On a graded notion of t-norm and dominance

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***Abstract*—The paper studies graded properties of MTL$_\triangle$-valued binary connectives, focusing on conjunctive connectives such as t-norms, uninorms, aggregation operators, or quasi-copulas. The graded properties studied include monotony, a generalized Lipschitz property, unit and null elements, commutativity, associativity, and idempotence. Finally, a graded notion of dominance is investigated and applied to transmission of graded properties of fuzzy relations. The framework of Fuzzy Class Theory (or higher-order fuzzy logic) is employed as a tool for easy derivation of graded theorems on the connectives.**

I. Introduction

In the early 1990's, Gottwald introduced what he called *graded properties of fuzzy relations* [1], [2], [3]: an approach in which it is possible to deal with partial—*graded*—fulfillment of properties like reflexivity, transitivity, etc. In this approach, it is not only possible to define properties in a graded way, but also to generalize theorems on fuzzy relations so that their conclusions hold to degrees depending on the degrees to which the relations fulfill the preconditions of the theorems.

Thus while traditional theory of fuzzy relations proves theorems of the following form:

If the assumptions $\varphi_1, \dots, \varphi_n$ are fully true
then the conclusion ψ is fully true,

under the graded approach theorems of the following form are proved:

The more the aggregation of (graded) assumptions
$\varphi_1, \dots, \varphi_n$ is true (even if partially),
the more the (graded) conclusion ψ is true.

The 'graded' theorems thus provide lower bounds on the truth degree of the conclusion, given by the aggregation (usually by a left-continuous t-norm) of the truth degrees of the graded assumptions.

Even though these ideas are sound and meaningful (and paralleling similar efforts, e.g., in fuzzy topology, cf. [4]), Gottwald's approach unfortunately found only little resonance (with a few exceptions, e.g., [5], [6]), possibly because the proofs were too complex in the traditional, non-axiomatic framework.

With the advent of Fuzzy Class Theory (FCT) [7], a formal axiomatic framework is available in which it is just natural to consider properties of fuzzy relations in a graded manner. Graded notions can in FCT be inspired by (and derived from) the corresponding notions of classical mathematics [8]; the syntax of FCT is close to the syntax of classical mathematical theories and the proofs in FCT thus resemble the proofs of the corresponding classical theorems. Therefore, it is technically easier to handle graded properties of fuzzy relations than in Gottwald's previous works and it is possible to access deeper results. For a treatment of basic graded properties of fuzzy relations (esp. of fuzzy preorders and similarities) in the framework of FCT see [9].

Here we apply the graded approach to binary fuzzy connectives, or truth-value operators. Many crisp classes of such operators (e.g., t-norms, uninorms, copulas, negations, etc.) can be defined by formulae of FCT. The apparatus, however, enables also *partial* satisfaction of their defining conditions. In the following, we therefore give several *fuzzy* conditions on truth-value operators and use them in theorems as graded preconditions that need not be satisfied to the full degree. This yields a completely new *graded* theory of truth-value operators and allows non-trivial generalizations of well-known theorems on such operators, including their consequences for properties of fuzzy relations.

Even though we do not have any particular application in mind, the theory developed here can be useful in many situations: whenever, e.g., a t-norm is slightly distorted, for instance by noise or just by rounding, it is actually no longer a t-norm and theorems on t-norms say nothing about the function. For example, a function resulting from the product t-norm by adding a random noise with the maximal amplitude 0.001 need not be commutative nor associative, so the well-known theorems on t-norms are not applicable. Nevertheless, it is obvious that the function approximates the product t-norm very closely, and that many (though not all) of its properties will be very close to the properties of the product t-norm itself. However, unless we can (i) measure the degree of corruption of its commutativity, associativity, etc., and (ii) derive theorems on how these degrees propagate to other properties, we possess no information on which properties of t-norms almost apply to the distorted function (and to what degrees). The formalism of FCT developed in this paper will allow us to make such estimates: with a suitable specification of parameters (namely taking the standard Łukasiewicz logic for the ground logical apparatus), we are able to capture the intuition that the above function is commutative at least to degree 0.999, and to estimate the degrees to which it shares other properties with t-norms—e.g., that it dominates the minimum t-norm at least to degree 0.997 (by Theorem 5.2 (D2) below).

We aim this paper at researchers in the theory and applications of fuzzy connectives and fuzzy relations to attract their interest to their graded properties. In this paper we focus on basic graded properties of binary fuzzy connectives (see Section II), namely the graded notions of commutativity, associativity, idempotence, unit and null elements, monotony, and a generalized Lipschitz property (Section III–IV). Moreover we study a graded notion of dominance (Section V) and apply it to transmission of graded properties of fuzzy relations (Section VI), generalizing the results of [10], [11]. Due to space restrictions, we omit all proofs; they can be found in the paper under preparation [12].

The definitions and results presented here are formulated in the framework of Fuzzy Class Theory, for which see the original paper [7] or the freely available primer [13]. The definitions of FCT notions used in this paper are given in the Appendix.

Even though the theorems of FCT are sound to a broader class of models, readers unfamiliar with the logic $\mathrm{MTL}_\triangle$ can always translate our results into the intended $[0,1]$-valued semantics via the following table, of an arbitrary universe of discourse U and a left-continuous t-norm $*$ (with the residuum $\Rightarrow_*$):

FCT	**Fuzzy relations**
object variable x	element $x \in U$
(fuzzy) class A	fuzzy set $A \in \mathcal{F}(U)$
unary predicate	fuzzy subset of U, $\mathcal{F}(U)$, etc.
binary predicate	binary f. rel. on U^2, $(\mathcal{F}(U))^2$, etc.
strong conjunction &	left-continuous t-norm $*$
implication $\rightarrow$	residual implication $\Rightarrow_*$
weak conjunction $\wedge$	minimum
weak disjunction $\vee$	maximum
negation $\neg$	the function $\neg x = (x \Rightarrow_* 0)$
equivalence $\leftrightarrow$	bi-residuum: $\min(x \Rightarrow_* y, y \Rightarrow_* x)$
universal quantifier $\forall$	infimum
existential quantifier $\exists$	supremum
predicate $=$	crisp identity
predicate $\in$	evaluation of membership function
class term $\{x \mid \varphi(x)\}$	f. set def. as $Ax = \varphi(x)$, for $x \in U$

The particular choice of a left-continuous t-norm in the above table should reflect the intended 'distance' of degrees in $[0,1]$, by which the defects of properties are measured. For example, the choice of the Łukasiewicz t-norm for the interpretation of & corresponds to the Euclidean distance of degrees (since the equivalence connective, which fuzzifies the equality of degrees in our formulae, comes out as one minus the Euclidean distance in standard Łukasiewicz logic). Other choices of the background fuzzy logic put different stress on different degrees: e.g., standard product logic is stricter on small degrees.

II. Binary fuzzy connectives

An important feature of FCT is the absence of variables for truth degrees: in FCT, truth degrees are the semantic values of formulae rather than objects of the theory (see [8] for an explanation of methodological advantages of this approach). However, many theorems of traditional fuzzy mathematics do speak about truth values or quantify over operators on truth values like aggregation operators, copulas, t-norms, etc. In order to be able to speak of truth values within FCT, truth values need be *internalized* in the theory. This is done in [14] by a rather standard technique, by representing truth values by subclasses of a crisp singleton.

The details of the representation are not important in the present paper; we refer the interested readers to [14, Sect. 3]. For our present purposes it is fully sufficient to assume that we do have variables $\alpha, \beta, \ldots$ for truth values in FCT, and that the ordering of truth values and the usual propositional connectives and the quantifiers $\forall, \exists$ are definable in FCT. The class of the internal truth values will denoted by L.

Binary operators on truth values (including propositional connectives $\&, \wedge, \vee, \ldots$) can be regarded as functions $\mathbf{c}\colon \mathrm{L} \times \mathrm{L} \rightarrow \mathrm{L}$ or, equivalently, as fuzzy relations $\mathbf{c} \sqsubseteq \mathrm{L} \times \mathrm{L}$. Consequently, graded class relations can be applied to such operators, e.g., fuzzy inclusion $\mathbf{c} \subseteq \mathbf{d} \equiv (\forall \alpha\beta)(\alpha \mathbf{c} \beta \rightarrow \alpha \mathbf{d} \beta)$, which means $\bigwedge_{\alpha,\beta}(\alpha\, \mathbf{c}\, \beta \Rightarrow_* \alpha\, \mathbf{d}\, \beta)$ in models.

Convention 2.1: We shall always use Greek letters for truth values, and the letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ for binary connectives. In formulae, internal binary connectives will by convention have the same priority as &: thus, e.g., $\neg\alpha\ \mathbf{c}\ \beta\ \rightarrow\ \gamma$ means $((\neg\alpha)\, \mathbf{c}\, \beta) \rightarrow \gamma$. (See Convention A.2 below in Appendix A for further abbreviations used in formulae, esp. $\Rightarrow$, $\longrightarrow$, $\longleftrightarrow$, and φ^n.)

Graded properties of connectives assign each connective a truth value indicating the degree to which the connective possesses the graded property. A graded property can thus be regarded as a crisp function from connectives to truth values, or equivalently as a fuzzy class of connectives.

III. Basic graded properties of binary connectives

Now we turn our attention to graded properties of binary connectives. We start with the simplest case, graded generalizations of unit and null elements. Following our general methodology, they are obtained by replacing $=$ in classical definitions by $\leftrightarrow$.

Definition 3.1: In FCT, we define the following graded properties of a binary connective $\mathbf{c}$:

$$\mathrm{Unit}(\mathbf{c}, \eta) \equiv_{\mathrm{df}} (\forall\alpha)((\eta\, \mathbf{c}\, \alpha \leftrightarrow \alpha)\ \&\ (\alpha\, \mathbf{c}\, \eta \leftrightarrow \alpha))$$
$$\mathrm{Null}(\mathbf{c}, \eta) \equiv_{\mathrm{df}} (\forall\alpha)((\eta\, \mathbf{c}\, \alpha \leftrightarrow \eta)\ \&\ (\alpha\, \mathbf{c}\, \eta \leftrightarrow \eta))$$

Furthermore, we define the single-sided variants:

$\mathrm{LUnit}(\mathbf{c}, \eta) \equiv_{\mathrm{df}} (\forall\alpha)(\eta\, \mathbf{c}\, \alpha \leftrightarrow \alpha)$	Left-unit element
$\mathrm{LNull}(\mathbf{c}, \eta) \equiv_{\mathrm{df}} (\forall\alpha)(\eta\, \mathbf{c}\, \alpha \leftrightarrow \eta)$	Left-null element
$\mathrm{RUnit}(\mathbf{c}, \eta) \equiv_{\mathrm{df}} (\forall\alpha)(\alpha\, \mathbf{c}\, \eta \leftrightarrow \alpha)$	Right-unit element
$\mathrm{RNull}(\mathbf{c}, \eta) \equiv_{\mathrm{df}} (\forall\alpha)(\alpha\, \mathbf{c}\, \eta \leftrightarrow \eta)$	Right-null element

When fully true, the properties yield the traditional non-graded properties. Unlike in classical case, the both-sided notions cannot be defined as the conjunction of both single-sided variants. Only the following estimates hold generally:

Theorem 3.2: FCT proves:

(B1) $\mathrm{LUnit}(\mathbf{c},\eta)\ \&\ \mathrm{RUnit}(\mathbf{c},\eta) \longrightarrow \mathrm{Unit}(\mathbf{c},\eta) \longrightarrow \mathrm{LUnit}(\mathbf{c},\eta) \wedge \mathrm{RUnit}(\mathbf{c},\eta)$

and analogously for Null.

The following theorem shows a graded uniqueness of unit and null elements and a graded incompatibility of the properties of being both a unit and a null of the same connective.

Theorem 3.3: FCT proves:

(B2) $\mathrm{LNull}(\mathbf{c},\eta), \mathrm{RNull}(\mathbf{c},\zeta) \ \Rightarrow\ \eta \leftrightarrow \zeta$

(B3) $\mathrm{LUnit}(\mathbf{c},\eta), \mathrm{RUnit}(\mathbf{c},\zeta) \ \Rightarrow\ \eta \leftrightarrow \zeta$

(B4) $\mathrm{LNull}(\mathbf{c},\eta), \mathrm{LUnit}(\mathbf{c},\eta) \ \Rightarrow\ \eta\wedge\neg\eta$, and analogously for RNull and RUnit

The following theorem indicates the degrees of null and unit elements of logical connectives:

Theorem 3.4: FCT proves:

(B5) $\mathrm{LUnit}(\wedge,\eta) \longleftrightarrow \eta \longleftrightarrow \mathrm{LNull}(\vee,\eta)$

(B6) $\mathrm{LNull}(\wedge,\eta) \longleftrightarrow \neg\eta \longleftrightarrow \mathrm{LUnit}(\vee,\eta)$

(B7) $\eta \longrightarrow \mathrm{LUnit}(\&,\eta) \longrightarrow (\neg\eta \rightarrow \eta)$

(B8) $\mathrm{LNull}(\&,\eta) \leftrightarrow \neg\eta$

(B9) $\eta \longrightarrow \mathrm{LUnit}(\rightarrow,\eta) \longrightarrow \neg\neg\eta$

(B10) $\mathrm{RUnit}(\rightarrow,\eta) \leftrightarrow 0$

(B11) $\mathrm{RNull}(\rightarrow,\eta) \leftrightarrow \eta$

Now we turn to the properties of congruence and monotony, whose both-sided variants (unlike their classical counterparts and unlike the previous case of unit and null elements) cannot in fuzzy logic be reduced to the component-wise ones. These properties will be crucial in further sections and are defined as follows:

Definition 3.5: In FCT, we define the following graded properties of binary connectives (see Convention A.2 below in Appendix A for the meaning of $\leq$ in formulae):

$$\mathrm{Cng}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta\gamma\delta)((\alpha{\leftrightarrow}\gamma)\&(\beta{\leftrightarrow}\delta) \rightarrow (\alpha\,\mathbf{c}\,\beta \leftrightarrow \gamma\,\mathbf{c}\,\delta))$$
$$\mathrm{Mon}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta\gamma\delta)((\alpha{\leq}\gamma)\&(\beta{\leq}\delta) \rightarrow (\alpha\,\mathbf{c}\,\beta \rightarrow \gamma\,\mathbf{c}\,\delta))$$
$$\mathrm{LMon}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta\gamma)((\alpha \leq \beta) \rightarrow (\gamma\,\mathbf{c}\,\alpha \rightarrow \gamma\,\mathbf{c}\,\beta))$$
$$\mathrm{RMon}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta\gamma)((\alpha \leq \beta) \rightarrow (\alpha\,\mathbf{c}\,\gamma \rightarrow \beta\,\mathbf{c}\,\gamma))$$

Notice that replacing both $\leq$'s by $\rightarrow$ in the definition of LMon (analogously in others) would not yield graded generalizations of monotony, as the resulting notion

$$(\forall\alpha\beta\gamma)((\alpha \rightarrow \beta) \rightarrow (\gamma\,\mathbf{c}\,\alpha \rightarrow \gamma\,\mathbf{c}\,\beta))$$

do not coincide with crisp left monotony when fully true and constitute a stronger property (see [12] for more details).

The graded property $\mathrm{Cng}(\mathbf{c})$ gives, roughly speaking, the degree to which $\mathbf{c}$ yields close values for close arguments, where closeness is evaluated in the sense of $\leftrightarrow$. In particular, in standard Łukasiewicz models of FCT, where $\leftrightarrow$ corresponds to the Euclidean distance, the property $\triangle\,\mathrm{Cng}(\mathbf{c})$ expresses the 1-Lipschitz property of $\mathbf{c}$. If $\mathbf{c}$ is regarded as a fuzzy class $\mathbf{c} \sqsubseteq \mathrm{L}$ rather than a crisp unary operation $\mathbf{c}\colon \mathrm{L} \rightarrow \mathrm{L}$, then $\mathrm{Cng}(\mathbf{c})$ expresses extensionality of $\mathbf{c}$ w.r.t. $\leftrightarrow$. The property will play an important role in many graded theorems on fuzzy connectives, as it denotes the largest guaranteed degree of intersubstitutivity of $\alpha\,\mathbf{c}\,\beta$ and $\gamma\,\mathbf{c}\,\delta$ for close (in the sense of $\leftrightarrow$) arguments α,γ and β,δ.

Finally we turn our attention to the graded versions of the properties of idempotence, commutativity, and associativity of binary connectives.

Definition 3.6: In FCT, we define the following graded properties for a binary connective $\mathbf{c} \sqsubseteq \mathrm{L}\times\mathrm{L}$:

$$\mathrm{Idem}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha)(\alpha\,\mathbf{c}\,\alpha \leftrightarrow \alpha)$$
$$\mathrm{Com}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta)(\alpha\,\mathbf{c}\,\beta \leftrightarrow \beta\,\mathbf{c}\,\alpha)$$
$$\mathrm{Ass}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta\gamma)((\alpha\,\mathbf{c}\,\beta)\,\mathbf{c}\,\gamma) \leftrightarrow (\alpha\,\mathbf{c}\,(\beta\,\mathbf{c}\,\alpha))$$

Notice that by (B12) of Theorem 3.7, it is immaterial whether we define graded commutativity with implication or equivalence. Theorems (B13) and (B14) furthermore show that all connectives with less than full "difference" (in the sense of $\rightarrow$) between their height and plinth are at least partially commutative and associative: thus, e.g., all subnormal connectives in Łukasiewicz models have non-zero degrees of commutativity and associativity.

Theorem 3.7: FCT proves:

(B12) $\mathrm{Com}(\mathbf{c}) \longleftrightarrow (\forall\alpha\beta)(\alpha\,\mathbf{c}\,\beta \rightarrow \beta\,\mathbf{c}\,\alpha)$

(B13) $\mathrm{Hgt}(\mathbf{c}) \rightarrow \mathrm{Plt}(\mathbf{c}) \ \Rightarrow\ \mathrm{Com}(\mathbf{c})$

(B14) $\mathrm{Hgt}(\mathbf{c}) \rightarrow \mathrm{Plt}(\mathbf{c}) \ \Rightarrow\ \mathrm{Ass}(\mathbf{c})$

IV. Graded t-norms and other classes of connectives

It can be observed that the traditional non-graded classes of truth-value operators can be defined by requiring the full satisfaction of some of the properties defined in Definition 3.6 and 3.1. In particular, a connective $\mathbf{c}$ is a (non-graded) *t-norm*, *uninorm*, or *binary aggregation operator* respectively iff it satisfies:

1) $\triangle\,\mathrm{Com}(\mathbf{c}), \triangle\,\mathrm{Ass}(\mathbf{c}), \triangle\,\mathrm{LMon}(\mathbf{c}), \triangle\,\mathrm{LUnit}(\mathbf{c},1)$
2) $\triangle\,\mathrm{Com}(\mathbf{c}), \triangle\,\mathrm{Ass}(\mathbf{c}), \triangle\,\mathrm{LMon}(\mathbf{c}), (\exists\eta)\triangle\,\mathrm{LUnit}(\mathbf{c},\eta)$
3) $\triangle\,\mathrm{Mon}(\mathbf{c}), \triangle(1\,\mathbf{c}\,1), \triangle\,\neg(0\,\mathbf{c}\,0)$

Furthermore, in standard Łukasiewicz logic, $\mathbf{c}$ is a (non-graded) *quasicopula* iff

$$\triangle\,\mathrm{Unit}(\mathbf{c},1),\ \triangle\,\mathrm{Null}(\mathbf{c},0), \triangle\,\mathrm{Mon}(\mathbf{c}), \triangle\,\mathrm{Cng}(\mathbf{c}).$$

Idempotent binary aggregation operators are those which also satisfy $\triangle\,\mathrm{Idem}(\mathbf{c})$, commutative quasicopulas those also satisfying $\triangle\,\mathrm{Com}(\mathbf{c})$; etc. The conditions $\triangle(1\,\mathbf{c}\,1), \triangle\,\neg(0\,\mathbf{c}\,0)$ in the definition of aggregation operators are shorter equivalents of the usual conditions $1\,\mathbf{c}\,1 = 1$ and $0\,\mathbf{c}\,0 = 0$, respectively. Quasicopulas can in our setting not only be generalized in a graded manner, but also to analogous operators that satisfy Cng w.r.t. an equivalence $\leftrightarrow$ other than standard Łukasiewicz as a measure of distance.

There are countless possibilities as to how the properties of being a t-norm, uninorm, etc. can be defined, which are all equivalent in the non-graded case: e.g., $\mathbf{c}$ is a t-norm also iff $\triangle\,\mathrm{Com}(\mathbf{c}), \triangle\,\mathrm{Ass}(\mathbf{c}), \triangle\,\mathrm{Mon}(\mathbf{c}), \triangle\,\mathrm{RUnit}(\mathbf{c},1)$. In the graded case, however, these definitions are no longer equivalent, since, e.g., the commutativity of $\mathbf{c}$, on which the equivalence of the latter two definitions depends, need not be satisfied to degree 1. Defining all of the countless notions of graded t-norm(ness) or uninorm(ness) would clearly be unmanageable: therefore we shall rather study the graded constituent properties of Com, Ass, LUnit, RUnit, etc. independent of each other, combining them freely as premises of theorems and not insisting on any particular predefined combinations thereof. Notice (cf. [15]) that this is just another feature of graded fuzzy mathematics which appears regularly when dealing with compound notions (i.e., those defined in the non-graded case as a conjunction of some conditions). Further we shall see that most of our theorems indeed require various combinations of the constituent properties, while still expressing the crisp properties of t-norms (or uninorms, quasicopulas, etc.) when the premises are fully true.

The following theorem provides us with samples of basic graded results generalizing the well-known basic properties of t-norms.

Theorem 4.1: FCT proves the following graded properties of truth-value operators:

(T1) $\mathrm{RMon}(\mathbf{c}), \mathrm{RUnit}(\mathbf{c}, 1) \;\Rightarrow\; \mathrm{LNull}(\mathbf{c}, 0)$

(T2) $(\mathrm{LMon}(\mathbf{c}) \,\&\, \mathrm{LUnit}(\mathbf{c}, 1)) \wedge (\mathrm{RMon}(\mathbf{c}) \,\&\, \mathrm{RUnit}(\mathbf{c}, 1)) \;\Rightarrow\; \mathbf{c} \subseteq \wedge$

(T3) $\mathrm{Mon}^2(\mathbf{c}), \mathrm{Unit}^2(\mathbf{c}, 1) \;\Rightarrow\; \mathbf{c} \subseteq \wedge$

(T4) $\mathrm{Idem}(\mathbf{c}), \mathrm{LMon}(\mathbf{c}) \wedge \mathrm{RMon}(\mathbf{c}) \;\Rightarrow\; \wedge \subseteq \mathbf{c}$

(T5) $(\mathrm{LMon}(\mathbf{c}) \,\&\, \mathrm{LUnit}(\mathbf{c}, 1)) \wedge (\mathrm{RMon}(\mathbf{c}) \,\&\, \mathrm{RUnit}(\mathbf{c}, 1)) \;\Rightarrow\; (\alpha \,\mathbf{c}\, \alpha \leftrightarrow \alpha) \leftrightarrow (\forall \beta)((\alpha \,\mathbf{c}\, \beta) \leftrightarrow (\alpha \wedge \beta))$

(T6) $\mathrm{Mon}^2(\mathbf{c}), \mathrm{Unit}^2(\mathbf{c}, 1) \;\Rightarrow\; (\alpha \,\mathbf{c}\, \alpha \leftrightarrow \alpha) \leftrightarrow (\forall \beta)((\alpha \,\mathbf{c}\, \beta) \leftrightarrow (\alpha \wedge \beta))$

Theorem (T1) generalizes the well-known fact that in t-norms, the nullness of 0 follows from the unitness of 1. Theorems (T2) and (T3) correspond to the fact that the minimum is the greatest (so-called strongest) t-norm. Theorem (T4) generalizes the basic fact that the minimum is the only idempotent t-norm, while (T5) and (T6) a graded characterization of the idempotents of $\mathbf{c}$ [16].

V. Graded dominance

Applying the definition of dominance between binary aggregation operators and making it graded by replacing crisp $\leq$ by $\rightarrow$, we obtain the following notion of graded dominance. As usually, the traditional notion of dominance is expressible as the graded notion satisfied to degree 1, i.e., prepended by $\triangle$.

Definition 5.1: The graded relation $\ll$ of *dominance* between binary connectives is defined as follows:

$$\mathbf{c} \ll \mathbf{d} \equiv_{\mathrm{df}} (\forall \alpha\beta\gamma\delta)((\alpha \,\mathbf{d}\, \gamma) \,\mathbf{c}\, (\beta \,\mathbf{d}\, \delta) \rightarrow (\alpha \,\mathbf{c}\, \beta) \,\mathbf{d}\, (\gamma \,\mathbf{c}\, \delta))$$

The following theorem shows how graded dominance is transmitted to $\approx$-close connectives:

Theorem 5.2: FCT proves:

(D1) $\mathbf{c} \ll \mathbf{d}, \mathbf{c} \approx^3 \mathbf{c}', \mathrm{Cng}(\mathbf{d}) \;\Rightarrow\; \mathbf{c}' \ll \mathbf{d}$

(D2) $\mathbf{c} \ll \mathbf{d}, \mathbf{d} \approx^3 \mathbf{d}', \mathrm{Cng}(\mathbf{c}) \;\Rightarrow\; \mathbf{c} \ll \mathbf{d}'$

Theorem 5.3: FCT proves, for any $i \in \{1, 2\}$:

(D3) $\triangle \mathrm{Com}(\mathbf{c}), \triangle \mathrm{Ass}(\mathbf{c}) \;\Rightarrow\; \mathbf{c} \ll \mathbf{c}$

(D4) $\mathrm{Com}(\mathbf{c}), \mathrm{Ass}^4(\mathbf{c}), \mathrm{Cng}(\mathbf{c}) \;\Rightarrow\; \mathbf{c} \ll \mathbf{c}$

(D5) $\triangle \mathrm{Com}(\mathbf{c}_i), \triangle \mathrm{Ass}(\mathbf{c}_i), \mathrm{Mon}(\mathbf{c}_i), \mathbf{c}_1 \subseteq \mathbf{c}_2, \mathbf{c}_2 \sqsubseteq \mathbf{c}_1 \;\Rightarrow\; \mathbf{c}_1 \ll \mathbf{c}_2$

(D6) $\mathrm{Com}(\mathbf{c}_i), \mathrm{Ass}^4(\mathbf{c}_i), \mathrm{Cng}(\mathbf{c}_i), \mathrm{Mon}(\mathbf{c}_i), \mathbf{c}_1 \subseteq \mathbf{c}_2, \mathbf{c}_1 \sqsubseteq \mathbf{c}_2 \;\Rightarrow\; \mathbf{c}_1 \ll \mathbf{c}_2$

Theorem (D3) is just a basic fact, that every t-norm dominates itself and (D4) is it graded generalization, which can be informally explained as saying that self-domination (or Aczél's property of bisymmetry), holds not only for t-norms, but to a fair degree also for connectives which are very associative and fairly commutative and monotone.

Theorems (D5)–(D6) have no correspondences among known results; they provide us with bounds for the degree to which $(\mathbf{c} \ll \mathbf{d})$ holds, where the assumption $(\mathbf{d} \sqsubseteq \mathbf{c})$ & $(\mathbf{c} \subseteq \mathbf{d})$ would be obviously useless in the crisp non-graded framework (as it necessitates that $\mathbf{c}$ and $\mathbf{d}$ coincide anyway). Notice that relaxing the assumption of full commutativity (associativity) from (D3) and (D5) necessitates the presence of *congruence* property in (D4) and (D6).

The following theorem is a graded version of another classical results (that dominance implies inclusion / pointwise order). Read contrapositively, it provides a bound to the degree of dominance from the (usually known or at least more easily calculable) degrees of subsethood of the connectives.

Theorem 5.4: FCT proves the following graded properties of dominance:

(D7) $\triangle \mathrm{LUnit}(\mathbf{c}, \eta), \triangle \mathrm{RUnit}(\mathbf{d}, \eta)), \mathbf{c} \ll \mathbf{d} \;\Rightarrow\; \mathbf{c} \subseteq \mathbf{d}$

The following theorem shows preservation of dominance under compositions and is a generalization of non-graded theorems of [11].

Theorem 5.5: Let $(\forall \alpha\beta)(\mathbf{e}(\alpha, \beta) = (\alpha \,\mathbf{a}\, \beta) \,\mathbf{c}\, (\alpha \,\mathbf{b}\, \beta))$ or $(\forall \alpha\beta)(\mathbf{e}(\alpha, \beta) = (\alpha \,\mathbf{a}\, \alpha) \,\mathbf{c}\, (\beta \,\mathbf{b}\, \beta))$. Then FCT proves:

(D8) $\mathbf{d} \ll \mathbf{c}, \triangle(\mathbf{d} \ll \mathbf{a}), \triangle(\mathbf{d} \ll \mathbf{b}), \mathrm{Mon}(\mathbf{c}) \;\Rightarrow\; \mathbf{d} \ll \mathbf{e}$

The following two theorems study graded properties of dominance w.r.t. both 'logical' conjunctions present in our language.

Theorem 5.6: FCT proves the following graded properties of dominance w.r.t. &:

(D9) $\& \ll \mathbf{c}, \mathrm{Mon}(\mathbf{c}) \;\Rightarrow\; (\alpha \rightarrow \beta) \,\mathbf{c}\, (\gamma \rightarrow \delta) \rightarrow (\alpha \,\mathbf{c}\, \gamma \rightarrow \beta \,\mathbf{c}\, \delta)$

(D10) $\& \ll \mathbf{c}, \mathrm{Mon}(\mathbf{c}) \;\Rightarrow\; (\alpha \leftrightarrow \beta) \,\mathbf{c}\, (\gamma \leftrightarrow \delta) \rightarrow (\alpha \,\mathbf{c}\, \gamma \leftrightarrow \beta \,\mathbf{c}\, \delta)$

(D11) $\& \ll \mathbf{c}, \mathrm{Mon}(\mathbf{c}), \mathrm{RUnit}(\mathbf{c}, 1) \;\Rightarrow\; \mathrm{LMon}(\mathbf{c})$

(D12) $\& \ll \mathbf{c}, \mathrm{Mon}(\mathbf{c}), \& \subseteq \mathbf{c} \;\Rightarrow\; \mathrm{Cng}(\mathbf{c})$

Theorem 5.7: FCT proves the following graded properties of dominance w.r.t. $\wedge$:

(D13) $\mathrm{Mon}(\mathbf{c}) \;\Rightarrow\; \mathbf{c} \ll \wedge$

(D14) $\triangle \mathrm{Unit}(\mathbf{c}, 1) \;\Rightarrow\; (\wedge \ll \mathbf{c}) \leq (\wedge \subseteq \mathbf{c})$

(D15) $\mathrm{Unit}(\mathbf{c}, 1), \mathrm{Cng}(\mathbf{c}), \wedge \ll \mathbf{c} \;\Rightarrow\; \wedge \subseteq \mathbf{c}$

(D16) $\triangle \mathrm{Mon}(\mathbf{c}), \triangle \mathrm{Unit}(\mathbf{c}, 1) \;\Rightarrow\; (\wedge \subseteq \mathbf{c}) = (\wedge \ll \mathbf{c})$

(D17) $\mathrm{LMon}(\mathbf{c}) \wedge \mathrm{RMon}(\mathbf{c}), \wedge \ll \mathbf{c} \;\Rightarrow\; (\alpha \,\mathbf{c}\, 1) \wedge (1 \,\mathbf{c}\, \beta) \leftrightarrow \alpha \,\mathbf{c}\, \beta$

Theorem (D13) is a graded generalization of the well-known fact that the minimum dominates any aggregation operator [11]. Theorem (D16) demonstrates a rather surprising fact: that the degree to which a monotonic binary operation with neutral element 1 dominates the minimum is nothing else but the degree to which it is larger. Theorem (D17) is (a part of) an alternative characterization of operators dominating the minimum; for its non-graded version see [11, Prop. 5.1].

Example 5.8: Theorem (D16) can easily be utilized to compute degrees to which standard t-norms on the unit interval dominate the minimum. It can be shown easily that

$$(\wedge \subseteq \mathbf{c}) = \inf_{x\in[0,1]} (x \Rightarrow \mathbf{c}(x,x))$$

holds, i.e. the largest "difference" of a t-norm $\mathbf{c}$ from the minimum can always be found on the diagonal. In standard Łukasiewicz logic, this is, for instance, 0.75 for the product t-norm and 0.5 for the Łukasiewicz t-norm itself. So we can infer that the product t-norm dominates the minimum with a degree of 0.75 (assuming that the underlying logic is standard Łukasiewicz!); with the same assumption, the Łukasiewicz t-norm dominates the minimum to a degree of 0.5.

VI. Applications to fuzzy relations

In this section we shall apply graded dominance to graded properties of $\mathbf{c}$-transitivity and $\mathbf{c}$-extensionality of fuzzy relations.

Definition 6.1: In FCT, we define the following graded properties of binary fuzzy relations:

$$\mathrm{Trans}_{\mathbf{c}}(R) \equiv_{\mathrm{df}} (\forall xyz)(Rxy \mathbin{\mathbf{c}} Ryz \to Rxz)$$
$$\mathrm{Ext}_{\mathbf{c}}(A,R) \equiv_{\mathrm{df}} (\forall xy)(Ax \mathbin{\mathbf{c}} Rxy \to Ay)$$

Furthermore we define the class operation $\underline{\mathbf{c}}$ given by the connective $\mathbf{c}$ as follows:

$$P \mathbin{\underline{\mathbf{c}}} Q =_{\mathrm{df}} \{\vec{x} \mid P\vec{x} \mathbin{\mathbf{c}} Q\vec{x}\},$$

for tuples $\vec{x}$ of an arbitrary arity. (Thus, e.g., $\underline{\mathbf{c}}$ is strong intersection if $\mathbf{c} = \&$, weak union if $\mathbf{c} = \vee$, etc., of fuzzy classes or fuzzy relations.)

The following theorems show the importance of graded dominance for graded properties of fuzzy relations. Theorem 6.2 is a graded generalization of the well-known theorem by De Baets and Mesiar that uses dominance to characterize preservation of transitivity by aggregation [10, Th. 2]. By (R4), in monotone operators with the null element 0 (e.g., t-norms), the degree of graded dominance $\mathbf{c} \ll \mathbf{d}$ is exactly the degree to which $\mathbf{c}$-transitivity is preserved by $\mathbf{d}$-intersections.

Theorem 6.2: FCT proves:

(R1) $\mathbf{c} \ll \mathbf{d}, \mathrm{Mon}(\mathbf{d}), \triangle\,\mathrm{Trans}_{\mathbf{c}}(R), \triangle\,\mathrm{Trans}_{\mathbf{c}}(S) \;\Rightarrow\; \mathrm{Trans}_{\mathbf{c}}(R \mathbin{\underline{\mathbf{d}}} S)$

(R2) $\mathbf{c} \ll \mathbf{d}, \mathrm{Mon}(\mathbf{d}) \wedge \mathrm{Cng}(\mathbf{d}), \mathrm{Trans}_{\mathbf{c}}(R), \mathrm{Trans}_{\mathbf{c}}(S) \;\Rightarrow\; \mathrm{Trans}_{\mathbf{c}}(R \mathbin{\underline{\mathbf{d}}} S)$

(R3) $(\forall RS)(\triangle(\mathrm{Trans}_{\mathbf{c}}(R) \mathbin{\&} \mathrm{Trans}_{\mathbf{c}}(S)) \to \mathrm{Trans}_{\mathbf{c}}(R \mathbin{\underline{\mathbf{d}}} S)), \triangle\,\mathrm{Null}(\mathbf{c},0) \;\Rightarrow\; \mathbf{c} \ll \mathbf{d}$

(R4) $\mathrm{Mon}(\mathbf{d}), \triangle\,\mathrm{Null}(\mathbf{c},0) \;\Rightarrow\; (\mathbf{c} \ll \mathbf{d}) \leftrightarrow (\forall RS)(\triangle(\mathrm{Trans}_{\mathbf{c}}(R) \mathbin{\&} \mathrm{Trans}_{\mathbf{c}}(S)) \to \mathrm{Trans}_{\mathbf{c}}(R \mathbin{\underline{\mathbf{d}}} S))$

The similitude between the defining formulae of $\mathrm{Trans}_{\mathbf{c}}(R)$ and $\mathrm{Ext}_{\mathbf{c}}(A,R)$ makes it possible to transfer the results of Theorem 6.2 to graded extensionality:

Theorem 6.3: FCT proves:

(R5) $\mathbf{c} \ll \mathbf{d}, \mathrm{Mon}(\mathbf{d}), \triangle\,\mathrm{Ext}_{\mathbf{c}}(A,R), \triangle\,\mathrm{Ext}_{\mathbf{c}}(B,S) \;\Rightarrow\; \mathrm{Ext}_{\mathbf{c}}(A \mathbin{\underline{\mathbf{d}}} B, R \mathbin{\underline{\mathbf{d}}} S)$

(R6) $\mathbf{c} \ll \mathbf{d}, \mathrm{Mon}(\mathbf{d}) \wedge \mathrm{Cng}(\mathbf{d}), \mathrm{Ext}_{\mathbf{c}}(A,R), \mathrm{Ext}_{\mathbf{c}}(B,S) \;\Rightarrow\; \mathrm{Ext}_{\mathbf{c}}(A \mathbin{\underline{\mathbf{d}}} B, R \mathbin{\underline{\mathbf{d}}} S)$

(R7) $\triangle\,\mathrm{Null}(\mathbf{c},0), (\forall ABRS)(\triangle\,\mathrm{Ext}_{\mathbf{c}}(A,R) \mathbin{\&} \triangle\,\mathrm{Ext}_{\mathbf{c}}(B,S) \to \mathrm{Ext}_{\mathbf{c}}(A \mathbin{\underline{\mathbf{d}}} B, R \mathbin{\underline{\mathbf{d}}} S)) \;\Rightarrow\; \mathbf{c} \ll \mathbf{d}$

(R8) $\mathrm{Mon}(\mathbf{d}), \triangle\,\mathrm{Null}(\mathbf{c},0) \;\Rightarrow\; (\mathbf{c} \ll \mathbf{d}) \leftrightarrow (\forall ABRS)(\triangle\,\mathrm{Ext}_{\mathbf{c}}(A,R) \mathbin{\&} \triangle\,\mathrm{Ext}_{\mathbf{c}}(B,S) \to \mathrm{Ext}_{\mathbf{c}}(A \mathbin{\underline{\mathbf{d}}} B, R \mathbin{\underline{\mathbf{d}}} S))$

Appendix

In this section, we present a self-contained list of definitions related to Fuzzy Class Theory (FCT). For a complete and detailed introduction to FCT, the reader is referred to the freely available primer [13]. Preprints of many papers on FCT, including most of those cited in the present paper, are (as of 2009) also available at the website of the FCT project, www.cs.cas.cz/hp.

Definition A.1: Fuzzy Class Theory (over $\mathrm{MTL}_\triangle$) is a theory over multi-sorted first-order logic $\mathrm{MTL}_\triangle$ with crisp equality. There are sorts for individuals of the zeroth order (i.e., atomic objects), denoted by lowercase variables $a, b, c, x, y, z, \ldots$; individuals of the first order (i.e., fuzzy classes), denoted by uppercase variables $A, B, X, Y, \ldots$; etc. Individuals $\xi_1, \ldots, \xi_k$ of each order can form k-tuples (for any $k \geq 0$), denoted by $\langle \xi_1, \ldots, \xi_k \rangle$; tuples are governed by the usual axioms known from classical mathematics (e.g., that tuples equal if and only if their respective constituents equal). Furthermore, for each variable x of any order n and for each formula φ there is a class term $\{x \mid \varphi\}$ of order $n+1$.

Besides the logical predicate of identity, the only primitive predicate is the membership predicate $\in$ between successive sorts (i.e., between individuals of the n-th order and individuals of the $(n+1)$-st order, for any n). The axioms for $\in$ are the following (for variables of all orders and all formulae φ):

($\in$1) $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$, (comprehension axioms)

($\in$2) $(\forall x)\triangle(x \in A \leftrightarrow x \in B) \to A = B$ (extensionality)

Besides the above specific axioms, FCT uses the axioms and deduction rules of multi-sorted first-order logic $\mathrm{MTL}_\triangle$ with crisp identity. Theorems, proofs, etc., are defined completely analogously as in classical logic.

The *models* of FCT are systems (closed under definable operations) of fuzzy sets (and fuzzy relations) of all orders over some crisp universe U, where the membership functions of fuzzy subsets take values in some $\mathrm{MTL}_\triangle$-chain. *Intended* models are those which contain *all* fuzzy subsets and fuzzy relations over U (of all orders). Models in which moreover the $\mathrm{MTL}_\triangle$-chain is standard (i.e., given by a left-continuous t-norm on the unit interval $[0,1]$) correspond to Zadeh's [17] original notion of fuzzy set, and are called *Zadeh models*. FCT is sound with respect to Zadeh models, therefore all theorems provable in FCT are true statements about fuzzy sets and relations in the traditional sense.

Convention A.2: For better readability of FCT formulae, we make the following conventions:

- We use Ax and $Rx_1 \dots x_n$ as synonyms for $x \in A$ and $\langle x_1, \dots, x_n \rangle \in R$, respectively.
- The formulae $\varphi \;\&\; \dots \;\&\; \varphi$ (n times) are abbreviated φ^n; instead of $(x \in A)^n$, we can write $x \in^n A$ (and analogously for other predicates).
- A chain of implications

$$\varphi_1 \to \varphi_2,\ \varphi_2 \to \varphi_3,\ \dots,\ \varphi_{n-1} \to \varphi_n$$

  is, for short, written as $\varphi_1 \longrightarrow \varphi_2 \longrightarrow \cdots \longrightarrow \varphi_n$ (and similarly for the equivalence connective).
- Formulae of the form $\varphi_1 \,\&\, \dots \,\&\, \varphi_n \to \psi$ can be written as $\varphi_1, \dots, \varphi_n \;\Rightarrow\; \psi$.
- Finally, $\varphi \leq \psi$ abbreviates $\triangle(\varphi \to \psi)$.

Definition A.3: We define the following elementary relations between fuzzy sets in FCT:

$$\begin{array}{rcl}
A \subseteq B & \equiv_{\mathrm{df}} & (\forall x)(x \in A \to x \in B) \\
A \sqsubseteq B & \equiv_{\mathrm{df}} & (\forall x)(x \in A \leq x \in B) \\
A \approxeq B & \equiv_{\mathrm{df}} & (A \subseteq B) \;\&\; (B \subseteq A) \\
A \approx B & \equiv_{\mathrm{df}} & (\forall x)(x \in A \leftrightarrow x \in B) \\
\mathrm{Hgt}(A) & \equiv_{\mathrm{df}} & (\exists x)Ax \\
\mathrm{Plt}(A) & \equiv_{\mathrm{df}} & (\forall x)Ax
\end{array}$$

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