Topology in Fuzzy Class Theory: Basic Notions

Libor Běhounek[1]* and Tomáš Kroupa[2]**

[1] Institute of Computer Science
Academy of Sciences of the Czech Republic
Pod Vodárenskou věží 2, 182 07 Prague, Czech Republic
behounek@cs.cas.cz, www.cs.cas.cz/hp

[2] Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic
kroupa@utia.cas.cz

Abstract. In the formal and fully graded setting of Fuzzy Class Theory (or higher-order fuzzy logic) we make an initial investigation into basic notions of fuzzy topology. In particular we study graded notions of fuzzy topology regarded as a fuzzy system of open or closed fuzzy sets and as a fuzzy system of fuzzy neighborhoods. We show their basic graded properties and mutual relationships provable in Fuzzy Class Theory and give some links to the traditional notions of fuzzy topology.

1 Introduction

Fuzzy topology is among the fundamental disciplines of fuzzy mathematics whose development was stimulated from the very beginning of the invention of fuzzy sets [1]. Following the role of topology in classical mathematics, fuzzy topology should capture the notions of openness, neighborhood, closure, etc., within the setting of fuzzy set theory. The paper [2] by Höhle and Šostak, which is contained in the special issue of Fuzzy Sets and Systems (1995) on fuzzy topology, mentions and classifies a number of conceptual frameworks (lattice-, model-, and category-theoretical) that have arisen during past decades. A detailed and up-to-date exposition of many-valued and fuzzy topologies, mostly based on a categorical viewpoint, is contained in the monograph [3] by Höhle.

This paper follows the footsteps of Ying's attempt [4] to establish fuzzy topology as a non-elementary theory over many-valued logic. We make initial steps towards understanding fuzzy topology as an axiomatic higher-order theory over Hájek-style [5] formal fuzzy logic, following the methodology for formal fuzzy mathematics described in [6]. According to the classification proposed in [2], the models of our theory are closest to "L-fuzzy topologies as characteristic

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morphisms". However, the apparatus of Fuzzy Class Theory, employed in this paper, makes our notions and the way in which they can be studied quite distinct from (and in some aspects more general than) other approaches to fuzzy topology.

The paper is organized as follows: Section 2 gives a brief exposition of Fuzzy Class Theory and the definitions needed in the paper. Section 3 studies the graded notion of fuzzy topology regarded as a fuzzy system of open (or closed) fuzzy sets. Section 4 then studies graded fuzzy topologies regarded as fuzzy systems of fuzzy neighborhoods.

2 Preliminaries

Fuzzy Class Theory FCT, introduced in [7], is an axiomatization of Zadeh's notion of fuzzy set in formal fuzzy logic. Here we use its variant defined over $\mathrm{IMTL}_\triangle$ [8], the logic of all left-continuous t-norms whose residual negation is involutive (we shall call them *IMTL t-norms;* the most important example is the Łukasiewicz t-norm $x * y =_{\mathrm{df}} \max(0, x + y - 1)$).

Remark 2.1. We have the following reasons for choosing $\mathrm{IMTL}_\triangle$ for the ground logic: the logic $\mathrm{MTL}_\triangle$ [8] of all left-continuous t-norms is arguably [9] the weakest fuzzy logic with good inferential properties for fully graded fuzzy mathematics in the framework of formal fuzzy logic [6]. $\mathrm{IMTL}_\triangle$ extends it with the law of double negation, which is in fuzzy topology needed for the correspondence between open and closed fuzzy sets. A generalization of fuzzy topology to the logic $\mathrm{MTL}_\triangle$ (with independent systems of open and closed fuzzy sets) will be the subject of some future paper.

We assume the reader's familiarity with $\mathrm{IMTL}_\triangle$; for details on this logic see [8]. Here we only recapitulate its standard $[0, 1]$ semantics:

$\&$	...	a left-continuous t-norm $*$ with involutive residual negation
$\to$	...	the residuum $\Rightarrow$ of $*$, defined as $x \Rightarrow y =_{\mathrm{df}} \sup\{z \mid z * x \leq y\}$
$\wedge$, $\vee$	...	min, max
$\neg$	...	$x \Rightarrow 0$; in $\mathrm{IMTL}_\triangle$ it is involutive, due to the axiom $\neg\neg\varphi \to \varphi$
$\veebar$	...	the t-conorm dual to $*$ (since $\varphi \veebar \psi$ is defined as $\neg(\neg\varphi \,\&\, \neg\psi)$)
$\leftrightarrow$	...	the bi-residuum: $\min(x \Rightarrow y, y \Rightarrow x)$
$\triangle$	...	$\triangle x = 1 - \mathrm{sgn}(1 - x)$
$\forall$, $\exists$	...	inf, sup; by involutiveness, $(\exists x)\neg\varphi \leftrightarrow \neg(\forall x)\varphi$

Definition 2.1. Fuzzy Class Theory FCT *is a formal theory over multi-sorted first-order fuzzy logic (in this paper,* $\mathrm{IMTL}_\triangle$*), with the sorts of variables for*

- *atomic objects (lowercase letters $x, y, \dots$)*
- *fuzzy classes of atomic objects (uppercase letters $A, B, \dots$)*
- *fuzzy classes of fuzzy classes of atomic objects (Greek letters $\tau, \sigma, \dots$)*
- *fuzzy classes of the third order (calligraphic letters $\mathcal{A}, \mathcal{B}, \dots$)*
- *etc., in general for fuzzy classes of the n-th order ($X^{(n)}, Y^{(n)}, \dots$)*

Table 1. Abbreviations used in the formulae of FCT

Ax	$\equiv_{\mathrm{df}}$	$x \in A$
$x_1 \dots x_k$	$=_{\mathrm{df}}$	$\langle x_1, \dots, x_k \rangle$
$x \notin A$	$\equiv_{\mathrm{df}}$	$\neg(x \in A)$, and similarly for other predicates
$(\forall x \in A)\varphi$	$\equiv_{\mathrm{df}}$	$(\forall x)(x \in A \rightarrow \varphi)$
$(\exists x \in A)\varphi$	$\equiv_{\mathrm{df}}$	$(\exists x)(x \in A \mathbin{\&} \varphi)$
$(\forall x, y \in A)\varphi$	$\equiv_{\mathrm{df}}$	$(\forall x \in A)(\forall y \in A)\varphi$, similarly for $\exists$
$\{x \in A \mid \varphi\}$	$=_{\mathrm{df}}$	$\{x \mid x \in A \mathbin{\&} \varphi\}$
$\{t(x_1, \dots, x_k) \mid \varphi\}$	$=_{\mathrm{df}}$	$\{z \mid z = t(x_1, \dots, x_k) \mathbin{\&} \varphi\}$
φ^n	$\equiv_{\mathrm{df}}$	$\varphi \mathbin{\&} \dots \mathbin{\&} \varphi$ (n times)

Besides the crisp identity predicate $=$*, the language of* FCT *contains:*

- *the membership predicate* $\in$ *between objects of successive sorts*
- *class terms* $\{x \mid \varphi\}$*, for any formula* φ *and any variable* x *of any order*
- *symbols* $\langle x_1, \dots, x_k \rangle$ *for* k*-tuples of individuals* $x_1, \dots, x_k$ *of any order*

FCT *has the following axioms (for all formulae* φ *and variables of all orders):*

- *the logical axioms of multi-sorted first-order logic* $\mathrm{IMTL}_\triangle$
- *the axioms of crisp identity: (i)* $x = x$*, (ii)* $x = y \rightarrow (\varphi(x) \rightarrow \varphi(y))$*, (iii)* $\langle x_1, \dots, x_k \rangle = \langle y_1, \dots, y_k \rangle \rightarrow x_1 = y_1 \mathbin{\&} \dots \mathbin{\&} x_k = y_k$
- *the comprehension axioms:* $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$
- *the extensionality axioms:* $(\forall x)\triangle(x \in A \leftrightarrow x \in B) \rightarrow A = B$

Remark 2.2. Notice that in FCT, fuzzy sets are rendered as a *primitive notion* rather than modeled by membership functions. In order to capture this distinction, fuzzy sets are in FCT called *fuzzy classes;* the name *fuzzy set* is reserved for membership functions in the models of the theory.

The models of FCT are systems of fuzzy sets of all orders over a fixed crisp universe of discourse, with truth degrees taking values in an $\mathrm{IMTL}_\triangle$-chain (e.g., the interval $[0, 1]$ equipped with an IMTL t-norm). Thus all theorems on fuzzy classes provable in FCT are true statements about **L**-valued fuzzy sets, for any $\mathrm{IMTL}_\triangle$-chain **L**. Notice however that the theorems of FCT have to be derived from its axioms by the rules of the fuzzy logic $\mathrm{IMTL}_\triangle$ rather than classical Boolean logic. For details on proving theorems of FCT see [10] or [11].

Convention 2.1 *In formulae of* FCT*, we employ usual abbreviations known from classical mathematics, including those listed in Table 1. Usual rules of precedence apply to the connectives of* $\mathrm{IMTL}_\triangle$*. Furthermore we define standard defined notions of* FCT*, summarized in Table 2, for all orders of fuzzy classes.*

Remark 2.3. Notice that in FCT, not only the membership predicate $\in$, but all defined notions are in general fuzzy (unless they are defined as provably crisp). FCT thus presents a fully graded approach to fuzzy mathematics. The

Table 2. Defined notions of FCT

$\emptyset$	$=_{\mathrm{df}}$	$\{x \mid 0\}$	empty class
V	$=_{\mathrm{df}}$	$\{x \mid 1\}$	universal class
$\mathrm{Ker}\, A$	$=_{\mathrm{df}}$	$\{x \mid \triangle Ax\}$	kernel
αA	$=_{\mathrm{df}}$	$\{x \mid \alpha \mathbin{\&} Ax\}$	α-resize
$-A$	$=_{\mathrm{df}}$	$\{x \mid \neg Ax\}$	complement
$A \cap B$	$=_{\mathrm{df}}$	$\{x \mid Ax \mathbin{\&} Bx\}$	(strong) intersection
$A \cup B$	$=_{\mathrm{df}}$	$\{x \mid Ax \veebar Bx\}$	(strong) union
$A \times B$	$=_{\mathrm{df}}$	$\{xy \mid Ax \mathbin{\&} By\}$	Cartesian product
$\bigcup \tau$	$=_{\mathrm{df}}$	$\{x \mid (\exists A \in \tau)(x \in A)\}$	class union
$\bigcap \tau$	$=_{\mathrm{df}}$	$\{x \mid (\forall A \in \tau)(x \in A)\}$	class intersection
$\mathrm{Pow}(A)$	$=_{\mathrm{df}}$	$\{X \mid X \subseteq A\}$	power class
$\mathrm{Crisp}(A)$	$\equiv_{\mathrm{df}}$	$(\forall x)\triangle(Ax \vee \neg Ax)$	crispness
$\mathrm{Ext}_E\, A$	$\equiv_{\mathrm{df}}$	$(\forall x, y)(Exy \mathbin{\&} Ax \to Ay)$	E-extensionality
$A \subseteq B$	$\equiv_{\mathrm{df}}$	$(\forall x)(Ax \to Bx)$	inclusion
$A \approxeq B$	$\equiv_{\mathrm{df}}$	$(A \subseteq B) \mathbin{\&} (B \subseteq A)$	(strong) bi-inclusion

importance of full gradedness in fuzzy mathematics is explained in [10, 12, 13]: its main merit lies in that it allows inferring relevant information even when a property of fuzzy sets is not fully satisfied. Fuzzy topology has a long tradition of attempting full gradedness, cf. graded definitions and theorems e.g. in [3, 4].

Remark 2.4. It should be noted that fully graded theories have some peculiar features in which they differ from both classical mathematics and traditional fuzzy mathematics. A detailed account of the unusual features of fully graded theories is given in [14]; some of them can also be found in [10] (available online). Here we only briefly stress the main features of graded mathematics:

- Since $\varphi \to \varphi \mathbin{\&} \varphi$ is not a generally valid law of fuzzy logic, premises may occur several times in theorems. A typical graded theorem has the form $\varphi_1^{k_1} \mathbin{\&} \ldots \mathbin{\&} \varphi_n^{k_n} \to \psi$, where φ^k abbreviates $\varphi \mathbin{\&} \ldots \mathbin{\&} \varphi$ (k times, where φ^0 is 1). The multiplicity k_i of the premise φ_i shows how strongly it influences (the lower bound for) the truth of ψ (when only partially true), and depends on how many times the premise is used in the derivation of ψ from $\varphi_1, \ldots, \varphi_k$. The exponent k in φ^k can also take the conventional value "$\triangle$", where $\varphi^\triangle$ is understood as $\triangle\varphi$ (recall that $\varphi^\triangle \to \varphi^n$ for all n).
- If a complex notion Φ is defined as a conjunction $\varphi_1 \mathbin{\&} \ldots \mathbin{\&} \varphi_n$, then the conjuncts φ_i will get different multiplicities in different theorems. It is therefore appropriate to parameterize Φ by the multiplicities of the components φ_i and define it as $\Phi^{k_1,\ldots,k_n} \equiv_{\mathrm{df}} \varphi_1^{k_1} \mathbin{\&} \ldots \mathbin{\&} \varphi_n^{k_n}$. (All graded topological notions in the following sections will be defined in this way.) We can write just Φ^k instead of $\Phi^{k_1,\ldots,k_n}$ if $k_i = k$ for all i, and just Φ if $k_i = 1$ for all i.

The following defined predicates will be employed in the next sections.

Definition 2.2. *We define the following (graded) unary predicates:*

$\cup$*-closedness:*	$\mathrm{uc}(\tau) \equiv_{\mathrm{df}} (\forall A, B \in \tau)(A \cup B \in \tau)$
$\cap$*-closedness:*	$\mathrm{ic}(\tau) \equiv_{\mathrm{df}} (\forall A, B \in \tau)(A \cap B \in \tau)$
$\bigcup$*-closedness:*	$\mathrm{Uc}(\tau) \equiv_{\mathrm{df}} (\forall \nu \subseteq \tau)\big(\bigcup \nu \in \tau\big)$
$\bigcap$*-closedness:*	$\mathrm{Ic}(\tau) \equiv_{\mathrm{df}} (\forall \nu \subseteq \tau)\big(\bigcap \nu \in \tau\big)$
$\subseteq$*-upperness:*	$\mathrm{Upper}(\tau) \equiv_{\mathrm{df}} (\forall A, B)(A \subseteq B \mathbin{\&} A \in \tau \rightarrow B \in \tau)$
being a filter:	$\mathrm{Filter}^{v,e,u,i}(\tau) \equiv_{\mathrm{df}} (\mathrm{V} \in \tau)^v \mathbin{\&} (\emptyset \notin \tau)^e \mathbin{\&} \mathrm{Upper}^u(\tau) \mathbin{\&} \mathrm{ic}^i(\tau)$

3 Topology as a System of Open (Closed) Fuzzy Classes

In classical mathematics, topology can be introduced in several equivalent ways—by open sets, closed sets, neighborhoods, closure, etc. In FCT, however, these approaches yield different concepts. In this paper, we make an initial investigation into two of them, namely the system of open (or closed) classes (in this section) and the system of neighborhoods (in Sect. 4). Due to the limited size of this paper we present only some of the initial results and have to omit all proofs.

The fuzzification of the concept of open (closed) fuzzy topology presented in Def. 3.1 follows the methodology sketched in [15, §5] and formally elaborated in [7, §7], i.e., reinterpreting the formulae of the classical definition in fuzzy logic.[1]

Definition 3.1. *We define an* (open) (e, v, i, u)–fuzzy topology *and a* closed (e, v, u, i)–fuzzy topology *respectively by the predicates*

$$\mathsf{OTop}^{e,v,i,u}(\tau) \equiv_{\mathrm{df}} (\emptyset \in \tau)^e \mathbin{\&} (\mathrm{V} \in \tau)^v \mathbin{\&} \mathrm{ic}^i(\tau) \mathbin{\&} \mathrm{Uc}^u(\tau)$$
$$\mathsf{CTop}^{e,v,u,i}(\sigma) \equiv_{\mathrm{df}} (\emptyset \in \sigma)^e \mathbin{\&} (\mathrm{V} \in \sigma)^v \mathbin{\&} \mathrm{uc}^u(\sigma) \mathbin{\&} \mathrm{Ic}^i(\sigma)$$

(see Remark 2.4 for the meaning of the parameters e, v, u, i).

Note that this concept of topology is graded, i.e., the predicate $\mathsf{OTop}^{e,v,i,u}$ determines the *degree* to which τ is an open (e, v, i, u)–fuzzy topology.

Example 3.1. Let $*$ be an IMTL t-norm and $\Rightarrow$ its residuum. The $*$-based Zadeh models of open $(1, 1, \triangle, \triangle)$–fuzzy topology, i.e., of the predicate

$$\mathsf{OTop}^{1,1,\triangle,\triangle}(\tau) \equiv \emptyset \in \tau \mathbin{\&} \mathrm{V} \in \tau \mathbin{\&} \triangle\,\mathrm{ic}(\tau) \mathbin{\&} \triangle\,\mathrm{Uc}(\tau)$$

are functions $\tau\colon [0,1]^{\mathrm{V}} \to [0,1]$ satisfying the following conditions:

(i) $\tau(A) * \tau(B) \leq \tau(A \cap B)$ for every $A, B \in [0,1]^{\mathrm{V}}$
(ii) $\bigwedge_{A \in [0,1]^{\mathrm{V}}} (\nu(A) \Rightarrow \tau(A)) \leq \tau\left(\bigcup \nu\right)$ for every $\nu\colon [0,1]^{\mathrm{V}} \to [0,1]$

[1] The requirement that both $\emptyset$ and the ground set be open can meaningfully be reinterpreted in fuzzy logic in several ways; here we restrict ourselves to the weakest one, requiring openness just for the two classes $\emptyset$ and V. Stronger notions of topology (e.g., *stratified* topology [3] with the condition $\alpha\mathrm{V} \in \tau$ for all truth degrees α) will be studied in subsequent papers.

where $(A \cap B)(x) = A(x) * B(x)$ and $\left(\bigcup \nu\right)(x) = \bigvee_{A \in [0,1]^{\mathrm{V}}} (\nu(A) * A(x))$. Since both (i) and (ii) are crisp, the degree to which τ is a $(1, 1, \triangle, \triangle)$-fuzzy topology equals $\tau(\emptyset) * \tau(\mathrm{V})$. These models cover fuzzy topologies studied under the name "L-fuzzy topologies of Höhle type" [2].

In $\mathrm{IMTL}_\triangle$, open and closed topologies are interdefinable:

Definition 3.2. *Let* $\tau_c =_{\mathrm{df}} \{A \mid -A \in \tau\}$.

Theorem 3.1. FCT *proves:* $\mathsf{OTop}(\tau) \leftrightarrow \mathsf{CTop}(\tau_c)$, $\mathsf{CTop}(\sigma) \leftrightarrow \mathsf{OTop}(\sigma_c)$.

Definition 3.3. *Given a class of classes* τ*, we define the* interior *and* closure *in* τ *as follows:*

$$\mathrm{Int}_\tau(A) =_{\mathrm{df}} \bigcup \{B \in \tau \mid B \subseteq A\}$$
$$\mathrm{Cl}_\tau(A) =_{\mathrm{df}} \bigcap \{B \in \tau_c \mid A \subseteq B\}$$

Theorem 3.2. *It is provable in* FCT*:*

(i) $\mathrm{Int}_\tau(A) \subseteq A$
(ii) $A \subseteq B \rightarrow \mathrm{Int}_\tau(A) \subseteq \mathrm{Int}_\tau(B)$
(iii) $A \in \tau \rightarrow \mathrm{Int}_\tau(A) \approxeq A$
(iv) $\mathrm{Int}_\tau(A \cap B) \cap \mathrm{Int}_\tau(A \cap B) \subseteq \mathrm{Int}_\tau(A) \cap \mathrm{Int}_\tau(B)$

Theorem 3.3 (OTop and the interior operator). *It is provable in* FCT*:*

(i) $\mathsf{OTop}^{0,0,0,1}(\tau) \rightarrow \mathrm{Int}_\tau(A) \in \tau$
(ii) $\mathsf{OTop}^{0,0,0,1}(\tau) \rightarrow \mathrm{Int}_\tau(\mathrm{Int}_\tau(A)) \approxeq \mathrm{Int}_\tau(A)$
(iii) $\mathsf{OTop}^{0,0,1,0}(\tau) \rightarrow \mathrm{Int}_\tau(A) \cap \mathrm{Int}_\tau(B) \subseteq \mathrm{Int}_\tau(A \cap B)$
(iv) $\mathsf{OTop}^{0,1,0,0}(\tau) \rightarrow \mathrm{Int}_\tau(\mathrm{V}) \approxeq \mathrm{V}$

Since $\mathrm{Cl}_\tau(A) = -\mathrm{Int}_\tau(-A)$ is provable in FCT, the next two theorems are just dual counterparts of Th. 3.2 and 3.3.

Theorem 3.4. *It is provable in* FCT*:*

(i) $A \subseteq \mathrm{Cl}_\tau(A)$
(ii) $A \subseteq B \rightarrow \mathrm{Cl}_\tau(A) \subseteq \mathrm{Cl}_\tau(B)$
(iii) $A \in \tau_c \rightarrow \mathrm{Cl}_\tau(A) \approxeq A$
(iv) $\mathrm{Cl}_\tau(A) \cup \mathrm{Cl}_\tau(B) \subseteq \mathrm{Cl}_\tau(A \cup B) \cup \mathrm{Cl}_\tau(A \cup B)$

Theorem 3.5 (OTop and the closure operator). *It is provable in* FCT*:*

(i) $\mathsf{OTop}^{0,0,0,1}(\tau) \rightarrow \mathrm{Cl}_\tau(A) \in \tau_c$
(ii) $\mathsf{OTop}^{0,0,0,1}(\tau) \rightarrow \mathrm{Cl}_\tau(\mathrm{Cl}_\tau(A)) \approxeq \mathrm{Cl}_\tau(A)$
(iii) $\mathsf{OTop}^{0,0,1,0}(\tau) \rightarrow \mathrm{Cl}_\tau(A \cup B) \subseteq \mathrm{Cl}_\tau(A) \cup \mathrm{Cl}_\tau(B)$
(iv) $\mathsf{OTop}^{0,1,0,0}(\tau) \rightarrow \mathrm{Cl}_\tau(\emptyset) \approxeq \emptyset$

Definition 3.4. *A predicate expressing that* A *is a* neighborhood *of* x *in* τ *is defined as*

$$\mathrm{Nb}_\tau(x, A) \equiv_{\mathrm{df}} (\exists B \in \tau)(B \subseteq A \mathbin{\&} x \in B)$$

The system of all neighborhoods of x *will be denoted by* $\nu_x =_{\mathrm{df}} \{A \mid \mathrm{Nb}_\tau(x, A)\}$.

Theorem 3.6 (OTop and neighborhoods). *It is provable in* FCT*:*

(i) $x \in \bigcap \nu_x$
(ii) $\mathrm{Nb}_\tau(x, A) \leftrightarrow x \in \mathrm{Int}_\tau(A)$
(iii) $\mathsf{OTop}(\tau) \to \mathrm{Filter}(\nu_x) \mathbin{\&} (\forall A \in \nu_x)(\exists B \in \nu_x)(B \subseteq A \mathbin{\&} (\forall y \in B)\,\mathrm{Nb}_\tau(y, B))$

In general, the system of all open fuzzy topologies is not closed under arbitrary intersections. Nevertheless, the system of all open $\triangle$–fuzzy topologies is at least closed under crisp intersections, which allows introducing the notion of open fuzzy topology generated by a subbase of fuzzy classes:

Theorem 3.7. *Let* $\mathcal{X}$ *be a fuzzy class of the third order. Then* FCT *proves:*

$$\mathrm{Crisp}(\mathcal{X}) \mathbin{\&} (\forall \tau \in \mathcal{X})\big(\triangle\, \mathsf{OTop}(\tau)\big) \to \triangle\, \mathsf{OTop}\big(\bigcap \mathcal{X}\big)$$

Definition 3.5. *Let* σ *be a fuzzy class of fuzzy classes. Then we define*

$$\tau_\sigma =_{\mathrm{df}} \bigcap \{\tau' \mid \triangle(\mathsf{OTop}(\tau') \mathbin{\&} \sigma \subseteq \tau')\}$$

By Th. 3.7, FCT proves that $\triangle\, \mathsf{OTop}(\tau_\sigma)$, and obviously also that τ_σ is the least open $\triangle$–fuzzy topology containing σ.

Example 3.2. Interval open fuzzy topology. Let $\leq$ be a crisp dense ordering (e.g., of real or rational numbers). The fuzzy properties of being an upper resp. lower class in $\leq$ are defined by the predicates

$$\mathrm{Upper}_{\leq}(A) \equiv_{\mathrm{df}} (\forall p, q)(p \leq q \mathbin{\&} Ap \to Aq)$$
$$\mathrm{Lower}_{\leq}(A) \equiv_{\mathrm{df}} (\forall p, q)(p \geq q \mathbin{\&} Bp \to Bq)$$

Fuzzy intervals $[A, B]$ in $\leq$ can be defined [16] as intersections $A \cap B$ of two fuzzy classes A, B, where $\triangle\, \mathrm{Upper}_{\leq}(A) \mathbin{\&} \triangle\, \mathrm{Lower}_{\leq}(B)$. An *open* fuzzy interval can be defined by the following fuzzy predicate:[2]

$$\begin{aligned}\mathrm{Op}([A, B]) \equiv_{\mathrm{df}} &\triangle(\mathrm{Upper}_{\leq}(A)) \mathbin{\&} (\forall p)(Ap \to (\exists q < p)Aq) \mathbin{\&} \\ &\triangle(\mathrm{Lower}_{\leq}(B)) \mathbin{\&} (\forall p)(Bp \to (\exists q > p)Bq)\end{aligned}$$

By Th. 3.7, the fuzzy system $\sigma = \{[A, B] \mid \mathrm{Op}([A, B])\}$ of open fuzzy intervals generates an open fuzzy topology τ_σ—the *interval open fuzzy topology* of $\leq$. It can be shown that σ itself is $\cap$-closed; since furthermore $\cap$ distributes over $\bigcup$, FCT proves that $\tau_\sigma = \{\bigcup \nu \mid \nu \subseteq \sigma\}$ (just like in the crisp interval topology).

[2] Observe that it generalizes the notion of crisp open interval, by the requirement of the appropriate left- or right- continuity of the characteristic function of the interval.

4 Topology Given by a Neighborhood Relation

The following definition of fuzzy topology is an internalization in fuzzy logic of the conditions required from the system of neighborhoods.[3]

Definition 4.1. *We define a* neighborhood (i, j, k, l)–fuzzy topology *by the predicate*

$$\begin{aligned}&\mathsf{NTop}^{i,j,k,l}(\mathrm{Nb}) \equiv_{\mathrm{df}} \triangle(\mathrm{Nb} \subseteq \mathrm{V} \times \mathrm{Ker}\,\mathrm{Pow}(\mathrm{V}))\ \&\\ &\quad ((\forall x, A)(\mathrm{Nb}(x, A) \to x \in A))^i\ \&\\ &\quad ((\forall x, A, B)(\mathrm{Nb}(x, A)\ \&\ A \subseteq B \to \mathrm{Nb}(x, B)))^j\ \&\\ &\quad ((\forall x, A, B)(\mathrm{Nb}(x, A)\ \&\ \mathrm{Nb}(x, B) \to \mathrm{Nb}(x, A \cap B)))^k\ \&\\ &\quad ((\forall x, A)(\mathrm{Nb}(x, A) \to (\exists B \subseteq A)(\mathrm{Nb}(x, B)\ \&\ (\forall y \in B)\,\mathrm{Nb}(y, B)))^l\end{aligned}$$

Definition 4.2. *Let* $\triangle(\mathrm{Nb} \subseteq \mathrm{V} \times \mathrm{Ker}\,\mathrm{Pow}(\mathrm{V}))$*. Then we define (as usual) the system of* Nb-open *classes:*

$$\tau_{\mathrm{Nb}} =_{\mathrm{df}} \{A \mid (\forall x \in A)\,\mathrm{Nb}(x, A)\}$$

It can be shown that even if Nb is a neighborhood fuzzy topology to degree one, τ_{Nb} still need not be an open fuzzy topology (in particular, it need not be closed under arbitrary unions). Only the following holds:

Theorem 4.1. FCT *proves:* $\triangle\,\mathsf{NTop}(\mathrm{Nb}) \to (\forall \sigma \subseteq \tau_{\mathrm{Nb}})\,(\bigcup(\sigma \cap \sigma) \in \tau_{\mathrm{Nb}})$.

This motivates a modified notion of open fuzzy topology:

Definition 4.3. *We define the following predicates:*

$$\begin{aligned}\mathrm{U_2c}(\tau) &\equiv_{\mathrm{df}} (\forall \sigma \subseteq \tau)\big(\bigcup(\sigma \cap \sigma) \in \tau\big)\\ \mathsf{O_2Top}^{e,v,i,u}(\tau) &\equiv_{\mathrm{df}} (\emptyset \in \tau)^e\ \&\ (\mathrm{V} \in \tau)^v\ \&\ \mathrm{ic}^i(\tau)\ \&\ \mathrm{U_2c}^u(\tau)\end{aligned}$$

Theorem 4.2. FCT *proves:*

$$(\exists x, A)\,\mathrm{Nb}(x, A)\ \&\ \mathsf{NTop}^{1,3,1,1}(\mathrm{Nb}) \to \mathsf{O_2Top}(\tau_{\mathrm{Nb}})\ \&\ (\mathrm{Nb}(x, A) \leftrightarrow \mathrm{Nb}_{\tau_{\mathrm{Nb}}}(x, A))$$

Thus, a "sufficiently monotone" non-empty neighborhood topology determines a corresponding open "topology" which is closed under the operation $\bigcup(\sigma \cap \sigma)$ rather than under usual unions $\bigcup \sigma$. Such systems are met quite often in fully graded fuzzy topology:

Example 4.1. It is well-known from traditional fuzzy mathematics that the system of fuzzy sets fully extensional w.r.t. a fuzzy relation R is closed under unions of arbitrary crisp systems of fuzzy sets and under min-intersections of crisp pairs of fuzzy sets (i.e., it forms a fuzzy topology in the traditional, non-graded sense

[3] The first condition only determines the type of the neighborhood predicate (i.e., that it is a relation between points and classes), therefore its full validity is required.

of [17]). In the *graded* framework of FCT it can be proved that the *fuzzy* system of R-extensional classes $\{A \mid \mathrm{Ext}_R\, A\}$ is closed under $\bigcup(\sigma \cap \sigma)$ (but not under arbitrary fuzzy unions), and provided $R \subseteq R \cap R$ (which holds e.g. if R is crisp), it satisfies $\mathsf{O}_2\mathsf{Top}$.

Both OTop and $\mathsf{O}_2\mathsf{Top}$ topologies are closed under crisp unions, which leads to a further generalization of the notion of open fuzzy topology:

Definition 4.4. *We define the following predicates:*

$$\mathrm{U}_\triangle \mathrm{c}(\tau) \equiv_{\mathrm{df}} (\forall \sigma \subseteq \tau)\big(\mathrm{Crisp}(\sigma) \to \bigcup \sigma \in \tau\big)$$
$$\mathsf{O}_\triangle\mathsf{Top}^{e,v,i,u}(\tau) \equiv_{\mathrm{df}} (\emptyset \in \tau)^e \mathbin{\&} (\mathrm{V} \in \tau)^v \mathbin{\&} \mathrm{ic}^i(\tau) \mathbin{\&} \mathrm{U}_\triangle \mathrm{c}^u(\tau)$$

The models of $\mathsf{O}_\triangle\mathsf{Top}$ are among frequently studied fuzzy topological structures called "L-fuzzy topologies of Šostak-type" according to the classification proposed in [2].

The definition of the interior operator needs to be modified to have good properties in neighborhood fuzzy topologies:

$$\mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A) =_{\mathrm{df}} \bigcup \{B \mid \triangle(B \in \tau_{\mathrm{Nb}} \mathbin{\&} B \subseteq A)\}$$

Theorem 4.3. *It is provable in* FCT*:*

(i) $\mathsf{NTop}^{0,1,0,0}(\mathrm{Nb}) \to \triangle(\mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A) \in \tau_{\mathrm{Nb}})$
(ii) $A \subseteq B \to \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A) \subseteq \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(B)$
(iii) $\triangle(A \in \tau_{\mathrm{Nb}}) \to \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A) = A$

Theorem 4.4 (NTop and interior operator). *It is provable in* FCT*:*

(i) $\triangle(\mathrm{V} \in \tau_{\mathrm{Nb}}) \to \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(\mathrm{V}) = \mathrm{V}$
(ii) $\mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A) \subseteq A$
(iii) $\mathsf{NTop}^{0,1,0,0}(\mathrm{Nb}) \to \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(\mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A)) = \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A)$
(iv) $\mathsf{NTop}^{0,0,1,0}(\mathrm{Nb}) \to \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A) \cap \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(B) \subseteq \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A \cap B)$
(v) $\mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A \cap B) \cap \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A \cap B) \subseteq \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A) \cap \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(B)$

The following theorem guarantees that neighborhoods defined from a (sufficiently union-closed) open fuzzy topology are exactly the neighborhoods in the sense of predicate NTop.

Theorem 4.5 (OTop and NTop). *It is provable in* FCT*:*

$$\mathsf{OTop}^{1,1,1,2}(\tau) \to \mathsf{NTop}(\mathrm{Nb}_\tau) \mathbin{\&} (A \in \tau \leftrightarrow (\forall x \in A)\, \mathrm{Nb}_\tau(x, A))$$

Example 4.2. Interval neighborhood fuzzy topology. The (fuzzy) system of open fuzzy intervals of Example 3.2 allows introducing the *interval neighborhood fuzzy topology* w.r.t. a crisp dense ordering $\leq$, by taking

$$\mathrm{Nb}(x, X) \equiv_{\mathrm{df}} (\exists A, B) \triangle (\mathrm{Op}([A, B]) \mathbin{\&} [A, B] \subseteq X \mathbin{\&} x \in [A, B])$$

Then it can be shown that FCT proves $\triangle\, \mathsf{NTop}(\mathrm{Nb})$, and in virtue of Th. 4.2, $\triangle\, \mathsf{O}_2\mathsf{Top}(\tau_{\mathrm{Nb}})$ and $\mathrm{Nb} = \mathrm{Nb}_{\tau_{\mathrm{Nb}}}$. Notice, however, that the interval open topology of Example 3.2 differs from the interval neighborhood topology introduced here, since in the latter all classes open to degree 1 are crisp.

5 Conclusions

We have introduced two notions of fuzzy topology in the graded framework of Fuzzy Class Theory and investigated their basic properties; where appropriate, we gave links to similar notions of fuzzy topology studied previously in traditional fuzzy mathematics. Most of our notions generalize usual concepts of fuzzy topology by allowing full gradedness of all defined predicates and functions. Proofs of the graded theorems, though omitted here due to the limited space, are rather simple and show the strength of the apparatus of higher-order fuzzy logic in fuzzy topology. The results open a wide area of fully graded topological theory and show the possibility of the investigation of more advanced graded topological notions by means of Fuzzy Class Theory.

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