Extending Cantor–Łukasiewicz Set Theory with Classes

Libor Běhounek

Institute of Computer Science, Academy of Sciences of the Czech Republic
Pod Vodárenskou věží 2, 182 07 Prague 8, Czech Republic
behounek@cs.cas.cz

As conjectured by Skolem [8] and proved by White [10], naïve set theory with the unrestricted axiom schema of comprehension, which is inconsistent over classical logic due to Russell's paradox, turns out to be consistent over infinite-valued Łukasiewicz logic. Hájek [5, 3] studied the theory under the name *Cantor–Łukasiewicz set theory* (denoted by CŁ further on)[1] and showed several negative results on arithmetic over CŁ. Additionally, some basic constructions (such as kernels of fuzzy sets) are in general undefinable in CŁ on pain of contradiction, as any bivalent or finitely-valued operator makes it possible to reproduce Russell's paradox. These facts cast serious doubts on Skolem's conjecture that a large part of mathematics could be formalized in the theory.

Here I suggest to remedy the drawbacks of CŁ by extending the theory with classes, in a similar manner as von Neumann–Bernays–Gödel's classical set theory NBG extends Zermelo–Fraenkel's ZF. Besides a few observations on the features and expressive power of the resulting theory CŁC, I discuss its motivational aspects and compare it with two set theories with classes over classical logic (NBG and Vopěnka's [9] AST).

1 Cantor–Łukasiewicz Set Theory with Classes

Cantor–Łukasiewicz set theory CŁ is a theory over first-order Łukasiewicz infinite-valued logic Ł$\forall$ (see, e.g., [4]) with the only primitive predicate $\in$ and the set comprehension terms $\{x \mid \varphi(x)\}$ governed by the comprehension axioms $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$, for all formulae φ. The extension CŁC of CŁ by classes can be defined as follows:

Definition 1. CŁC *is a theory over two-sorted first-order Łukasiewicz logic with the connective* $\triangle$ *(*Ł$\forall_\triangle$*, see, e.g., [4]). The language of* CŁC *consists of:*

- *The sort of variables for* sets *(lowercase letters)*
- *The sort of variables for* classes *(uppercase letters)*
- *The primitive membership predicate* $\in$ *between sets (*set membership predicate, *or* set-in-set membership*)*
- *The primitive predicate of membership of sets in classes (*class membership predicate, *or* set-in-class membership*, denoted also by* $\in$*, as the two are always distinguishable by the type of arguments)*
- Set comprehension terms $\{x \mid \varphi\}$ *(of the set sort) for any set formula (see below)* φ
- Class comprehension terms $[x \mid \varphi]$ *(of the class sort) for any formula* φ *of* CŁC

[1] In [5] and several follow-up articles, the theory is denoted by $CŁ_0$, while CŁ denotes a certain inconsistent extension of $CŁ_0$. For notational simplicity, we shall use the name CŁ for Hájek's $CŁ_0$, since the inconsistent theory is of a very limited interest.

Set formulae *are those that contain no* $\triangle$ *nor any class term. The axioms of* CŁC *are the following, for any set formula* φ *and any formula* ψ*:*

- Set comprehension axioms: $\varphi(y) \leftrightarrow y \in \{x \mid \varphi(x)\}$
- Class comprehension axioms: $y \in [x \mid \psi(x)] \leftrightarrow \psi(y)$
- Class extensionality axioms: $(\forall x)\triangle(x \in A \leftrightarrow x \in B) \rightarrow (\psi(A) \leftrightarrow \psi(B))$

Classes of CŁC are intended to represent crisp or fuzzy subsets of models of CŁ: class comprehension axioms ensure the existence of any class delimited by a property expressible in the language of CŁC. Notice that the logical vocabulary of CŁC contains the connective $\triangle$, which allows us, i.a., to speak about crisp collections of objects in models. Unrestricted set comprehension, however, only applies to set formulae, in which $\triangle$ is forbidden. In fact, the set fragment of CŁC coincides with CŁ:

Theorem 1. CŁC *is a conservative extension of* CŁ *(therefore is consistent).*

Proof. Every model M of CŁ can be extended to a model M' of CŁC by interpreting class variables as ranging over fuzzy classes of set-objects (i.e., membership functions from the universe of M to the algebra of truth values) and realizing the set-in-class membership predicate accordingly (namely, defining the values of set-in-class membership as the degrees provided by these membership functions): the validity of the axioms of CŁC in M' is easily seen. The conservativeness then follows (by the strong completeness of Ł$\forall$ and Ł$\forall_\triangle$, see, e.g., [6]) from the fact that the truth values of set formulae only regard the elements of M (as set formulae cannot contain class terms and the semantics is compositional).

It can be seen that the axioms for classes are the same as those of Henkin-style monadic second-order fuzzy logic Ł$\forall$, analogous to that of [1, §3]. CŁC can thus be understood as a fuzzy class theory over the universe of CŁ.

Even though a hierarchy of higher-order classes over the CŁ-universe could be introduced in the same way as in [1, §5], many classes of classes (e.g., the partition of Theorem 2(5) below) can be encoded in a rather standard way (cf. [9, §I.5–6] for AST) by first-order relations,[2] understanding a (class) binary relation R together with a class A as encoding the class $\mathcal{K}$ of classes X with $X \in \mathcal{K} \equiv_{\mathrm{df}} (\exists i \in A)(X = [j \mid Rij])$. Obviously, tuples (or set-indexed systems) of classes and usual higher-order class operations (e.g., class intersection or union) can be encoded in CŁC as well.

2 Extensionality and intensionality

In CŁC, classes are construed as extensional (i.e., determined by their membership functions), as they are intended to represent (crisp or fuzzy) collections of objects in models. The axiom of class extensionality indeed ensures that any two classes with the same membership function (i.e., with the same degrees of membership of all elements) are intersubstitutable salva veritate. Since intersubstitutivity (which in Ł$\forall_\triangle$ is a crisp relation) can be regarded as the logical identity (as factoring a model of CŁC by the intersubstitutivity relation does not change the truth values of formulae), we can define:

[2] See [5] for handling ordered pairs in CŁ.

Definition 2. *In* CŁC*, we define:* $A = B \equiv_{\mathrm{df}} (\forall x)\triangle(x \in A \leftrightarrow x \in B)$.

On the other hand, CŁ-sets are not extensional. Recall from [5] that two different set equalities are introduced in CŁ: the provably crisp *Leibniz equality* $=$ and the (provably fuzzy) *extensional equality* $\approx$ (denoted by $=_{\mathrm{e}}$ in [5] and its follow-ups), defined as $x = y \equiv_{\mathrm{df}} (\forall u)(x \in u \leftrightarrow y \in u)$ and $x \approx y \equiv_{\mathrm{df}} (\forall u)(u \in x \leftrightarrow u \in y)$. Leibniz equality ensures intersubstitutivity salva veritate (so it can be identified with the logical identity predicate), while extensional equality (which will be also called *co-extensionality* further on) does not (though it is also a fuzzy equivalence relation). Leibniz equality implies extensional equality, $x = y \rightarrow x \approx y$, but it is inconsistent to assume $x = y \leftrightarrow x \approx y$ in CŁ. Hájek has actually proved in [3] that there are infinitely many set terms which are all provably co-extensional with (e.g.) $\emptyset =_{\mathrm{df}} \{x \mid 0\}$ while being Leibniz non-identical.

Even though CŁ-sets are not extensional, in CŁC we can define their *extensions,* i.e., the classes of their elements:

Definition 3. *In* CŁC *we define the* extension *of a set x as the class* $\mathrm{Ext}\,x =_{\mathrm{df}} [q \mid q \in x]$.

The definitions of extension and co-extensionality can be extended to classes by setting $\mathrm{Ext}A =_{\mathrm{df}} A$; $A \approx x \equiv_{\mathrm{df}} (\forall q)(q \in A \leftrightarrow q \in x)$ and analogously for $x \approx A$ and $A \approx B$. The following observations are easily obtained:

Theorem 2. CŁC *proves:*

1. $A = B \leftrightarrow \triangle(A \approx B)$*, by the axiom of class extensionality*[3]
2. $x \approx y \leftrightarrow \mathrm{Ext}\,x \approx \mathrm{Ext}\,y$*, and similarly for* $A \approx x$ *and* $A \approx B$
3. $\mathrm{Ext}\,\{x \mid \varphi\} = [x \mid \varphi]$
4. $\approx$ *is a fuzzy equivalence relation which partitions the set universe into fuzzy blocks* $\{x\}_{\approx} =_{\mathrm{df}} \{q \mid q \approx x\}$ *that satisfy* $\{x\}_{\approx} \approx \{y\}_{\approx} \leftrightarrow x \approx y$ *and* $x \in \{x\}_{\approx}$
5. *The crisp equivalence relation of full co-extensionality* $\triangle(x \approx y)$ *partitions the set universe into crisp class blocks* $[x]_{\approx} =_{\mathrm{df}} [q \mid \triangle(q \approx x)]$

In contrast to NBG or AST, it is not the case in CŁC that all sets are classes and only some classes are sets. Nevertheless, every set is in CŁC fully co-extensional with a class (namely, its extension), and only some classes are fully co-extensional with sets. This motivates the following definition of (im)proper classes in CŁC:

Definition 4. *In* CŁC*, we say that a class A is* proper *if* $\neg(\exists x)\triangle(x \approx A)$*, and* improper *(or a* set extension*) if* $(\exists x)\triangle(a \approx A)$.

Examples of improper classes are the empty class $\Lambda =_{\mathrm{df}} [x \mid 0]$, the universal class $\mathrm{V} = [x \mid 1]$, and generally $\mathrm{Ext}\,x$ for any set x. By Yatabe's overspill theorem [11], an example of a proper class is the class FN of standard natural numbers (similarly as in AST;[4] details are omitted here for space restrictions).

Though not yet proved for CŁ, a claim analogous to one valid for naïve set theory over the logic BCK (see [7]) has been conjectured by Terui:[5]

$$\text{CŁ} \vdash \{x \mid \varphi\} = \{x \mid \psi\} \text{ iff } \varphi \text{ and } \psi \text{ are syntactically identical}$$

[3] Though only $x = y \rightarrow \triangle(x \approx y)$ is provable for sets by Hájek's result of [3] cited above.

[4] Although the theories differ in that AST has, so to speak, a 'strange' structure of classes over 'common' finite sets, while CŁC has 'common' fuzzy classes over a 'strange' structure of sets.

[5] Yatabe, pers. comm.

Even though this feature might be viewed as a defect that trivializes CŁ, it would nevertheless make a good sense in CŁC, as it would make the distinction in CŁC between sets and classes parallel Frege's [2] distinction between *Sinn* (sense, or intension) and *Bedeutung* (meaning, or extension): indeed, the extensional CŁC-class $[x \mid \varphi(x)]$ represents the *collection of instances* of the property $\varphi(x)$—or its *extension;* while the intensional CŁC-set $\{x \mid \varphi(x)\}$ represents (by Terui's conjecture, exactly; otherwise partly) the *way* the property φ is defined—i.e., its *sense* (or intension). Thus it is not counter-intuitive if, e.g., CŁ proves $\{x \mid \varphi \vee \psi\} \neq \{x \mid \psi \vee \varphi\}$, as the two sets, though co-extensional, are presented in different ways. (Arguably, this is a *desired* feature in naïve set theories.)

3 On the motivation of CŁC

It may be objected that classes destroy the appealing simplicity of the full comprehension principle in CŁ. Nevertheless, they only represent fuzzy or crisp classes that are anyway present in the models of CŁ, and they make it possible to handle many natural constructions (such as kernels of fuzzy sets, including, e.g., FN and Ker(ω) as models of arithmetic) within the theory. The features of CŁC (the existence of a universal set, the distinction between intensional sets and extensional classes, the properness of the class of standard natural numbers, etc.) suggest that CŁ-sets may provide a sufficiently rich ground structure for a mathematically non-trivial class theory over CŁ.

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