Extensionality in Graded Properties of Fuzzy Relations

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Abstract

New definitions of graded reflexivity, symmetry, transitivity, antisymmetry, and functionality of fuzzy relations are proposed which are relative to an indistinguishability relation E on the universe of discourse. It is shown that if considered non-graded (i.e., either fully present or else fully absent), the new definitions reduce to the usual ones under full extensionality of the relation w.r.t. E. However, if *graded* properties of R (e.g., transitivity to some degree) are taken into account, the new definitions have to be distinguished from the conjunction of the original property and E-extensionality of R. Some arguments and results are given which suggest that the new concepts are well-motivated.

Keywords: Fuzzy relations, Extensionality, Similarity, Graded properties.

1 Graded properties of fuzzy relations

In traditional fuzzy mathematics, fuzzy relations are defined as binary functions from some universe of discourse U to $[0,1]$ (or another suitable lattice L of truth values). The usual properties of fuzzy relations are then defined as follows:

Definition 1.1 *Let T be a (left-continuous) t-norm. We say that a fuzzy relation R is* reflexive *iff $R(x,x) = 1$ for all x;* symmetric *iff $R(x,y) \le R(y,x)$ for all x, y; T-*transitive *iff $T(R(x,y), R(y,z)) \le R(x,z)$ for all x, y, z; etc.*

These conditions, formulated in ordinary mathematics over classical logic, can also be expressed by certain formulae of fuzzy logic. Let us work in the first-order fuzzy logic MTLΔ with crisp identity predicate $=$, or in any of its extensions.[1] In its usual semantics, binary predicates of its formal language are interpreted as fuzzy relations over the domain of discourse. A suitable defining formula for the reflexivity of R is then $\forall x Rxx$; for symmetry, $\forall xy(Rxy \to Ryx)$; for transitivity, $\forall xyz(Rxy \mathbin{\&} Ryz \to Rxz)$; etc. Each of these formulae has the truth value 1 iff the respective condition of Definition 1.1 is satisfied.

If the conditions of Definition 1.1 are not satisfied, then the property of R simply does not hold (its truth value is 0). The defining formulae in first-order fuzzy logic, however, may even in such cases yield meaningful non-zero truth values. For instance, if $R(x,x) = 0.999$ for all x, then the truth value of $\forall x Rxx$ is 0.999. It is clear that such a relation is "al-

[1] MTL, introduced in [5], is the logic of left-continuous t-norms; see [9] for its most important extensions and an exposition of the semantics of first-order fuzzy logic. We use extensions that contain the Δ connective as we need to express the full truth of some statements. The crisp identity predicate is inessential in this paper and is only used for expository purposes: it will consistently be replaced by a fuzzy predicate E.

most reflexive" (all pairs xx are almost fully in R), even though it is not reflexive according to Definition 1.1. Since furthermore the formula $\forall x Rxx$ has the same form as the formula which defines reflexivity in classical mathematics, it is natural to take its truth value for the degree of *graded reflexivity* of R, and say that R is 0.999-reflexive. (Similarly for symmetry, transitivity, and other properties of fuzzy relations.)

The graded properties of fuzzy relations have been introduced in Gottwald's paper [6] and systematically studied in his monograph [7]; more recently they have been elaborated in Gottwald's [8, §18.6], Bělohlávek's [3, §4.1], and Jacas and Recasens' [12]. The graded approach to the properties of fuzzy relations is important for several reasons:

- Graded properties generalize the traditional (non-graded) ones: R is reflexive (in the traditional sense) iff the truth value of graded reflexivity is exactly 1. In all other cases, the graded properties provide a fine-grained scale of the degrees of their validity, while the non-graded properties are then simply false.

- The graded approach allows to infer relevant information when the traditional conditions are almost, but still not completely, fulfilled. E.g., in the example above, R is 0.999-reflexive: if we prove that (graded) reflexivity of R implies (in the sense of fuzzy logic) some property φ, we shall know that φ holds at least to the degree 0.999. On the contrary, from the non-graded reflexivity of Definition 1.1 we cannot infer anything as it is simply false.

- Graded properties can easily be handled by first-order fuzzy logic: valid inferences about them can be proved by the formal rules of fuzzy logic. The semantics of fuzzy logic (relative to a particular t-norm) then translates the formal theorems into the laws valid for "real" fuzzy relations.

- Graded properties are "fuzzier" than their non-graded counterparts: if we take seriously the idea of general fuzziness of concepts, there is no reason to presuppose that the properties of fuzzy relations should only be crisp (i.e., either true or false as in Definition 1.1).

In the rest of this paper we shall always work with *graded* properties of fuzzy relations. Suspending Definition 1.1, we now define (graded) reflexivity, symmetry, transitivity, antisymmetry,[2] and functionality in the first-order logic MTL as follows:

Definition 1.2

$$
\begin{aligned}
\mathrm{Refl}\, R &\equiv \forall x Rxx\\
\mathrm{Sym}\, R &\equiv \forall xy(Rxy \to Ryx)\\
\mathrm{Trans}\, R &\equiv \forall xyz(Rxy \mathbin{\&} Ryz \to Rxz)\\
\mathrm{Asym}\, R &\equiv \forall xy(Rxy \mathbin{\&} Ryx \to x{=}y)\\
\mathrm{Fnc}\, R &\equiv \forall xyy'(Rxy \mathbin{\&} Rxy' \to y{=}y')
\end{aligned}
$$

These definitions can be combined (by strong conjunction), yielding more complex graded notions of proximity, similarity (fuzzy equivalence), fuzzy preorder, and fuzzy order:

Definition 1.3

$$
\begin{aligned}
\mathrm{Prox}\, R &\equiv \mathrm{Refl}\, R \mathbin{\&} \mathrm{Sym}\, R\\
\mathrm{Sim}\, R &\equiv \mathrm{Prox}\, R \mathbin{\&} \mathrm{Trans}\, R\\
\mathrm{Preord}\, R &\equiv \mathrm{Refl}\, R \mathbin{\&} \mathrm{Trans}\, R\\
\mathrm{Ord}\, R &\equiv \mathrm{Preord}\, R \mathbin{\&} \mathrm{Asym}\, R
\end{aligned}
$$

It can be observed that the defining formulae in Definition 1.2 are exactly the same as the definitions of these properties for crisp relations in classical mathematics. This correlates with the motivation of fuzzy logic as the generalization of classical logic to non-sharp predicates: classical mathematical notions are then fuzzified in a natural way just by interpreting the classical definitions in fuzzy logic. This methodology has been foreshadowed in [11, §5] by Höhle, much later formalized in [1,

[2]Even though many authors (e.g., [3], [11]) use min-conjunction in the definition of antisymmetry, arguments can be given that strong conjunction is in order here.

§7], and suggested as an important guideline for formal fuzzy mathematics in [2].[3]

2 Indistinguishability-relative properties

The adoption of graded properties of fuzzy relations can be viewed as part of the pursuit of a full-blown (rather than half-way) fuzzification of classical notions: the semi-classical (bivalent) notions of Definition 1.1 have been replaced by fuzzy notions of Definition 1.2. In general, according to the methodology of [2], one wants to fuzzify as much as one can, and find and eliminate hidden crispness in definitions wherever possible.

A case of such hidden crispness can be descried in the above definitions of antisymmetry and functionality: they refer to the (crisp) identity predicate =. In the fuzzy world, we should be ready to admit that not only crisp identity, but also a fuzzy similarity relation can play the role here.[4] The corrected definitions of these two notions will therefore replace = with a similarity relation E:

Definition 2.1

$$\begin{aligned} \mathrm{Asym}_{(E)}\, R &\equiv \forall xy(Rxy \mathbin{\&} Ryx \to Exy) \\ \mathrm{Fnc}_{(E)}\, R &\equiv \forall xyy'(Rxy \mathbin{\&} Rxy' \to Eyy') \end{aligned}$$

Indeed, such definitions of E-antisymmetry and E-functionality can be found in the literature (e.g., [3], [4], [11]).

These two cases of "hidden" crispness were patent—the crisp identity was explicitly present in the formula. What I want to propose in this paper is to avoid another, less explicit case of hidden crispness present in the definitions of properties of fuzzy relations. The kind of hidden crispness I address is caused by multiple occurrences of the same variable in the defining formula: in such cases, a hidden identity predicate is present, which should again be eliminated by replacing it with fuzzy similarity.

Consider reflexivity, $\forall x Rxx$. If we suppose that there is a relation E which measures the degree of *indistinguishability* of individuals, we find that the formula $\forall x Rxx$ is no longer adequate for the intuitive notion of reflexivity. The reason is that it only takes into account R on pairs xx, even though Rxy should also be taken into consideration on reflexivity if y is indistinguishable from x (i.e., on condition Exy). The need for this is often obvious: if the value of R is, for example, obtained by some independent measurements on its two arguments, we may often fail to recognize whether the two arguments independently presented to us (e.g., by Nature) are indeed identical or just indistinguishable. Thus we should rather define:

$$\mathrm{Refl}_E\, R \equiv \forall xy(Exy \to Rxy) \qquad (1)$$

From the formal point of view, the reason why the original definition ceased to be adequate in the presence of indistinguishability was that the double occurrence of x in $\forall x Rxx$ contained a hidden identity predicate: it was in fact $\forall xy(x{=}y \to Rxy)$, in which (1) has replaced = by E, just as did Definition 2.1.

The same considerations can be carried out for other properties of fuzzy relations, and the hidden crispness caused by multiple occurrences of variables in the defining formulae be cured in the same way: by first making the hidden identity predicates explicit, and then replacing them with the (fuzzy) indistinguishability relation E. This leads to the following definitions:

Definition 2.2

$$\begin{aligned} \mathrm{Refl}_E\, R &\equiv \forall xx'(Exx' \to Rxx') \\ \mathrm{Sym}_E\, R &\equiv \forall xx'yy'(Exx' \mathbin{\&} Eyy' \mathbin{\&} \\ &\qquad Rxy \to Ry'x') \\ \mathrm{Trans}_E\, R &\equiv \forall xx'yy'zz'(Exx' \mathbin{\&} Eyy' \mathbin{\&} \\ &\qquad Ezz' \mathbin{\&} Rxy \mathbin{\&} Ry'z \to Rx'z') \\ \mathrm{Asym}_E\, R &\equiv \forall xx'yy'(Exx' \mathbin{\&} Eyy' \mathbin{\&} \\ &\qquad Rxy \mathbin{\&} Ry'x' \to Exy) \end{aligned}$$

[3]Of course, the method cannot be applied mechanically: but due to the motivation of fuzzy logical connectives and quantifiers, it often yields intuitive notions, and only occasionally a deeper analysis is required; an example of the latter situation are the new definitions presented in this paper.

[4]The intuitions behind the definitions of antisymmetry and functionality will be preserved especially if the similarity is interpreted as the indistinguishability of individuals.

$$
\begin{aligned}
\mathrm{Fnc}_E\,R &\equiv \forall xx'yy'(Exx' \mathbin{\&} \\
&\quad Rxy \mathbin{\&} Rx'y' \to Eyy') \\
\mathrm{Preord}_E\,R &\equiv \mathrm{Refl}_E\,R \mathbin{\&} \mathrm{Trans}_E\,R \\
\mathrm{Ord}_E\,R &\equiv \mathrm{Preord}_E\,R \mathbin{\&} \mathrm{Asym}_E\,R \\
\mathrm{Prox}_E\,R &\equiv \mathrm{Refl}_E\,R \mathbin{\&} \mathrm{Sym}_E\,R \\
\mathrm{Sim}_E\,R &\equiv \mathrm{Preord}_E\,R \mathbin{\&} \mathrm{Sym}_E\,R
\end{aligned}
$$

Generally we do not impose any restriction on E in this definition: so any assumptions regarding the properties of E will always be explicitly stated in theorems. By convention, the index E can be dropped if E is the identity (this accommodates Definitions 1.2 and 1.3).

It can be objected that the main motivation of these definitions is not yet (and generally can never be) accomplished: the formulae in Definition 2.2 still contain two occurrences of each variable, and by the same argument as above we cannot be sure whether the individuals denoted by them are indeed identical or just E-indistinguishable. In order to eliminate the double occurrences in the new definitions, we would have to make the same trick again, ending up in an infinite regress:

$$
\begin{aligned}
\mathrm{Refl}_{0E}\,R &\equiv \forall x Rxx \\
\mathrm{Refl}_{1E}\,R &\equiv \forall xy(Exy \to Rxy) \\
\mathrm{Refl}_{2E}\,R &\equiv \forall xx'yy'(Exx' \mathbin{\&} Eyy' \mathbin{\&} \\
&\quad Exy \to Rx'y') \\
&\dots
\end{aligned}
$$

There are at least three possible answers to this objection:[5]

First, in the formula $\mathrm{Refl}_{1E}\,R$, each variable occurs only *once under* R. Conceivably, establishing the truth value of the indistinguishability E can be much easier than the measurement of R (e.g., E can be obvious, intuitive, etc.). Thus in some cases it may only be necessary to distinguish the arguments of R, not E.

Second, observe that $\mathrm{Refl}_{2E}\,R \leftrightarrow \mathrm{Refl}_{1E'}\,R$, where[6]

$$E'xy \equiv \exists x'y'(Ex'x \mathbin{\&} Ey'y \mathbin{\&} Ex'y') \qquad (2)$$

Thus the iterated E-properties have the same form as the non-iterated ones (only with a different E'). The *theory* of iterated properties (abstracting from particular E's) is therefore the same as that of non-iterated ones.

Finally, under the reasonable assumption that E is a similarity (to degree 1), the iterated notions coincide with the non-iterated ones:[7]

Lemma 2.3 $\Delta\,\mathrm{Sim}\,E \to E' = E$

Proof: Observe that by (2),[8]

$$E' = E^{-1} \circ E \circ E$$

By known results (see, e.g., [3] or [8]) which can be transferred to MTLΔ, if E is fully symmetric, then $E^{-1} = E$; and if E is fully reflexive and fully transitive, then $E = E \circ E$. Thus if $\Delta\,\mathrm{Sim}\,E$, then $E' = E$. QED

Corollary 2.4 $\Delta\,\mathrm{Sim}\,E \to$
$\to (\mathrm{Refl}_{2E}\,R \leftrightarrow \mathrm{Refl}_{1E}\,R)$

(Similarly for Sym_{2E}, Trans_{2E}, Asym_{2E}, *and* Fnc_{2E}*.)*

This ensures that under the assumption that the indistinguishability relation is a (full) similarity, all of the iterated notions coincide with those of Definition 2.2.

[5] We present them for the case of reflexivity; for other properties they are fully analogous.

[6] Since by the rules of MTL,

$$
\begin{aligned}
&\forall xx'yy'(Exx' \mathbin{\&} Eyy' \mathbin{\&} Exy \to Rx'y') \\
&\quad\leftrightarrow\ \forall x'xy'y(Ex'x \mathbin{\&} Ey'y \mathbin{\&} Ex'y' \to Rxy) \\
&\quad\leftrightarrow\ \forall xy(\exists x'y'(Ex'x \mathbin{\&} Ey'y \mathbin{\&} Ex'y') \to Rxy)
\end{aligned}
$$

[7] Recall that we work formally in the logic MTLΔ or some of its extensions; therefore, by stating a lemma or a theorem in this paper we mean that it is provable in MTLΔ.

[8] E^{-1} is the inverse relation and $\circ$ denotes relational composition:

$$
\begin{aligned}
E^{-1}xy &\equiv Eyx \\
(R \circ S)xy &\equiv \exists z(Rxz \mathbin{\&} Szy)
\end{aligned}
$$

The identity of fuzzy relations is defined as the identity of their membership functions (which ensures their intersubstitutivity *salva veritate*):

$$R = S \quad\equiv\quad \forall xy\Delta(Rxy \leftrightarrow Sxy)$$

Remark 2.5 By the same argument as above, one should prefer $\Delta\,\mathrm{Sim}_E\,E$ as the precondition for Lemma 2.3 and Corollary 2.4. However, by Proposition 3.5 below,

$$\Delta\,\mathrm{Sim}_E\,E \leftrightarrow \Delta\,\mathrm{Sim}\,E$$

Thus the simpler precondition $\Delta\,\mathrm{Sim}\,E$ is sufficient.

Remark 2.6 In a completely graded approach to fuzzy relations we should not be satisfied with the non-graded results of Lemma 2.3 and Corollary 2.4 (as they do not allow to infer anything unless E is a similarity *to degree* 1). Graded variants of Lemma 2.3 and Corollary 2.4 can indeed be derived by a more careful proof.[9] For instance,

$$\begin{aligned}
\mathrm{Sim}^2\,E &\rightarrow (\mathrm{Refl}_{2E}\,R \leftrightarrow \mathrm{Refl}_{1E}\,R)\\
\mathrm{Sim}^4\,E &\rightarrow (\mathrm{Sym}_{2E}\,R \leftrightarrow \mathrm{Sym}_{1E}\,R)\\
\mathrm{Sim}^6\,E &\rightarrow (\mathrm{Trans}_{2E}\,R \leftrightarrow \mathrm{Trans}_{1E}\,R),\ \text{etc.,}
\end{aligned}$$

where $\mathrm{Sim}^n\,E$ stands for $\mathrm{Sim}\,E\,\&\ldots\&\,\mathrm{Sim}\,E$ (n times). (The same abbreviations for multiple conjunctions will also be used for Refl, Sym, etc.)[10]

3 E-relative properties vs. extensionality w.r.t. E

In the non-graded approach, the motivation of our E-properties leads to the notion of *extensionality* of a relation R w.r.t. a relation (usually a similarity) E. Indeed, the definition of extensionality expresses the same idea of the congruence of R w.r.t. E. The *graded* definition of extensionality (of which the non-graded version is obtained by requiring its 1-validity) reads as follows:

Definition 3.1

$$\mathrm{Ext}_E\,R \;\equiv\; \forall xx'yy'(Exx'\,\&\,Eyy'\,\&\,Rxy \rightarrow Rx'y')$$

It can be shown that in the non-graded approach, extensionality is a sufficient substitute for E-properties (see Corollary 3.4 below). However, if graded properties are taken into account, E-properties can only partially be reduced to the conjunction of the usual properties and extensionality:

Theorem 3.2 *1.* $\mathrm{Refl}^2\,E\,\&\,\mathrm{Ext}_E\,R \rightarrow (\mathrm{Refl}_E\,R \leftrightarrow \mathrm{Refl}\,R)$

2. $\mathrm{Prox}^2\,E\,\&\,\mathrm{Ext}_E\,R \rightarrow (\mathrm{Sym}_E\,R \leftrightarrow \mathrm{Sym}\,R)$

3. $\mathrm{Prox}^3\,E\,\&\,\mathrm{Ext}^2_E\,R \rightarrow (\mathrm{Trans}_E\,R \leftrightarrow \mathrm{Trans}\,R)$

4. $\mathrm{Prox}^2\,E\,\&\,\mathrm{Ext}_E\,R \rightarrow (\mathrm{Asym}_E\,R \leftrightarrow \mathrm{Asym}_{(E)}\,R)$

5. $\mathrm{Prox}\,E\,\&\,\mathrm{Ext}_E\,R \rightarrow (\mathrm{Fnc}_E\,R \leftrightarrow \mathrm{Fnc}_{(E)}\,R)$

Proof: We shall show e.g. the proof for antisymmetry, the other proofs are analogous.

First we prove in first-order MTL that

$$\mathrm{Refl}^2\,E \rightarrow (\mathrm{Asym}_E\,R \rightarrow \mathrm{Asym}_{(E)}\,R)$$

By specifying x for x' and y for y' in $\mathrm{Asym}_E\,R$ we get $Exx\,\&\,Eyy\,\&\,Rxy\,\&\,Ryx \rightarrow Exy$. Detaching Exx and Eyy by double use of $\mathrm{Refl}\,E$, we get $\mathrm{Asym}_{(E)}\,R$ by generalization on xy.

Next we prove that

$$\mathrm{Sym}^2\,E\,\&\,\mathrm{Ext}_E\,R \rightarrow (\mathrm{Asym}_{(E)}\,R \rightarrow \mathrm{Asym}_E\,R)$$

Clearly $Ex'x\,\&\,Ey'y\,\&\,Ry'x'$ implies Ryx by $\mathrm{Ext}_E\,R$, which together with Rxy implies Exy by $\mathrm{Asym}_{(E)}\,R$. Thus by $\mathrm{Sym}\,E\,\&\,Exx' \rightarrow Ex'x$ and $\mathrm{Sym}\,E\,\&\,Eyy' \rightarrow Ey'y$ we have

[9] From graded variants of the statements used in the proof of Lemma 2.3, see [8, Prop. 18.6.1] or [3, L. 4.21], it follows that $\mathrm{Sim}^2\,E \rightarrow (E'xy \leftrightarrow Exy)$. This is then used once for each variable occurring in the defining formula of Refl_{1E}, Sym_{1E}, etc.

In fact, the precondition $\mathrm{Sim}^2\,E$ can be weakened a bit: it suffices if $\mathrm{Refl}^2\,E\&\mathrm{Sym}\,E\&\mathrm{Trans}^2\,E$, i.e., if E is a *preorder similarity,* $\mathrm{Sim}\,E\,\&\,\mathrm{Preord}\,E$. (Notice that in the graded approach, notions like *transitive similarity* or *reflexive preorder* are meaningful and strengthen non-trivially the respective conditions.)

[10] By [10], $\varphi\,\&\,\varphi$ can be interpreted as "very φ". Thus informally, $\mathrm{Refl}^2\,E$ can be understood as requiring E to be *very* reflexive, $\mathrm{Refl}^3\,E$ even more reflexive, etc. One must, however, be careful here, since $\varphi\,\&\,\varphi$ is not the only possible interpretation of "very", and the meaning of "very" in natural language usually differs from this particular one. Therefore this kind of reading of the exponents can only be understood as a rough, 'heuristic' aid.

$\mathrm{Sym}^2 E$ & $\mathrm{Ext}_E R$ & $\mathrm{Asym}_{(E)} R \to (Exx'$ & Eyy' & Rxy & $Ry'x' \to Exy)$, whence the required formula follows by generalization. QED

Notice that for the reduction of $\mathrm{Trans}_E R$ to $\mathrm{Trans}\, R$ we needed $\mathrm{Ext}_E R$ twice. The following counter-example shows that single $\mathrm{Ext}_E R$ is not sufficient.

Example 3.3 *Let the universe of discourse comprise six elements a, a', b, b', c, c' with $Eaa = Ea'a' = Ebb = Eb'b' = Ecc = Ec'c' = 1$, $Eaa' = Ea'a = Ebb' = Eb'b = Ecc' = Ec'c = 0.9$,*
$Rab = Rb'c = 1$,
$Rab' = Ra'b = Rac = Rbc = Rb'c' = 0.8$,
$Ra'b' = Rac' = Ra'c = Rbc' = 0.7$,
$Ra'c' = 0.5$, and $Exy = Rxy = 0$ otherwise. Then for the Łukasiewicz t-norm, the truth value of $\mathrm{Prox}\, E$ is 1, that of $\mathrm{Ext}_E R$ is 0.9, and that of $\mathrm{Trans}\, R$ is 1; thus the truth value of $\mathrm{Prox}^3 E$ & $\mathrm{Ext}_E R$ & $\mathrm{Trans}\, R$ is 0.9, while that of $\mathrm{Trans}_E R$ is only 0.8.

Similarly, single $\mathrm{Ext}_E R$ is not enough for complex notions like similarity or preorder, where we must sum up the exponents; thus, e.g., $(\mathrm{Sim}_E R \leftrightarrow \mathrm{Sim}\, R) \leftarrow \mathrm{Refl}^7 E$ & $\mathrm{Sym}^5 E$ & $\mathrm{Ext}^3_E R$.

As a corollary to Theorem 3.2 we get the reduction of E-properties to the usual ones by E-extensionality for non-graded notions:

Corollary 3.4 *Let $\Delta\,\mathrm{Prox}\, E$ & $\Delta\,\mathrm{Ext}_E R$. Then the following equivalences are 1-valid:*

$$\begin{aligned}
\mathrm{Refl}_E R &\leftrightarrow \mathrm{Refl}\, R \\
\mathrm{Sym}_E R &\leftrightarrow \mathrm{Sym}\, R \\
\mathrm{Trans}_E R &\leftrightarrow \mathrm{Trans}\, R \\
\mathrm{Asym}_E R &\leftrightarrow \mathrm{Asym}_{(E)} R \\
\mathrm{Fnc}_E R &\leftrightarrow \mathrm{Fnc}_{(E)} R \\
\mathrm{Preord}_E R &\leftrightarrow \mathrm{Preord}\, R \\
\mathrm{Prox}_E R &\leftrightarrow \mathrm{Prox}\, R \\
\mathrm{Sim}_E R &\leftrightarrow \mathrm{Sim}\, R
\end{aligned}$$

Proof: By Δ-necessitation applied to Theorem 3.2 and the appropriate distribution of the Δ's. QED

As a corollary we can see that if E is a full similarity, it is already a full similarity w.r.t. itself:

Proposition 3.5 $\Delta\,\mathrm{Sim}\, E \leftrightarrow \Delta\,\mathrm{Sim}_E E$

Proof: From Corollary 3.4 and the following Lemma 3.6. QED

Lemma 3.6 $\mathrm{Trans}^2 E$ & $\mathrm{Sym}\, E \to \mathrm{Ext}_E E$

Proof: Exx'&Exy&$Eyy' \to Ex'y'$ by applying symmetry to the first conjunct and then transitivity twice. QED

Corollary 3.4 explains why most of the E-properties have not yet been defined in the fuzzy literature: in the (prevalent) non-graded approach, in order to satisfy the natural idea of congruence w.r.t. E it is sufficient for R to be (fully) E-extensional, provided E is a (full) proximity (or even similarity).

There have been three notable exceptions to the absence of E-properties from the literature: the (E)-notions of Definition 2.1 (e.g., in [3], [4], [11]), and E-reflexivity whose non-graded variant sometimes occurs as one of the axioms of similarity-based fuzzy ordering (e.g., in [4]). Corollary 3.4 sheds some light on why this is so:

First, notice that full extensionality reduces $\mathrm{Asym}_E R$ and $\mathrm{Fnc}_E R$ only to $\mathrm{Asym}_{(E)} R$ resp. $\mathrm{Fnc}_{(E)} R$; thus the two notions of Definition 2.1 are indispensable even in the non-graded approach.[11]

Second, the E-reflexivity has been explained in the non-graded theory of fuzzy orders as a *combination* of ordinary reflexivity and extensionality, to which it is indeed equivalent under certain conditions:

Theorem 3.7 $\Delta\,\mathrm{Prox}\, E$ & $\Delta\,\mathrm{Trans}\, R \to$
$\to (\Delta\,\mathrm{Refl}_E R \leftrightarrow \Delta\,\mathrm{Refl}\, R$ & $\Delta\,\mathrm{Ext}_E R)$

Proof: By Δ-necessitation from the following easy lemma, which shows how the situation changes under gradedness; its proof is similar to that of Theorem 3.2. QED

[11] In [3] they are already generalized to their *graded* versions.

Lemma 3.8 *1.* $\mathrm{Prox}\, E \,\&\, \mathrm{Trans}^2 R \to$
$\to (\mathrm{Refl}^2_E R \to \mathrm{Ext}_E R \,\&\, \mathrm{Refl}\, R)$

2. $\mathrm{Refl}\, E \to (\mathrm{Ext}_E R \,\&\, \mathrm{Refl}\, R \to \mathrm{Refl}_E R)$

Thus, since the preconditions of Theorem 3.7 are always presupposed in the non-graded definition of similarity-based fuzzy ordering, $\mathrm{Refl}_E R$ indeed plays the role of both reflexivity and E-extensionality there; and since therefore E-extensionality is already ensured by $\mathrm{Refl}_E R$, it is not necessary to introduce it into the definition of transitivity in the non-graded theory of fuzzy orderings.

This explains why E-transitivity and E-symmetry[12] have not been defined in the theory of fuzzy relations, even though Refl_E, $\mathrm{Asym}_{(E)}$, and $\mathrm{Fnc}_{(E)}$ have. Theorem 3.2 further shows that in *graded* properties of fuzzy relations, E-notions have nevertheless to be distinguished from the simple presence of E-extensionality.

4 Some generalizations

In this paper we restricted our attention to reflexivity, symmetry, transitivity, antisymmetry, and functionality (and some combinations thereof), as they are the most usual properties of fuzzy relations found in the literature. The definitions and results can, however, be easily extended to a wider class of graded properties of fuzzy relations.[13]

A further generalization of Fnc_E might consist in considering different indistinguishability relations on the domain and codomain of the fuzzy relation. Thus we could define $\mathrm{Fnc}_{E_1,E_2} R \equiv \forall x x' y y' (E_1 x x' \,\&\, Rxy \,\&\, Rx'y' \to E_2 y y')$. However, this definition can always be reduced to Definition 2.2 by taking the disjoint union of E_1 and E_2 on the disjoint union of (the supports of) the domain and codomain of R.[14]

Finally, the E-relative properties treated in this paper require that indistinguishable individuals behave uniformly, just as if they were equal. In some situations, however, it may be sufficient that *any* (rather than all) objects among those indiscernible are in relation R. Thus we can define the 'existential' versions of E-relative properties as follows:

Definition 4.1

$$\begin{aligned}
\mathrm{Refl}^{\exists}_E R &\equiv \forall x \exists x' (Exx' \,\&\, Rxx') \\
\mathrm{Sym}^{\exists}_E R &\equiv \forall xy (Rxy \to \\
&\qquad \exists x'y' (Exx' \,\&\, Eyy' \,\&\, Ry'x')) \\
\mathrm{Trans}^{\exists}_E R &\equiv \forall xyz (Rxy \,\&\, \exists y' (Eyy' \,\&\, Ry'z) \\
&\qquad \to \exists x'z' (Exx' \,\&\, Ezz' \,\&\, Rx'z'))
\end{aligned}$$

These notions are weaker than those of Definition 2.2 if E is reflexive enough:

Observation 4.2

$$\begin{aligned}
\mathrm{Refl}\, E &\leftrightarrow (\mathrm{Refl}_E R \to \mathrm{Refl}^{\exists}_E R) \\
\mathrm{Refl}^2 E &\leftrightarrow (\mathrm{Sym}_E R \to \mathrm{Sym}^{\exists}_E R) \\
\mathrm{Refl}^2 E &\leftrightarrow (\mathrm{Trans}_E R \to \mathrm{Trans}^{\exists}_E R)
\end{aligned}$$

The differences between both variants of E-properties are shown by their characterizations in terms of relational compositions:[15]

Observation 4.3

$$\begin{aligned}
\mathrm{Refl}_E R &\leftrightarrow E \subseteq R \\
\mathrm{Refl}^{\exists}_E R &\leftrightarrow \mathrm{I} \subseteq R \circ E^{-1} \\
\mathrm{Sym}_E R &\leftrightarrow E^{-1} \circ R^{-1} \circ E \subseteq R \\
\mathrm{Sym}^{\exists}_E R &\leftrightarrow R \subseteq E \circ R^{-1} \circ E^{-1} \\
\mathrm{Trans}_E R &\leftrightarrow E^{-1} \circ R \circ E \circ R \circ E \subseteq R \\
\mathrm{Trans}^{\exists}_E R &\leftrightarrow R \circ E \circ R \subseteq E \circ R \circ E^{-1}
\end{aligned}$$

[12] $\Delta\, \mathrm{Refl}_E R$ has also been used in the non-graded definition of E-extensional *similarity,* in which the preconditions of Theorem 3.7 are satisfied as well; therefore it supplies the definition with the needed $\Delta\, \mathrm{Ext}_E R$, and thus it is not necessary to build extensionality into the definitions of symmetry or transitivity there, either.

[13] The methods shown here work at least for properties given by formulae $\forall x_1 \ldots x_n \big(\&\varphi_i \to \psi\big)$, where all φ_i and ψ are atoms of the form $Rx_k x_l$ or $x_k{=}x_l$.

[14] Still, Fnc_{E_1,E_2} can sometimes be a convenient notation; e.g., $\mathrm{Fnc}_{(E)}$ of Definition 2.1 is in fact $\mathrm{Fnc}_{=,E}$.

[15] $R \subseteq S$ is defined as $\forall xy (Rxy \to Sxy)$ and $\mathrm{I}xy \equiv x{=}y$.

Notice that even the notion of relational composition should be made E-relative under the presence of indistinguishability E, namely

$$(R \circ_E S)xy \equiv \exists zz' (Rxz \,\&\, Ezz' \,\&\, Sz'y)$$

Obviously $R \circ_E S = R \circ E \circ S$.

The choice of the appropriate variant of an E-property depends on the context; in particular, whether the objects are given to us (e.g., by Nature) or we can choose them; or alternatively, whether indistinguishable objects must all behave as required, or only one object suffices to witness the property.

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