic.* Kluwer, 1998.
[6] Hájek P., Novák V.: The sorites paradox and fuzzy logic. *International Journal of General Systems* 32: 373–383, 2003.
[7] Novák V.: A formal theory of intermediate quantifiers. *Fuzzy Sets and Systems* 159: 1229–1246, 2008.
[8] Prade H., Schockaert S.: Handling borderline cases using degrees: an information processing perspective. This volume, 2011.
[9] Sauerland U.: Vagueness in language: the case against fuzzy logic revisited. This volume, 2011.
[10] Smith N.J.J.: Vagueness as closeness. *Australasian Journal of Philosophy* 83: 157–183, 2005.
[11] Smith N.J.J.: *Vagueness and Degrees of Truth.* Oxford University Press, 2008.
[12] Smith N.J.J.: Fuzzy logic and higher-order vagueness. This volume, 2011.
[13] Vopěnka, P.: *Mathematics in the Alternative Set Theory.* Teubner, 1979.
[14] Weatherson B.: Three objections to Smith on vagueness. Manuscript, published online at http://brian.weatherson.org/totsov.pdf (accessed October 2011), 2004.
[15] Zadeh L.: Fuzzy logic and approximate reasoning. *Synthese* 30: 407–428, 1975.

Libor Běhounek
Institute of Computer Science
Academy of Sciences of the Czech Republic
Pod Vodárenskou věží 2
182 07 Prague 8, Czech Republic
Email: behounek@cs.cas.cz