FUZZY CLASS THEORY AS FOUNDATIONS FOR FUZZY MATHEMATICS *

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ABSTRACT: $Ł\Pi_\omega$ is a deductive first-order theory over the fuzzy logic ŁΠ, which axiomatically captures Zadeh's notion of fuzzy set and aims at giving a unified formal framework for a large part of fuzzy mathematics. An overview of the concepts expressible in the theory is given and informal proof methods for doing fuzzy mathematics in $Ł\Pi_\omega$ are sketched.

Keywords: Axiomatic fuzzy set theory, fuzzy logic ŁΠ, fuzzy mathematics, proof methods in fuzzy logic.

1 INTRODUCTION

This paper explains a new unified formalism for fuzzy set theory that emerged during last years in the Prague working group in fuzzy logic. It follows the methodology described in [1] and employs the apparatus introduced in [2]. In this paper we focus on explanations of how to actually *work* in the proposed framework, and the way it can help in the development of fuzzy mathematics. In consequence, we proceed rather informally; the relevant technical details can be found in the follow-up paper [3].

Traditionally, fuzzy set theory is a generalization of the concept of *characteristic function:* fuzzy sets are identified with their membership functions; operations on fuzzy sets (union, intersection, etc.) are defined by means of some operations on membership functions, which yield other membership functions; relations between fuzzy sets (equality, inclusion, etc.) are defined as relations between their membership functions. Thus, traditional fuzzy set theory captures the notion of fuzzy set only indirectly, by means of the classical (crisp) notion of $[0,1]$-valued (or lattice-valued) function, and can therefore be viewed as part of real analysis with a specific motivation.

The new formalism described in this paper, on the other hand, tries to capture fuzzy sets axiomatically, as a primitive notion. Since the axiomatic method is very fruitful in many parts of mathematics, there had been various attempts at axiomatization of the notion of fuzzy set already in 1970's. These early attempts (most notably [4]) used a *ternary* membership predicate, in which the third argument represented the membership degree. The theorems of such formal theories expressed the laws valid for fuzzy sets and could formally be derived by the rules of *classical* logic from a set of suitably chosen axioms.

Our approach, on the other hand, follows more recent attempts ([13], [12], or [9]), which instead take a *binary* membership predicate and construct a theory over *fuzzy* logic; consequently, not only the membership predicate, but most of the defined notions are naturally graded. In such formal systems, membership degrees are not mentioned explicitly as objects of the theory, but rather are 'hidden' in the semantical metalevel as the meanings of atomic formulae. There are several reasons for this approach, both of philosophical and technical nature; for its advantages over more traditional methods see [2] or [3].

It turns out that if the background fuzzy logic and the axioms for the fuzzy membership predicate are appropriately chosen, the resulting theory can serve as a unified framework for a large part of fuzzy mathematics. The requirements on the background fuzzy logic seem to be best satisfied by the logic ŁΠ, as it contains definable logical connectives of a wide class of t-norm fuzzy logics. The theory of fuzzy membership, on the other hand, need not be a full-fledged set theory like those of [12] or [9]; it must however be sufficiently rich to admit usual operations on fuzzy sets. A fuzzy analogue of Russell and Whitehead's simple type theory of [14], rendered as a first-order theory over fuzzy logic, is sufficient for almost all practical needs. In what follows, the theory will be denoted by $Ł\Pi_\omega$, as it is in fact the Henkin-style logic ŁΠ of order ω. The axiomatic system seems to be equivalent (maybe up to some minor details) to Vilém Novák's fuzzy type theory of [11], if the latter is defined over the logic ŁΠ.

The models of $Ł\Pi_\omega$ are systems (closed under definable operations) of fuzzy subsets (of all orders) of some crisp universe U, where the membership functions take values in some ŁΠ-algebra. *Intended models* are those in which the ŁΠ-algebra is standard (i.e., the interval $[0,1]$ with the usual operations for connectives), and the system contains *all* fuzzy subsets of U (of all orders). These models correspond exactly to Zadeh's [15] original notion of fuzzy set (we call them *Zadeh models*).

The theory $Ł\Pi_\omega$ is sound w.r.t. Zadeh models, thus whatever we prove in $Ł\Pi_\omega$ is true about the "flesh and blood" fuzzy sets. Although the full theory of Zadeh models is not axiomatizable, the axiomatic system of $Ł\Pi_\omega$ approximates it very well: the comprehension axioms secure the existence of every fuzzy set which is definable by a formula of $Ł\Pi_\omega$, and the extensionality axioms insure that identical membership functions represent the same fuzzy set. (For the axioms of $Ł\Pi_\omega$ see [2] or [3].)

2 FORMAL EXPRESSIONS OF $Ł\Pi_\omega$

2.1 Atomic expressions

The language of the theory $Ł\Pi_\omega$ contains variables for atomic objects (lowercase letters $x, y, \ldots$), fuzzy sets of these objects (uppercase letters $A, B, \ldots$), fuzzy sets of fuzzy sets of objects

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(calligraphic letters $\mathcal{A}, \mathcal{B}, \ldots$) also called fuzzy sets of the 2nd order, fuzzy sets of the 3rd order, etc. If necessary, the order of a variable can be marked by a parenthesized superscript, e.g. $x^{(0)}, Y^{(1)}, \mathcal{Z}^{(3)}, W^{(n)}$. Objects or sets of order n can be construed as belonging to all types $m \geq n$ as well.

There are no variables for truth degrees; the degree in which x belongs to A is expressed by the atomic formula $x \in A$ (which can alternatively be written in the more traditional way as Ax). The theory is typed, so only such atomic formulae are well-formed which express the membership of an object of a lesser type to an object of a higher type: $X^{(n)} \in A^{(m)}$ is a well-formed formula iff $n < m$.

Crisp identity of objects or fuzzy sets is expressed by the predicate $=$. For objects, $x = y$ holds iff x, y are the same object in the model; for (fuzzy) sets, $A = B$ iff the membership functions of A and B are identical. Identical objects or sets are freely intersubsitutable in formulae. The identity of truth degrees (e.g., $Ax = Bx$) is also expressible in $Ł\Pi_\omega$ (though not by means of atomic formulae—see below in Section 2.2). Many non-crisp equality notions are definable in $Ł\Pi_\omega$ as well, as shown in Section 3.

In order to express statements about fuzzy relations, $Ł\Pi_\omega$ contains the usual apparatus of tuples of objects or fuzzy sets (of any order). The type of the tuple $\langle X_1, \ldots, X_n \rangle$ is the maximal type of all X_i's it contains. As usual, the identity of tuples is component-wise.

2.2 Logical connectives

Membership degrees, expressed by atomic formulae, can be combined by means of definable connectives of ŁΠ. Among these, the following means for manipulation with truth degrees can be found (see [6] for details on propositional ŁΠ):

- *T-Norms.* For any continuous t-norm T which is a finite ordinal sum of the minimum (G), product (Π), and/or Łukasiewicz (L) t-norms, ŁΠ contains a definable conjunction $\&_T$ whose standard semantics is T. (Many other t-norms, such as the nilpotent minimum or the drastic t-norm, are also available; for details see [5].) In this paper, we often use $*$ as a sign representing an arbitrary continuous t-norm. The traditional sign $\wedge$ can alternatively be used for the min-conjunction $\&_{\mathrm{G}}$; thus, for example, the minimum of the membership degrees expressed by the formulae Ax and Bx is expressed as usual, by the formula $Ax \wedge Bx$.

- *T-Conorms.* For any continuous t-norm T representable in ŁΠ, its dual t-conorm S is definable in ŁΠ as a disjunction $\veebar_T$ (the maximum $\veebar_{\mathrm{G}}$ can be denoted by $\vee$ as usual).

- *R-Implications.* For any left-continuous t-norm T representable in ŁΠ, its corresponding residuum (R-implication) $\vec{T}$ is definable in ŁΠ; it is denoted by $\rightarrow_T$. Infix notation is used for logical connectives, thus instead of

$$\vec{T}(T(Ax, T(Ax, Bx)), T(Ax, Bx)) \tag{1}$$

we write

$$Ax \,\&_*\, Ax \,\&_*\, Bx \;\rightarrow_*\; Ax \,\&_*\, Bx \tag{2}$$

(associativity of $\&_*$ and the precedence rules are used to avoid unnecessary brackets); the algebraic-style notation

$$Ax * Ax * Bx \;\Rightarrow_*\; Ax * Bx \tag{3}$$

can be used as well. R-equivalence connectives are defined as $\varphi \leftrightarrow_T \psi \equiv_{\mathrm{df}} (\varphi \rightarrow_T \psi) \,\&_T\, (\psi \rightarrow_T \varphi)$, with the standard semantics of the biresidua $T(\vec{T}(\varphi, \psi), \vec{T}(\psi, \varphi))$. S-implications are definable in ŁΠ as well, by means of negations (see below) and t-conorm disjunctions.

- *Truth constants.* Propositional truth constants 0 and 1 are definable in ŁΠ. If an axiom postulating $\frac{1}{2}$ is added to $Ł\Pi_\omega$, all *rational* truth constants $\frac{m}{n}$ become definable (see [6], [2], or [3]).

- *Arithmetical operations.* There are definable connectives of ŁΠ which in the standard semantics realize arithmetical operations on the truth values of their operands. Available arithmetical operations include, i.a., the bounded sum $\oplus$, the difference $-$, the product $\cdot$, the bounded ratio $\rightarrow_\Pi$, etc.

- *Comparison of truth degrees.* The logic ŁΠ contains definable connectives $\leq, <, =, \neq, \geq, >$ whose standard semantics is that of (crisp) comparison of truth degrees. Thus, for example, $Ax < Bz$ is evaluated to 1 iff the membership degree of x in A is strictly less than that of z in B. Similarly, the formula $Ax = Bx$ expresses the fact that the membership degrees of x in A and B are the same (in which case it has the truth value 1, otherwise 0). It is important to understand that $=$ here is a defined logical *connective* joining two *formulae* (and yielding a truth value 0 or 1), while in $x = y$ or $A = B$ it is the identity *predicate* between *terms*. The comparison with rational truth constants (e.g., $Ax \leq \frac{1}{2}$) is also available, even in the absence of rational truth constants (then "$\leq \frac{1}{2}$" must be regarded as a unary defined connective of ŁΠ). By means of arithmetical connectives, the comparison with arithmetically definable irrationals is also definable (for example, $Ax < \frac{1}{\sqrt{2}}$ as $Ax \cdot Ax < \frac{1}{2}$).

- *Negations.* Both strict negation and standard involutive negation are available in ŁΠ, respectively as $\neg_{\mathrm{G}}$ (or $\neg_\Pi$) and $\neg_{\mathrm{L}}$. Using the above-mentioned connectives, we can write $\neg_{\mathrm{L}} Ax = 1 - Ax$, which is a provable formula of ŁΠ. Similarly, $\neg_{\mathrm{G}} Ax = (Ax \neq 0)$. Generally, for any residuum $\rightarrow_*$ its corresponding negation $\neg_*$ is defined as $\neg_* \varphi \equiv_{\mathrm{df}} \varphi \rightarrow_* 0$.

2.3 Quantifiers

Infima and suprema of truth degrees are symbolized by the logical symbols $\forall$ and $\exists$, respectively. Thus, for example, instead of $\sup_x Ax$ we write $(\exists x)Ax$, and instead of $\inf_y(1 - Ay)$ we write $(\forall y)(1 - Ay)$, or $(\forall x)\neg_{\mathrm{L}} Ax$.

It should be noticed that unless φ is crisp, the expressions of the form $(\forall x)\varphi$ should not be read "for all x it holds that φ", since the meaning of the formula is a (possibly intermediate) truth degree, rather than a statement which either holds or not. Similarly, $(\exists x)\varphi$ must be understood as *the degree* to which there is an x such that φ.

2.4 Set comprehension terms

In virtue of the comprehension axioms of $Ł\Pi_\omega$, fuzzy sets can be denoted by the comprehension terms $\{x \mid \varphi(x)\}$ (for any formula $\varphi(x)$). If A is defined by the rule $Ax = \varphi(x)$, then it can be written as $A = \{x \mid \varphi(x)\}$; vice versa, if $A = \{x \mid \varphi(x)\}$, then for any x it holds that $Ax = \varphi(x)$. The formula $y \in \{x \mid \varphi(x)\}$ is thus equivalent simply to $\varphi(y)$, and the formula $A = \{x \mid \varphi(x)\}$ to $(\forall x)(Ax = \varphi(x))$. The same notation can be used for fuzzy sets of higher types; the term $\{X^{(n)} \mid \ldots\}$ always denotes a fuzzy set of type $n+1$.

Unless the formula φ expresses a crisp condition, the term $\{x \mid \varphi(x)\}$ should not be read "the set of all those x for which φ holds", but rather "the (fuzzy) set to which any x belongs in the same degree in which φ is true about x".

2.5 Abbreviations and conventions

Various abbreviations which are common in classical mathematics or traditional fuzzy set theory can be used in the formulae of $Ł\Pi_\omega$. This makes many of them look quite similar to the usual statements about fuzzy sets.

The defined connectives of $Ł\Pi$ are themselves such abbreviations. Besides those mentioned above, the formula $\varphi = 1$ is often abbreviated as $\Delta\varphi$ (the connective is known as Baaz delta). Further we can abbreviate $(\exists x)(x \in A \mathbin{\&_*} \varphi(x))$ and $(\forall x)(x \in A \rightarrow_* \varphi(x))$ respectively by $(\exists x \in A)_*\varphi(x)$ and $(\forall x \in A)_*\varphi(x)$, etc.

By convention, we discard t-norm indices whenever they do not matter. Most often this is when they are applied to crisp subformulae, for instance $\neg[(Ax = 0) \rightarrow (Bx = \frac{1}{2})]$. The indices can also be omitted in the principal connectives of theorems of $Ł\Pi_\omega$, since all their $*$-variants are equiprovable.

3 DEFINED NOTIONS OF $Ł\Pi_\omega$

It could be seen in the previous section that the apparatus of $Ł\Pi_\omega$ is rich enough to express many concepts of the usual fuzzy set theory. Further notions can be introduced by means of defined constants, predicates, and functors.

3.1 Set constants

We introduce the constant $\emptyset$ to denote the *empty set* $\{x \mid 0\}$. Similarly, the *universe of discourse* V is defined as $\{x \mid 1\}$. Thus A is empty iff $Ax = 0$ for any x, and $(\forall x)(Vx = 1)$.

3.2 Elementary set operations

The usual fuzzy set operations like unions and intersections can be defined in $Ł\Pi_\omega$ by means of simple comprehension terms, like in classical mathematics. For instance, we define the $*$-intersection of two fuzzy sets as

$$A \cap_* B \;=_{\mathrm{df}}\; \{x \mid x \in A \mathbin{\&_*} x \in B\} \tag{4}$$

It can be seen that if $C = A \cap_* B$, then $Cx = Ax * Bx$ as usual.

Similarly we can define a variety of other operations like the $*$-unions, $*$-complements, etc.: any operation on fuzzy sets given by an arithmetically definable operation can be defined by the comprehension term with the corresponding arithmetical connectives.

Specifically, the supports, kernels, α-cuts, and α-levels of fuzzy sets are defined by the familiar-looking set terms $\{x \mid Ax = 1\}$, $\{x \mid Ax > 0\}$, $\{x \mid Ax \geq \alpha\}$, and $\{x \mid Ax = \alpha\}$, respectively (where α is a *formula* expressing a truth degree). Since these fuzzy sets are (in consequence of their definitions) crisp, they can be (as usual in the traditional fuzzy set theory) identified with the corresponding classical subsets of the universe of discourse in the models of $Ł\Pi_\omega$.

3.3 Properties of fuzzy sets

Many usual properties of fuzzy sets are expressible by suitable formulae of $Ł\Pi_\omega$. For example, the *normality* of A is expressed by the formula $(\exists x)(Ax = 1)$. We can define that A is *crisp* iff $(\forall x)[(Ax = 0) \vee (Ax = 1)]$, and *fuzzy* iff $(\exists x)[(Ax \neq 0) \mathbin{\&} (Ax \neq 1)]$. The usual facts about these properties are easily provable in $Ł\Pi_\omega$ (e.g., that the only crisp subnormal set is $\emptyset$). It can be noticed that these properties are themselves crisp: any set either is, or is not normal (crisp, fuzzy), as can be proved in $Ł\Pi_\omega$.

Besides such crisp properties of fuzzy sets, traditional fuzzy set theory defines some functions that assign a truth degree to a fuzzy set. For example the *height* of a fuzzy set is the supremum of the membership degrees it hits. In $Ł\Pi_\omega$, these notions are expressed by *fuzzy properties* of fuzzy sets. Thus the height of a fuzzy set A is defined by the formula $(\exists x)(x \in A)$, or its notational variant $(\exists x)Ax$. Again this is not to be read "there is an x in A", but interpreted as the *degree* to which there is an x in A.

One can notice that in $Ł\Pi_\omega$, any property of fuzzy sets is itself an individual of the theory, viz. a fuzzy (or crisp) set of the 2nd order (a fuzzy set of fuzzy sets). For instance the property of normality delimits the (crisp) 2nd-order set $Norm = \{X \mid (\exists x)(Ax = 1)\}$, and the property of height delimits the (fuzzy) 2nd-order set $Height = \{X \mid (\exists x)(x \in A)\}$ (to which a fuzzy set belongs to the degree of its height).

3.4 Relations between fuzzy sets

The usual relations between fuzzy sets (like inclusion, disjointness, etc.) can be defined by formulae of $Ł\Pi_\omega$ as well. An important feature of $Ł\Pi_\omega$, however, is that not only membership, but many defined predicates can naturally be introduced as *graded*. Unlike traditional fuzzy mathematics, this does not complicate things too much; very often it even *simplifies* the definitions and proofs.

To demonstrate this point, let us consider the notion of *inclusion*. In traditional fuzzy mathematics, a fuzzy set A is often said to be a subset of a fuzzy set B iff for all x, the membership degree of x in A does not exceed that of x in B. The corresponding $Ł\Pi_\omega$ formula has the familiar look $(\forall x)(Ax \leq Bx)$. Both in traditional fuzzy set theory and $Ł\Pi_\omega$, this is a crisp notion of inclusion: a fuzzy set A either is, or is not a subset of B. Nevertheless, in $Ł\Pi_\omega$ the subsethood can be more naturally be defined with $\rightarrow_*$ instead of $\leq$:

$$A \subseteq_* B \;\equiv_{\mathrm{df}}\; (\forall x)(Ax \rightarrow_* Bx) \tag{5}$$

Both notions are true in degree 1 iff the membership function of A is majorized by that of B. The latter, however, has a high degree of truth also whenever A does not exceed B too much (where the exact meaning of "too much" depends on the t-norm used). This definition has been proposed in traditional fuzzy mathematics as well (though somewhat later). The point is that in $Ł\Pi_\omega$ it is a *natural* fuzzy counterpart of classical inclusion, which is *more general,* but *easy to handle.* It is natural, since the definition is the exact copy of the classical definition of inclusion, rewritten in ŁΠ (cf. [10, Sect. 5]). It is more general, since the former subsethood relation can easily be recovered from it (by Δ), but not vice versa. And it is easy to handle, because its form admits directly to translate many proofs that work for subsethood of crisp sets (more on this in Section 4). Similarly we can introduce a variety of fuzzy equalities $\approx_*$, defined by the formula $(\forall x)(Ax \leftrightarrow_* Bx)$.

Still, the search for fuzzy counterparts of classical notions should not blindly follow the forms of classical formulae. Often a (classically equivalent) reformulation of the classical notion gives better results. One can, for example, observe that $\approx_G$ is highly true only if the membership functions are identical on *low* truth values; product equality $\approx_\Pi$ is also more restrictive on lower truth values. Since intuitively the difference in the *high* degrees of membership (on the "prototypes") should matter more, equality of involutive complements, $-_L X \approx_* -_L Y$ may give a better measure of the similarity of fuzzy sets than $X \approx_* Y$.

3.5 Properties of fuzzy relations

$Ł\Pi_\omega$ naturally accommodates *graded* properties of fuzzy relations, as studied in [7]. To get the graded notion of, for instance, *reflexivity,* one defines it simply as $(\forall x)Rxx$ instead of $(\forall x)(Rxx = 1)$. The graded notion then arises in the same manner as in Section 3.3.

The formalism of $Ł\Pi_\omega$ can furthermore help us to spot and eliminate various sources of hidden crispness in definitions; surprisingly, this usually does not complicate the proofs too much. For example, antisymmetry and other properties that classically refer to identity are better defined w.r.t. some fuzzy equality relation E so as to avoid the crispness of $=$. Thus we can define $*$-antisymmetry w.r.t. E as

$$(\forall x)(\forall y)(\forall z)(Rxy \,\&_*\, Rxz \rightarrow_* Eyz) \tag{6}$$

instead of

$$(\forall x)(\forall y)(\forall z)(Rxy \,\&_*\, Rxz \rightarrow_* y = z) \tag{7}$$

(which is a $*$-antisymmetry w.r.t. $=$). Proofs which work for $*$-antisymmetry w.r.t. $=$ do often work for $*$-antisymmetry w.r.t. E as well, since they usually employ only those properties of $=$ which hold of all fuzzy equalities. The generalization is thus obtained at low cost.

Thus by usual definitions, the graded notions of fuzzy equivalence (better known as similarity), fuzzy ordering, fuzzy function, etc., are obtained, and can be studied in our formal theory. For functions (w.r.t. $=$ and in degree 1) we can use the functional notation $y = F(x)$ instead of Fxy.

3.6 Structure on the universe of discourse

In traditional fuzzy set theory, the universe of discourse is often equipped with some crisp structure, for instance an ordering, metric, or measure. It has been shown in [2] that such crisp structures can be introduced by additional axioms about set constants in $Ł\Pi_\omega$. Since this exceeds the scope of this introductory paper, the reader is referred to [2] and [3] for details.

4 PROOF METHODS IN $Ł\Pi_\omega$

We have seen in Sections 2 and 3 that many concepts of traditional fuzzy mathematics are expressible as formulae of $Ł\Pi_\omega$. Many of these formulae look very similar to those of classical set theory. This was made possible by the design of the theory which hid the references to the truth degrees into the atomic formulae $x \in A$ and their combinations. It is the *meaning* of the formulae in which $Ł\Pi_\omega$ differs from classical set theory: they need not have only the truth values 0 and 1, but also the values between these two. As such, the formulae are not subject to the laws of classical two-valued logic, but those of *fuzzy logic,* namely the logic ŁΠ. Thus the formulae which express the laws valid for fuzzy sets can be formally deduced from the axioms of $Ł\Pi_\omega$, but the derivation must use the logic ŁΠ instead of classical Boolean logic.

Making formal derivations according to the formal laws of ŁΠ is of course quite cumbersome. We will hint here, however, that it is possible to make informal proofs in $Ł\Pi_\omega$ in a similar manner an Intuitionist makes informal constructive proofs which are governed by intuitionistic logic (and which, if it were necessary, could be translated into formal proofs). In what follows, we give some examples of correct informal proof methods in $Ł\Pi_\omega$; all of them are based on provable theorems of $Ł\Pi_\omega$ (or the fuzzy logic ŁΠ itself) and its metatheory.

- *Elementary fuzzy set theory.* In [2], elementary fuzzy set theory has been effectively reduced to the propositional logic ŁΠ. The theorems on fuzzy set operations and relations of certain simple forms thus can be proved (or disproved) by simple propositional calculations. For example, $X \cap_* Y \subseteq X$ follows directly from the validity of the propositional formula $p \,\&_*\, q \rightarrow p$, while the converse $*$-inclusion is disproved by any counterexample to the converse implication. Although the form of propositionally derivable theorems is restricted, it covers most of the mathematically interesting properties of fuzzy set operations and relations (in fact, virtually any theorem on the first couple of dozens pages of any textbook on fuzzy sets can be proved in this simple way). The methods described further are only needed for more complex theorems, e.g. on fuzzy relations.

- ŁΠ-*valid equivalences.* In classical mathematics, we usually prove theorems not by applying the rules of some logical calculus for Boolean logic (e.g., modus ponens), but by transforming the statements according to equivalences which are known to be valid. Thus, e.g., we use de Morgan laws, the rules for the negation of quantifiers, etc. *Most of these rules are valid in* ŁΠ *as well.* One must only be careful not to use a rule which does not hold generally for fuzzy sets. However, such rules are few and one can easily learn to eschew them in proofs: one of such forbidden rules is the transformation of $(\forall x)\neg_*\varphi$ to $\neg_*(\exists x)\varphi$ (though it can be used if $* = L$ or if φ is known to be crisp).

- *Substitution of provably equivalent subformulae.* The previous method is applicable to transformations of subformulae as well, as the following metatheorem holds:

  > If $\varphi \leftrightarrow \psi$ is provable in a theory over the first-order logic ŁΠ, then so is $\chi[\varphi/\psi] \leftrightarrow \chi$ (where $\chi[\varphi/\psi]$ is the result of replacing an occurrence of the subformula φ by ψ).

- *Using Boolean logic for crisp subformulae.* If a subformula is provably crisp, then the rules of classical logic can be applied for its transformation.

- *Using a fragment of* ŁΠ. When working with a formula in which every connective is indexed by $*$, G, Π, or Ł, the rules of the respective logics BL, G, Π, or Ł (possibly with Δ) are applicable. A comprehensive list of valid and invalid laws for these fuzzy logics is found in [8]. Further derivative laws can be verified by means of these, e.g. the commutativity of restricted quantifiers:

  $$(\forall x \in A)_*(\forall y \in B)_*\varphi \;\leftrightarrow\; (\forall y \in B)_*(\forall x \in A)_*\varphi \qquad (8)$$

  and the analogous law for $\exists$.

  In what follows we shall assume all connectives indexed by the same $*$ (for simplicity and because it is the most frequent case), thus working in fact in BLΔ.

- *Generalization.* In mathematics (both fuzzy and classical), one often proves theorems of the form

  $$(\forall x_1 \in A_1)\ldots(\forall x_k \in A_k)(\varphi_1 \,\&\, \ldots \,\&\, \varphi_n \to \psi) \qquad (9)$$

  By the rules of the generalization and the distribution laws of $\forall$, it is sufficient to prove its instance

  $$x_1 \in A_1 \,\&\, \ldots \,\&\, x_k \in A_k \,\&\, \varphi_1 \,\&\, \ldots \,\&\, \varphi_n \to \psi \qquad (10)$$

- *The premises and the conclusion.* If there is an existential quantifier in the prefix of the demonstrandum, e.g.,

  $$(\forall x_1 \in A_1)(\forall x_2 \in A_2)(\exists x_3 \in A_3)(\forall x_4 \in A_4)\psi \qquad (11)$$

  then anything after the first $\exists$ is the conclusion to be arrived at. Thus to prove (11), we want to establish

  $$x_1 \in A_1 \,\&\, x_2 \in A_2 \to (\exists x_3 \in A_3)(\forall x_4 \in A_4)\psi \qquad (12)$$

  In what follows we call the conjuncts before the principal implication the *premises*. The premises are sometimes presented in the form of nested implications.

- *Proving in steps.* By the transitivity of implication, one can prove in steps. Thus to establish $\varphi \to \psi$, it is for instance possible first to prove $\varphi \to \chi$, then $\chi \to \xi$, and finally $\xi \to \psi$.

- *Proving a conjunction.* If the conclusion is a conjunction, it is correct to prove each conjunct separately, *but from disjoint sets of premises.* Thus it is correct to prove

  $$\varphi_1 \,\&\, \varphi_2 \,\&\, \varphi_3 \to \psi_1 \,\&\, \psi_2 \qquad (13)$$

  by establishing $\varphi_1 \,\&\, \varphi_2 \to \psi_2$ and $\varphi_3 \to \psi_1$, but not by $\varphi_1 \,\&\, \varphi_3 \to \psi_1$ and $\varphi_2 \,\&\, \varphi_3 \to \psi_2$ (the latter only proves $\varphi_1 \,\&\, \varphi_2 \,\&\, \varphi_3 \,\&\, \varphi_3 \to \psi_1 \,\&\, \psi_2$). Only crisp premises can be used repeatedly.

- *Marking used premises.* Thus if any premise is used in the proof of the conclusion or an intermediary step, it should be marked and never used again (or the theorem be weakened to contain it twice). The same holds for lemmata in the form of implication: they can be applied if their premises are presupposed or have already been proved, but then these premises must generally never be used again in the proof. If any non-crisp premise has been used more than once in the proof, it must be added to the premises of the theorem, even if it is already there.

- *Proof by cases.* The proof can be split to several cases whose max-disjunction is (already) proved. One must be, however, careful that for example the disjunction of $\varphi \vee \neg\varphi$ is generally *not* provable in fuzzy logic, so one must avoid taking cases on a condition and its negation, *unless the condition is crisp.*

- *Checking the classical proof against the rules of fuzzy logic.* Fuzzy logic is not much weaker than classical logic. Since also many formulae of $Ł\Pi_\omega$ have the same form as in classical mathematics, classical proofs often work for them. Sometimes the classical proof has to be slightly adapted so as to avoid a forbidden rule, or a premise must be multiplied in the theorem, but in many cases the classical proofs just work.

- *Example of an informal proof.* We prove the theorem

  $$\mathrm{Refl}(R) \to R \subseteq R \circ R \qquad (14)$$

  which says that any fuzzy relation is a subset of its self-composition at least to the degree of its reflexivity.

  Proof: It is sufficient to prove (by the rules of $Ł\Pi_\omega$, for arbitrary x,y) that $\langle x,y\rangle \in R \circ R$ from the premises (i) that R is reflexive, and (ii) that $\langle x,y\rangle \in R$. The conclusion is by definition equivalent to $(\exists z)(Rxz \,\&\, Rzy)$. Now x can be taken for the required z, since by (i), Rxx, and by (ii), Rxy. QED

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