GENERAL LOGICAL FORMALISM FOR FUZZY MATHEMATICS: METHODOLOGY AND APPARATUS*

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ABSTRACT: There is a programme in the formal foundations of fuzzy mathematics proposed by the authors, the goal of which is to encompass a large part of existing fuzzy mathematics within a general logical formalism. This paper presents the methodology behind this programme and reviews the technical aspects of a particular apparatus for this enterprise.

Keywords: Fuzzy logic, fuzzy mathematics, axiomatization, formalization, higher-order logic, ŁΠ-logic.

1 INTRODUCTION

Classical mathematics and logic can model the concept of vagueness only indirectly. Even though many-valued logics were developed (for other purposes) already during the first half of the XX century, a systematic study of vagueness by means of the many-valued approach began only after L.A. Zadeh [16] proposed to investigate fuzzy sets in 1965. Since then, the notion of fuzziness spread to nearly all mathematical disciplines: fuzzy arithmetic, fuzzy logic, fuzzy probability, fuzzy relations, fuzzy topology, etc. For a long time, however, fuzzy logic and fuzzy mathematics were engineering tools rather than well-designed mathematical theories. Driven mainly by applications, they lacked (meta)theoretical grounding and general results; developed mostly by engineers for particular purposes, they suffered from arbitrariness in definitions and often even mathematical imprecision. Moreover, it has been objected that it was just a theory of [0,1]-valued functions and thus a part of real analysis. On the other hand, fuzzy logic describes the laws of truth preservation in reasoning under (a certain form of) vagueness, and its interpretation in terms of truth degrees is only a *model*—a classical (i.e., crisp) rendering of vague phenomena. If one wants to reason about fuzzy predicates in a *direct* way, not mediated by a crisp model, one should do so according to the logical laws that hold for fuzzy predicates, i.e., in fuzzy logic.

The need for axiomatization of fuzzy mathematics is beyond doubt—axiomatization has always aided the development of mathematical theories. There have been many (more or less successful) attempts to formalize or even axiomatize some areas of fuzzy mathematics. However, these axiomatic systems are usually designed ad hoc. The authors select some concepts in their area of interest and change them into vague ones. This selection is usually based mainly on intuition or on the desired application. Another problem with these axiomatic attempts lies in their fragmentation; it is nearly impossible to combine two of them into one theory. It would certainly be better if fuzzy mathematics as a whole could employ a unified methodology in building its axiomatic theories, because it would facilitate the exchange of results between its branches. We propose such a unified methodology for the axiomatization of fuzzy mathematics.

Our work, in this paper, has two levels. First, we sketch a proposal for this unified methodology (the full text of this proposal is found in our paper [1]) and then we sketch a proposal for *particular logical system* following these methodological guidelines (this part is based on our paper [2]). The present paper is preceded by the paper [3] (in this volume), which gives an introductory overview of concepts of the presented theory and concentrates on informal proof methods for doing fuzzy mathematics within its framework.[1]

2 METHODOLOGICAL PROGRAMME

In the axiomatic construction of classical mathematics, a three-layer architecture proved worthy, with the layers of logic, foundations, and only then individual mathematical disciplines. Individual disciplines are thus developed within the framework of a unifying formal theory, be it some variant of set theory, type theory, category theory, or another sufficiently rich and general kind of theory.

In fuzzy mathematics, the level of *logic* seems to be advanced far enough so as to support sufficiently strong formal theories (by the works of Gottwald [7], Hájek [8], Novák et al. [12], Mundici et al. [4], and others). There are many existing formal systems of fuzzy logic, and thus we first need to make some design choices. We—following Hájek—believe a certain style of logical systems to be a most suitable formalism for representing fuzzy inference. For brevity's sake, in what follows we shall call them *Hájek-style fuzzy logics.* Put in a nutshell, they are fuzzy logics retaining the syntax of classical logic (preferably without truth constants), defined as axiomatic systems (rather than non-axiomatizable sets of tautologies). A prototypical example is Hájek's Basic Logic BL, propositional or predicate.

There is a pragmatic motive for retaining as much of classical syntax as possible. The way of working in theories over Hájek-style fuzzy logics resembles closely the way of work-

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[1] This work is part of the research programme for formalization of fuzzy mathematics. The interested reader can check the web page of our project: www.cs.cas.cz/hp.

ing in classical logic: Hájek-style fuzzy logics are often just weaker variants of Boolean logic—syntactically fully analogous, just lacking some of its laws. Therefore, many theoretical and metatheoretical methods developed for classical logic can be mimicked and employed, resulting in a quick and sound development of the theory. This feature has already been utilized in metamathematics of fuzzy logic—the proofs of the completeness, deduction, and other metatheorems have often been obtained by adjustments of classical proofs. For more reasons for this restriction see [1].

The search for a suitable *foundational theory* is the task of the day. The close analogy between Hájek-style fuzzy logics and classical logic gives rise to a hope that fuzzy analogues of classical foundational theories will be able to harbour all (or at least nearly all) parts of existing fuzzy mathematics.

As conceivable candidates for a foundational theory, several ZF-style fuzzy set theories has already arisen ([9], [13]). Many of them are certainly capable of doing the job. Nevertheless, the axiomatics of most ZF-style fuzzy set theories savour of a similar ad hoc axiom choices as other hitherto attempts at axiomatization of fuzzy mathematics. By large this is induced by the fact that such theories have to deal with a specific set-theoretical agenda and take into the account the structure of the whole set universe (expressed, e.g., by the axiom of well-foundedness). Moreover, for many of them it is not clear whether they can straightforwardly be generalized to other fuzzy logics than the one in which they were developed; thus they are only capable of providing the foundation for a limited part of fuzzy mathematics.

After a universal foundational theory is successfully found, the development of *individual concepts of fuzzy mathematics* has to proceed in a systematic way, taking into the account the dependencies between them as in classical mathematics. For example, the notion of cardinality should only be defined after the introduction and investigation of the notion of function, upon which it is based (and which in turn is based upon the concept of fuzzy equality, i.e., similarity). The coherence of defined notion from the point of view of category theory should also be checked. Only this kind of systematic approach can avoid giving ad hoc definitions of fuzzy concepts, which often suffer from arbitrariness and hidden crispness. Furthermore, if the background logic is sufficiently strong, there is a general method (to be described later) of embedding any classical theory, and even of its natural fuzzification (as well as conscious and controlled 'defuzzification' of its concepts if some of their features are to be left crisp). The method has already been foreshadowed in U. Höhle's 1987 paper [10, Section 5]:

> "It is the opinion of the author that from a mathematical viewpoint the important feature of fuzzy set theory is the replacement of the two-valued logic by a multiple-valued logic. [. . . I]t is now clear how we can find for every mathematical notion its 'fuzzy counterpart'. Since every mathematical notion can be written as a formula in a formal language, we have only to internalize, i.e. to interpret these expressions by the given multiple-valued logic."

As a concrete implementation of the general programme sketched above we propose a specific foundational theory described below. We do not claim it to be the only possible way neither of doing the foundations of fuzzy mathematics, nor of fulfilling our foundational programme; nevertheless, it seems to be a viable foundation for fuzzy mathematics of the present day.

By inspecting the existing approaches and having in mind the need for generality and simplicity, it becomes obvious that a full-fledged set theory is not necessary for the foundations of fuzzy mathematics. What is necessary is only the ability to perform within the theory the basic constructions of fuzzy mathematics. On the other hand, a great variability of the background fuzzy logic is required in order to encompass the whole of fuzzy mathematics.

Most notions of classical mathematics can be defined within the first few levels of a simple type theory. The similarity between classical and Hájek-style fuzzy logics hints that this could be true for fuzzy concepts defined in a fuzzified simple type theory as well. Indeed, many important notions can be defined already at the first level, which is in fact second-order fuzzy logic. Most notably, elementary fuzzy set theory, or the axiomatization of *Zadeh's notion of fuzzy set,* is contained in second-order fuzzy logic (second-order models are exactly Zadeh's universes of fuzzy sets). Some theories (e.g., topology), however, need more levels of type hierarchy, thus we employ higher-order fuzzy logic.

Unfortunately, fuzzy higher-order logic is not recursively axiomatizable. Since we prefer axiomatic deductive theories over non-axiomatizable sets of tautologies, we choose its Henkin-style variant, even though it admits non-intended models. We thus get a *first-order theory,* axiomatized very naturally by the extensionality, comprehension, and tuple-handling axioms for each order. Moreover, the construction works for virtually all imaginable fuzzy logics (and many non-fuzzy logics as well). The bunch of foundational theories we propose thus can be called *Henkin-style higher-order fuzzy logic* (for an individual fuzzy logic of one's choice; expressively rich logics like ŁΠ seem to be sufficient for all practical purposes). The details of this formalism can be found in the following sections (or in our paper [2]).

3 LOGIC

In what follows, we assume the basic knowledge of Hájek's Basic fuzzy logic BL and its three important extensions (Łukasiewicz, Gödel, and product logics) or their extensions by the projector Δ (see [8] for details). We however recall the definition of the logic ŁΠ and some of its properties. This logic has some unique features, which seem to be important to achieve our goal. It is common to use different (non-corresponding) connectives in existing fuzzy mathematics; e.g., for some application we need the product t-norm as conjunction and the Łukasiewicz residuum for implication. The logic ŁΠ combines all three basic logics into one and thus it offers huge expressive power (see Theorem 6 for more details). The definitions and theorems in this section are from papers [6] and [5].

Definition 1 *The logic* ŁΠ *has the following* basic connectives *(they are listed together with their standard semantics in* $[0,1]$; *we use the same symbols for logical connectives and the corresponding algebraic operations):*

0	0	*truth constant falsum*
$\varphi \to_{\mathrm{L}} \psi$	$x \to_{\mathrm{L}} y = \min(1, 1-x+y)$	*Łukasiewicz implication*
$\varphi \to_{\Pi} \psi$	$x \to_{\Pi} y = \min(1, \frac{x}{y})$	*product implication*
$\varphi \&_{\Pi} \psi$	$x \&_{\Pi} y = x \cdot y$	*product conjunction*

The logic ŁΠ$\frac{1}{2}$ *has one additional truth constant* $\frac{1}{2}$ *with the standard semantics* $\frac{1}{2}$. *We define the following* derived connectives:

$\neg_{\mathrm{L}}\varphi$	*is*	$\varphi \to_{\mathrm{L}} 0$	$\neg_{\mathrm{L}} x = 1 - x$
$\neg_{\Pi}\varphi$	*is*	$\varphi \to_{\Pi} 0$	$\neg_{\Pi} x = 1$ *if* $x = 0$, *otherwise* 0
1	*is*	$\neg_{\mathrm{L}} 0$	1
$\Delta\varphi$	*is*	$\neg_{\Pi}\neg_{\mathrm{L}}\varphi$	$\Delta x = 1$ *if* $x = 1$, *otherwise* 0
$\varphi \&_{\mathrm{L}} \psi$	*is*	$\neg_{\mathrm{L}}(\varphi \to_{\mathrm{L}} \neg_{\mathrm{L}}\psi)$	$x \&_{\mathrm{L}} y = \max(0, x+y-1)$
$\varphi \oplus \psi$	*is*	$\neg_{\mathrm{L}}\varphi \to_{\mathrm{L}} \psi$	$x \oplus y = \min(1, x+y)$
$\varphi \ominus \psi$	*is*	$\varphi \&_{\mathrm{L}} \neg_{\mathrm{L}}\psi$	$x \ominus y = \max(0, x-y)$
$\varphi \wedge \psi$	*is*	$\varphi \&_{\mathrm{L}} (\varphi \to_{\mathrm{L}} \psi)$	$x \wedge y = \min(x, y)$
$\varphi \vee \psi$	*is*	$\neg_{\mathrm{L}}(\neg_{\mathrm{L}}\varphi \wedge \neg_{\mathrm{L}}\psi)$	$x \vee y = \max(x, y)$
$\varphi \to_{\mathrm{G}} \psi$	*is*	$\Delta(\varphi \to_{\mathrm{L}} \psi) \vee \psi$	$x \to_{\mathrm{G}} y = 1$ *if* $x \leq y$, *othw.* y
$\varphi = \psi$	*is*	$\Delta(\varphi \leftrightarrow_{\mathrm{L}} \psi)$	1 *if* $x = y$, *otherwise* 0
$\varphi \leq \psi$	*is*	$\Delta(\varphi \to_{\mathrm{L}} \psi)$	1 *if* $x \leq y$, *otherwise* 0
$\varphi \leq \frac{1}{2}$	*is*	$\varphi \leq \neg_{\mathrm{L}}\varphi$	1 *if* $x \leq \frac{1}{2}$, *otherwise* 0

Occasionally we may write $\neg_{\mathrm{G}}$ *and* $\&_{\mathrm{G}}$ *as synonyms for* $\neg_{\Pi}$ *and* $\wedge$, *respectively. We further abbreviate* $(\varphi \to_{*} \psi) \&_{*} (\psi \to_{*} \varphi)$ *by* $\varphi \leftrightarrow_{*} \psi$ *for* $* \in \{\mathrm{G}, \mathrm{L}, \Pi\}$ *and* $\varphi \veebar_{*} \psi$ *for* $\neg_{\mathrm{L}}(\neg_{\mathrm{L}}\varphi \&_{*} \neg_{\mathrm{L}}\psi)$.

The *standard* ŁΠ*-algebra* $[\mathbf{0}, \mathbf{1}]$ has the domain $[0,1]$ and the operations as stated in Definition 1 above (analogously for the standard ŁΠ$\frac{1}{2}$-algebra).

Definition 2 *The logic* ŁΠ *is given by the following axioms and deduction rules:*

(Ł)	*The axioms of Łukasiewicz logic*
(Π)	*The axioms of product logic*
(ŁΔ)	$\Delta(\varphi \to_{\mathrm{L}} \psi) \to_{\mathrm{L}} (\varphi \to_{\Pi} \psi)$
(ΠΔ)	$\Delta(\varphi \to_{\Pi} \psi) \to_{\mathrm{L}} (\varphi \to_{\mathrm{L}} \psi)$
(Dist)	$\varphi \&_{\Pi} (\chi \ominus \psi) \leftrightarrow_{\mathrm{L}} (\varphi \&_{\Pi} \chi) \ominus (\varphi \&_{\Pi} \psi)$

The deduction rules are modus ponens and Δ*-necessitation (from* φ *infer* $\Delta\varphi$*). The logic* ŁΠ$\frac{1}{2}$ *results from* ŁΠ *by adding the axiom* $\frac{1}{2} \leftrightarrow \neg_{\mathrm{L}} \frac{1}{2}$.

The notions of proof, derivability $\vdash$, theorem, and theory over ŁΠ and ŁΠ$\frac{1}{2}$ are defined as usual.

Theorem 3 (Completeness) *Let* φ *be a formula of* ŁΠ *(*ŁΠ$\frac{1}{2}$ *respectively). Then the following conditions are equivalent:*

- φ *is a theorem of* ŁΠ *(*ŁΠ$\frac{1}{2}$ *resp.)*
- φ *is a* $[\mathbf{0}, \mathbf{1}]$*-tautology.*

The following definitions and theorems demonstrate the expressive power of ŁΠ and ŁΠ$\frac{1}{2}$. In particular, Corollary 7 shows that each propositional logic based on an arbitrary continuous t-norm of a certain simple form is contained in ŁΠ$\frac{1}{2}$.

Definition 4 *A function* $f : [0,1]^n \to [0,1]$ *is called a* rational ŁΠ-function *iff there is a finite partition of* $[0,1]^n$ *such that each block of the partition is a semi-algebraic set and* f *restricted to each block is a fraction of two polynomials with rational coefficients.*

Furthermore, a rational n-ary ŁΠ*-function* f *is* integral *iff all the coefficients are integer and* $f(\{0,1\}^n) \subseteq \{0,1\}$.

Definition 5 *Let* f *be a function* $f : [0,1]^n \to [0,1]$ *and* $\varphi(v_1, \ldots, v_n)$ *be a formula. We say that the function* f *is* represented by the formula φ *(*φ *is a representation of* f*) iff* $e(\varphi) = f(e(v_1), e(v_2), \ldots, e(v_m))$ *for each evaluation e.*

Theorem 6 (Functional representation) *A function* f *is an integral (rational respectively)* ŁΠ *function iff it is represented by some formula of* ŁΠ *(*ŁΠ$\frac{1}{2}$ *resp.).*

The following theorem was proved in [5], but it can be viewed as a corollary of the previous theorem.

Corollary 7 *Let* $*$ *be a continuous t-norm which is a finite ordinal sum of the three basic ones (i.e., of* G, L *and* Π*), and* $\Rightarrow$ *be its residuum. Then there are derived connectives* $\&_{*}$ *and* $\to_{*}$ *of the* ŁΠ$\frac{1}{2}$ *logic such that their standard* $[\mathbf{0}, \mathbf{1}]$*-semantics are* $*$ *and* $\Rightarrow$ *respectively. The logic* $PC(*)$ *of the t-norm* $*$ *(see [8]) is contained in* ŁΠ$\frac{1}{2}$ *if* & *and* $\to$ *of* $PC(*)$ *are interpreted as* $\&_{*}$ *and* $\to_{*}$*. Furthermore, if* φ *is provable in* $PC(*)$ *(and a fortiori, if it is provable in Hájek's logic* BLΔ*, see [8]), then the formula* φ_{*} *obtained from* φ *by replacing the connectives* & *and* $\to$ *of* $PC(*)$ *(or* BLΔ*) by* $\&_{*}$ *and* $\to_{*}$ *is provable in* ŁΠ$\frac{1}{2}$.

Corollary 8 *Let* $r \in [0,1]$ *be a rational number; then there is a formula* φ *of* ŁΠ$\frac{1}{2}$ *such that* $e(\varphi) = r$ *for any* $[\mathbf{0}, \mathbf{1}]$*-evaluation e.*

We assume the reader to be familiar with the notions of the multi-sorted predicate language and semantics of predicate fuzzy logics. We only recall the axioms of the predicate ŁΠ with crisp equality.

Definition 9 (First-order ŁΠ**)** *First-order* ŁΠ *logic adds the deduction rule of generalization and the following axioms for quantifiers and (crisp) identity:*

(∀1)	$(\forall x)\varphi(x) \to \varphi(t)$, *if t is substitutable for x in* ψ
(∀2)	$(\forall x)(\chi \to_{\mathrm{L}} \varphi) \to (\chi \to_{\mathrm{L}} (\forall x)\varphi)$, *x not free in* χ
(=1)	$x = x$
(=2)	$x = y \to \Delta(\varphi(x) \leftrightarrow \varphi(y))$

$(\exists x)\varphi$ *is defined as* $\neg_{\mathrm{L}}(\forall x)\neg_{\mathrm{L}}\varphi$.

4 FOUNDATIONAL THEORY

Now we sketch our foundational theory. Its first step is a fuzzy class theory (FCT, see [2] for details). We formulate this theory over arbitrary fuzzy logic (stronger than BLΔ). However, keep in mind that the logic ŁΠ is the intended background logic. To put our theory into the context we notice that the FCT is nothing else than a Henkin-style second-order fuzzy logic.

Definition 10 (Henkin-style second-order fuzzy logic) *Let* $\mathcal{F}$ *be a fuzzy logic which extends* BLΔ*. The Henkin-style second-order fuzzy logic* $\mathcal{F}$ *is a theory over multi-sorted first-order* $\mathcal{F}$ *with sorts for objects (lowercase variables) and fuzzy sets (uppercase variables). Both of the sorts subsume subsorts for n-tuples, for all* $n \geq 1$*. Apart from the obvious necessary function symbols and axioms for tuples (tuples equal iff their respective constituents equal), the only primitive symbol is the membership predicate* $\in$ *between objects and (fuzzy) sets. The axioms for* $\in$ *are the following:*

1. *The comprehension axioms $(\exists X)\Delta(\forall x)(x \in X \leftrightarrow \varphi)$, φ not containing X, which enable the (eliminable) introduction of comprehension terms $\{x \mid \varphi\}$ with the axioms $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$ (where φ may be allowed to contain other comprehension terms).*

2. *The extensionality axiom $(\forall x)\Delta(x \in X \leftrightarrow x \in Y) \rightarrow X = Y$.*

Convention 11 *The usual precedence of connectives is assumed. The formulae $(\forall x)(x \in X \rightarrow \varphi)$ and $(\exists x)(x \in X \mathbin{\&} \varphi)$ are abbreviated $(\forall x \in X)\varphi$ and $(\exists x \in X)\varphi$, respectively (similar notation can be used for defined binary predicates). The formulae $\varphi \mathbin{\&} \ldots \mathbin{\&} \varphi$ (n times) are abbreviated φ^n. Furthermore, $\{\langle x_1, \ldots, x_k\rangle \mid \varphi\}$ is shorthand for $\{x \mid (\exists x_1)\ldots(\exists x_k)(x = \langle x_1, \ldots, x_k\rangle \mathbin{\&} \varphi)\}$. An alternative notation for $x \in A$ and $\langle x_1, \ldots, x_n\rangle \in R$ is simply Ax and $Rx_1 \ldots x_n$, respectively.*

To get our foundational theory we just iterate the previous definition to get third-(fourth-, ω-)order fuzzy logic.

Definition 12 (Henkin-style higher-order fuzzy logic) *Henkin-style fuzzy logic of higher orders is obtained by repeating the previous definition on each level of the type hierarchy. Obviously, defined symbols of any type can then be shifted to all higher types as well. (Consequently, all theorems are preserved by uniform upward type-shifts.) Types may be allowed to subsume all lower types.*

Henkin-style fuzzy logic $\mathcal{F}$ of order n will be denoted by $\mathcal{F}_n$, the whole hierarchy by $\mathcal{F}_\omega$. The types of terms are either denoted by a superscripted parenthesized type (e.g., $X^{(3)}$), or understood from the context.

It should be stressed that despite the name, Henkin-style higher-order fuzzy logics are *theories* over *first-order* fuzzy logics (see [8]). Observe that Henkin-style ω-order fuzzy logic can also be viewed as a fuzzy type theory (recall that a fuzzy type theory for the logic IMTLΔ was introduced by Novák in [11]).

Now we present a uniform way for fuzzifying crisp theories. Since our theory contains classical type theory, we can introduce arbitrary relations and functions on the universe of objects which are definable in classical type theory. As they can be described by formulae, their existence is guaranteed by the comprehension axiom. So the only thing we need to add is a constant of the appropriate sort and the instance of the comprehension axiom. The following definition is the formalization of this approach for the first-order theories.

Definition 13 *Let Γ be a classical one-sorted predicate language and T be a Γ-theory. For each n-ary predicate symbol P of Γ let us introduce a new constant $\bar{P}$ for a set of n-tuples, and for each n-ary function symbol F we take a new constant $\bar{F}$ for a fuzzy set of (n+1)-tuples. We define the language $\mathcal{F}_2(\Gamma)$ as the language of $\mathcal{F}_2$ extended by the symbols $\bar{Q}$ for each symbol $Q \in \Gamma$. The translation $\bar{\varphi}$ of a Γ-formula φ to $\mathcal{F}_2(\Gamma)$ is obtained as the result of replacing all occurrences of all Γ-symbols Q in φ by $\bar{Q}$.*

We define the theory $\mathcal{F}_2(T)$ in the language $\mathcal{F}_2(\Gamma)$ as the theory with the following axioms:

- *The axioms of $\mathcal{F}_2$*
- *The translations $\bar{\varphi}$ of all axioms φ of T*
- $\mathrm{Crisp}(\bar{Q})$ *for each symbol $Q \in \Gamma$ (for the definition of* Crisp*, see Table 2)*
- $\langle x_1, \ldots x_n, y\rangle \in \bar{F} \mathbin{\&} \langle x_1, \ldots x_n, z\rangle \in \bar{F} \rightarrow y = z$ *for each n-ary function symbol $F \in \Gamma$.*

Lemma 14 *Let Γ be a classical predicate language, T a Γ-theory. If $\mathbf{M}$ is a model of $\mathcal{F}_2(T)$, then $\mathbf{M}^c = (M, (Q_{\mathbf{M}^c})_{Q\in\Gamma})$, where $Q_{\mathbf{M}^c} = \bar{Q}_{\mathbf{M}}$ for each $Q \in \Gamma$, is a model (in the sense of classical logic) of the theory T.*

Vice versa, for each model $\mathbf{M}$ of T there is a model $\mathbf{N}$ of $\mathcal{F}_2(T)$ such that $\mathbf{N}^c$ is isomorphic to $\mathbf{M}$.

Therefore, $T \vdash \varphi$ iff $\mathcal{F}_2(T) \vdash \bar{\varphi}$, for any Γ-formula φ.

Example 15 *Let R be a constant for a set of pairs. Then in each model of the theory* $\mathrm{Crisp}(R)$, $\mathrm{Refl}(R)$, $\mathrm{Trans}(R)$, *$(\forall x, y)(Rxy \mathbin{\&} Ryx \rightarrow x = y)$, the constant R is represented by a crisp ordering on the universe of objects. (For the definitions of* Refl *and* Trans*, see Definition 24.)*

Example 16 *If T is a classical theory of the real closed field, then in each* **L***-model $\mathbf{M}$ of the theory $\mathcal{F}_2(T)$, the universe of objects with $\bar{\leq}_{\mathbf{M}}, \bar{+}_{\mathbf{M}}, \bar{-}_{\mathbf{M}}, \bar{\cdot}_{\mathbf{M}}, \bar{0}_{\mathbf{M}}, \bar{1}_{\mathbf{M}}$ is a real closed field.*

In Lemma 14 we speak of first-order theories only. Nevertheless, it can be extended to any theory formalizable in classical type theory. Here we present only one example.

Example 17 *Let τ be a constant for a (fuzzy) set of (fuzzy) sets and T the theory with the following axioms:*

- $\mathrm{Crisp}(\tau)$
- $(\forall X)(X \in \tau \rightarrow \mathrm{Crisp}(X))$
- $(\forall \mathcal{X})(\mathrm{Crisp}(\mathcal{X}) \mathbin{\&} \mathcal{X} \subseteq \tau \rightarrow \{x \mid (\exists X \in \mathcal{X})x \in X\} \in \tau)$
- $(\forall X_1)\ldots(\forall X_n)(X_1 \in \tau \mathbin{\&} \ldots \mathbin{\&} X_n \in \tau \rightarrow X_1 \cap \ldots \cap X_n \in \tau)$ *for each $n \in N$*

Then in each model of the theory T, the constant τ is represented by a classical topology on the universe of objects.

5 ELEMENTARY FUZZY SET THEORY

Now we demonstrate the power of our theory by showing how it can handle the elementary fuzzy set-theoretic operation and relations. Due to the restricted length of this paper we present only very basic notion, the details can be found in [2].

Convention 18 *Let $\varphi(p_1, \ldots, p_n)$ be a propositional formula and $\psi_1, \ldots, \psi_n$ be any formulae. By $\varphi(\psi_1, \ldots, \psi_n)$ we denote the formula φ in which all occurrences of p_i are replaced by ψ_i (for all $i \leq n$).*

Definition 19 *Let $\varphi(p_1, \ldots, p_n)$ be a propositional formula. We define the* n-ary set operation induced by φ *as*

$$\mathrm{Op}_\varphi(X_1, \ldots, X_n) =_{\mathrm{df}} \{x \mid \varphi(x \in X_1, \ldots, x \in X_n)\}.$$

We give examples of operations defined in this way in ŁΠ$_2$ in Table 1.

Table 1: Elementary set operations

φ	$\mathrm{Op}_\varphi(X_1,\ldots,X_n)$	Name
0	$\emptyset$	empty set
1	V	universal set
$\Delta(\alpha\to p)$	X_α	α-cut
$\Delta(\alpha\leftrightarrow p)$	$X_{=\alpha}$	α-level
$\neg_{\mathrm{G}} p$	$\backslash X$	strict complement
$\neg_{\mathrm{L}} p$	$-X$	involutive compl.
$\neg_{\mathrm{G}}\neg_{\mathrm{L}} p$ (or Δp)	$\mathrm{Ker}(X)$	kernel
$\neg\neg_{\mathrm{G}} p$ (or $\neg\Delta\neg_{\mathrm{L}} p$)	$\mathrm{Supp}(X)$	support
$p \mathbin{\&_*} q$	$X\cap_* Y$	$*$-intersection
$p \veebar_* q$	$X\cup_* Y$	$*$-union
$p \mathbin{\&_{\neg_{\mathrm{G}}}} q$	$X\setminus Y$	strict difference
$p \mathbin{\&_*} \neg_{\mathrm{L}} q$	$X -_* Y$	involutive $*$-diff.

Table 2: Set properties and relations

Relation	Notation	Name
$\mathrm{Rel}^{\exists}_{p}(X)$	$\mathrm{Hgt}(X)$	height
$\mathrm{Rel}^{\exists}_{\Delta p}(X)$	$\mathrm{Norm}(X)$	normality
$\mathrm{Rel}^{\forall}_{\Delta(p\vee\neg p)}(X)$	$\mathrm{Crisp}(X)$	crispness
$\mathrm{Rel}^{\exists}_{\neg\Delta(p\vee\neg p)}(X)$	$\mathrm{Fuzzy}(X)$	fuzziness
$\mathrm{Rel}^{\forall}_{p\to_* q}(X,Y)$	$X\subseteq_* Y$	$*$-inclusion
$\mathrm{Rel}^{\forall}_{p\leftrightarrow_* q}(X,Y)$	$X\approx_* Y$	$*$-equality
$\mathrm{Rel}^{\exists}_{p\&_* q}(X,Y)$	$X\|_* Y$	$*$-compatibility

Definition 20 (Uniform and supremal relations) *Let $\varphi(p_1,\ldots,p_n)$ be a propositional formula. The n-ary* uniform relation *between $X_1,\ldots,X_n$ induced by φ is defined as*

$$\mathrm{Rel}^{\forall}_{\varphi}(X_1,\ldots,X_n) \equiv_{\mathrm{df}} (\forall x)\varphi(x\in X_1,\ldots,x\in X_n).$$

The n-ary supremal relation *between $X_1,\ldots,X_n$ induced by φ is defined as*

$$\mathrm{Rel}^{\exists}_{\varphi}(X_1,\ldots,X_n) \equiv_{\mathrm{df}} (\exists x)\varphi(x\in X_1,\ldots,x\in X_n).$$

The advantage of this general definition of the notion of operation and relation is demonstrated by the following theorem (and its corollaries), which reduce provability in our theory to provability in the background propositional logic. Of course, the facts we prove this way are known (they can be found in any textbook about fuzzy sets), but we can prove all of them at once (provided we know propositional theorems of the background fuzzy logic). In fact we are proving more: we are proving *graded* theorems (which appeared for the first time in Gottwald's monograph [7]). For example theorem $X\subseteq_* Y \to X\cap_* Z\subseteq_* Y\cap_* Z$ says the that X intersected with Z is more a subset of Y intersected with Z than X is a subset of Y (written in a "traditional fuzzy" way, $\inf_x \overrightarrow{T}(Ax,Bx) \le \inf_x \overrightarrow{T}(T(Ax,Cx),T(Bx,Cx))$. Results such as this one can in $\mathcal{F}_\omega$ be obtained without explicit calculations of membership functions.

Notice that the following theorem and its corollary are for simplicity written for arbitrary logic $\mathcal{F}$, thus we can avoid the indexing by $*$.

Theorem 21 *Let $\varphi,\psi_1,\ldots,\psi_n$ be propositional formulae. Then $\mathcal{F}\vdash\varphi(\psi_1,\ldots,\psi_n)$*

$$\textit{iff}\quad \mathcal{F}_2\vdash \mathrm{Rel}^{\forall}_{\varphi}(\mathrm{Op}_{\psi_1}(X_{1,1},\ldots X_{1,k_1}),\ldots \mathrm{Op}_{\psi_n}(X_{n,1},\ldots X_{n,k_n}))$$
$$\textit{iff}\quad \mathcal{F}_2\vdash \mathrm{Rel}^{\exists}_{\varphi}(\mathrm{Op}_{\psi_1}(X_{1,1},\ldots X_{1,k_1}),\ldots \mathrm{Op}_{\psi_n}(X_{n,1},\ldots X_{n,k_n}))$$

Corollary 22 *Let φ and ψ be propositional formulae.*
If $\vdash\varphi\to\psi$ then $\vdash \mathrm{Op}_\varphi(X_1,\ldots,X_n)\subseteq \mathrm{Op}_\psi(X_1,\ldots,X_n)$.
If $\vdash\varphi\leftrightarrow\psi$ then $\vdash \mathrm{Op}_\varphi(X_1,\ldots,X_n)= \mathrm{Op}_\psi(X_1,\ldots,X_n)$.
If $\vdash\varphi\vee\neg\varphi$ then $\vdash \mathrm{Crisp}(\mathrm{Op}_\varphi(X_1,\ldots,X_n))$.

By virtue of Theorem 21, the properties of propositional connectives directly translate to the properties of relations and operations. For example:

$\vdash\Delta p\to p$	proves	$\vdash\mathrm{Ker}(X)\subseteq X$
$\vdash p\to p\vee q$	"	$\vdash X\subseteq X\cup Y$
$\vdash 0\to p$	"	$\vdash\emptyset\subseteq X$
$\vdash p\mathbin{\&} q\to p\wedge q$	"	$\vdash X\cap_* Y\subseteq X\cap_{\mathrm{G}} Y$
$\vdash\neg_{\mathrm{G}} p\vee\neg\neg_{\mathrm{G}} p$	"	$\vdash\mathrm{Crisp}(\backslash X)$

We can prove more complicated general theorems (for details see [2]) which give us proofs of more complicated facts. Here we present only some of their consequences in $\text{Ł}\Pi_2$:

$X\subseteq_* Y\to X\cap_* Z\subseteq_* Y\cap_* Z$
$(X\subseteq_* Y \mathbin{\&_*} Y\subseteq_* X)\to X\approx_* Y$
$(X\subseteq_* Z \mathbin{\&_*} Y\subseteq_* Z)\to X\cup Y\subseteq_* Z$
$\Delta(X\subseteq Y)\to X_\alpha\subseteq Y_\alpha$
transitivity of $\subseteq_*,\approx_*$
$\mathrm{Hgt}(X)\mathbin{\&_*}(X\subseteq_* Y)\to\mathrm{Hgt}(Y)$
$\mathrm{Norm}(X\cup Y)\to\mathrm{Norm}(X)\vee\mathrm{Norm}(Y)$
$X\subseteq_* Z\mathbin{\&_*} X\|_* Y\to Y\|_* Z$

Definition 23 *The union and intersection of a (fuzzy) set of (fuzzy) sets are the functions $\bigcup^{(n+3)}$ and $\bigcap^{(n+3)}$, respectively, assigning a set $A^{(n+1)}$ to a set of sets $\mathcal{A}^{(n+2)}$, where:*

$$\bigcup\mathcal{A} =_{\mathrm{df}} \{x \mid (\exists A\in\mathcal{A})(x\in A)\}$$
$$\bigcap\mathcal{A} =_{\mathrm{df}} \{x \mid (\forall A\in\mathcal{A})(x\in A)\}$$

6 FUZZY PROPERTIES OF FUZZY RELATIONS

This section shows some very basic facts about fuzzy relations provable in our theory. Recall that we work with *graded* (fuzzy) properties of fuzzy relation, i.e., a relation can be partially reflexive, symmetric, etc. Again, the goal of this section is just to *demonstrate* the possibilities of our theory, details are subject of the consequent papers.

Definition 24 *In $\mathcal{F}_2$, we define the following operations:*

$$\begin{aligned}
X\times Y &=_{\mathrm{df}} \{\langle x,y\rangle \mid x\in X \mathbin{\&} y\in Y\}\\
\mathrm{Dom}(R) &=_{\mathrm{df}} \{x \mid \langle x,y\rangle\in R\}\\
\mathrm{Rng}(R) &=_{\mathrm{df}} \{y \mid \langle x,y\rangle\in R\}\\
R''A &\equiv_{\mathrm{df}} \{x \mid (\exists y)(y\in A \mathbin{\&} Ryx)\}\\
R\circ S &=_{\mathrm{df}} \{\langle x,y\rangle \mid (\exists z)(\langle x,z\rangle\in R \mathbin{\&} \langle z,y\rangle\in S)\}\\
R^{-1} &=_{\mathrm{df}} \{\langle x,y\rangle \mid \langle y,x\rangle\in R\}\\
\mathrm{Id} &=_{\mathrm{df}} \{\langle x,y\rangle \mid x=y\}
\end{aligned}$$

We can also define the usual properties of relations:

$$\begin{array}{rcl} \mathrm{Ext}_E(R) & \equiv_{\mathrm{df}} & (\forall x,x',y,y')(Exx' \mathbin{\&} Eyy' \mathbin{\&} Rxy \to Rx'y') \\ \mathrm{Refl}(R) & \equiv_{\mathrm{df}} & (\forall x)(Rxx) \\ \mathrm{Sym}(R) & \equiv_{\mathrm{df}} & (\forall x,y)(Rxy \to Ryx) \\ \mathrm{Trans}(R) & \equiv_{\mathrm{df}} & (\forall x,y,z)(Rxy \mathbin{\&} Ryz \to Rxz), \textit{etc.} \end{array}$$

There are many theorems on relations easily provable in our theory (some of them we list below). Recall that the properties of relations (e.g., reflexivity) are graded. Thus the implications in the following theorems are generally stronger than the corresponding statements about entailment.

Theorem 25 *The following properties of relations are provable in* $\mathcal{F}_2$*:*

1. $\mathrm{Refl}(R) \leftrightarrow \mathrm{Id} \subseteq R$
2. $\mathrm{Sym}(R) \leftrightarrow R^{-1} \subseteq R$
3. $\mathrm{Trans}(R) \leftrightarrow R \circ R \subseteq R$
4. $\mathrm{Refl}(R) \to R \subseteq R \circ R$
5. $\mathrm{Trans}(R) \mathbin{\&} \mathrm{Trans}(Q) \to \mathrm{Trans}(R \cap Q)$
6. $R \subseteq S \to (R \circ T \subseteq S \circ T) \wedge (T \circ R \subseteq T \circ S)$

Thus every relation 1. is reflexive to the same degree as it contains identity, 2. is symetric to the same degree as it contains its own inverse, 4. is contained in the composition with itself at least in the degree of its reflexivity, etc.

Theorem 26 *For an arbitrary binary relation R and arbitrary (fuzzy) sets* A, B *we have:*

1. $A \subseteq B \to R''A \subseteq R''B$
2. $\mathrm{Refl}(R) \to A \subseteq R''A$
3. $\mathrm{Trans}(R) \to \mathrm{Ext}_R(R''A)$
4. $A \subseteq B \mathbin{\&} \mathrm{Ext}_R(B) \to R''A \subseteq B$
5. $\mathrm{Trans}(R) \to R''(R''A) \subseteq R''A$
6. $\mathrm{Refl}(R) \mathbin{\&} \mathrm{Ext}_R(A) \to R''A \approx A$

Theorem 27 *If* $\mathcal{A}$ *is a crisp set of sets, then* $(\forall X \in \mathcal{A})\,\mathrm{Ext}_E(X) \to \mathrm{Ext}_E(\bigcap \mathcal{A}) \wedge \mathrm{Ext}_E(\bigcup \mathcal{A})$.

REFERENCE

[1] L. Běhounek and P. Cintula (2004) "From fuzzy logic to fuzzy mathematics: A methodological manifesto", submitted to Fuzzy Sets and Systems.

[2] L. Běhounek and P. Cintula (2004) "Fuzzy class theory", to appear in Fuzzy Sets and Systems.

[3] L. Běhounek and P. Cintula (2005) "Fuzzy class theory as foundations for fuzzy mathematics", in *Congress of International Fuzzy Systems Association (IFSA 2005)*, Beijing.

[4] R. Cignoli, I. d'Ottaviano and D. Mundici (1999) Algebraic Foundations of Many-Valued Reasoning, Kluwer, Dordercht.

[5] P. Cintula (2003) Advances in the ŁΠ and ŁΠ$\frac{1}{2}$ logics, Arch. Math. Logic, 42:449–468.

[6] F. Esteva, L. Godo and F. Montagna (2001) "The ŁΠ and ŁΠ$\frac{1}{2}$ logics: Two complete fuzzy systems joining Łukasiewicz and product logics", Arch. Math. Logic, 40:39–67.

[7] S. Gottwald (1993) Fuzzy Sets and Fuzzy Logic: Foundations of Application—from a Mathematical Point of View, Vieweg, Wiesbaden.

[8] P. Hájek (1998) Metamathematics of Fuzzy Logic, Kluwer, Dordercht.

[9] P. Hájek and Z. Haniková (2003) "A development of set theory in fuzzy logic", in Beyond Two: Theory and Applications of Multiple-Valued Logic, Eds. M. Fitting and E. Orlowska, 273–285, Physica-Verlag, Heidelberg.

[10] U. Höhle (1987) "Fuzzy real numbers as Dedekind cuts with respect to a multiple-valued logic", Fuzzy Sets and Systems, 24:263–278.

[11] V. Novák (2004) "On fuzzy type theory", Fuzzy Sets and Systems, 149:235–273.

[12] V. Novák, I. Perfilieva and J. Močkoř (2000) Mathematical Principles of Fuzzy Logic, Kluwer, Dordrecht.

[13] G. Takeuti and S. Titani (1992) "Fuzzy logic and fuzzy set theory", Arch. Math. Logic, 32:1–32.

[14] R.B. White (1979) "The consistency of the axiom of comprehension in the infinite-valued predicate logic of Łukasiewicz", J. Phil. Logic, 8:509–534.

[15] A.N. Whitehead and B. Russell (1910, 1911, 1913) Principia Mathematica, Cambridge UP, Cambridge.

[16] L. Zadeh (1965) "Fuzzy sets", Information and Control 8:338–353.