From Fuzzy Logic to Fuzzy Mathematics: A Methodological Manifesto

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Abstract

The paper states the problem of fragmentation of contemporary fuzzy mathematics and the need of a unified methodology and formalism. We formulate several guidelines based on Hájek's methodology in fuzzy logic, which enable us to follow closely the constructions and methods of classical mathematics recast in a fuzzy setting. As a particular solution we propose a three-layer architecture of fuzzy mathematics, with the layers of formal fuzzy logic, a foundational theory, and individual mathematical disciplines developed within its framework. The ground level of logic being sufficiently advanced, we focus on the foundational level; the theory we propose for the foundations of fuzzy mathematics can be characterized as Henkin-style higher-order fuzzy logic. Finally we give some hints on the further development of individual mathematical disciplines in the proposed framework, and proclaim it a research programme in formal fuzzy mathematics.

Key words: Non-classical logics, formal fuzzy logic, formal fuzzy mathematics, higher-order fuzzy logic
1991 MSC: 03A05, 03B52, 03E70, 03E72

One of the motives for theoretical studies in fuzzy mathematics is the pursuit of formal reconstruction of the methods commonly used in applied fuzzy mathematics. The greatest success in such investigations is undoubtedly the area of formal fuzzy logic: this discipline has recently reached the point when

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[1] Support of grant B100300502 of the Grant Agency of the Academy of Sciences of the Czech Republic is acknowledged.
[2] Partial support of grant B100300502 of the Grant Agency of the Academy of Sciences of the Czech Republic and of the Institutional Research Plan AV0Z10300504 is acknowledged.

it is reasonable to attempt to use it as a ground theory for the formalization of other branches of fuzzy mathematics.

This paper tries to provide certain guidelines for such a transition from formal fuzzy logic to formal fuzzy mathematics. The guidelines are based upon doctrines observed by the Prague workgroup on fuzzy logic founded and led by Petr Hájek. We attempt to formulate explicitly some distinct features of Petr Hájek's approach, which we reconstruct from his scattered remarks and the general direction of his papers, and implement them in the form of a research programme. We hope that Petr Hájek will find our reconstruction of his doctrine faithful enough; or else that he will enter into a fruitful dispute with his own disciples over the methodological foundations of our discipline. If the former is the case, then we deem that the best label for the enterprise described in this paper would be *Hájek's Programme* in the foundations of fuzzy mathematics.

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The cornerstone of Hájek's approach to fuzzy mathematics is the doctrine of working in a formal axiomatic theory over a fuzzy logic, rather than investigating particular models. For ease of reference, let us call it the *formalistic imperative.* Good reasons for such an approach can be found, both of a philosophical and pragmatic nature.

A philosophical reason is found in the following argument. Fuzzy logic describes the laws of truth preservation in reasoning under (a certain form of) vagueness. Its interpretation in terms of truth degrees and membership functions is just a mathematical *model*—a classical rendering of vague phenomena. Of course, the laws of fuzzy logic were originally discovered with the help of this model, and truth degrees form its principal semantics; but once we believe that the laws capture fuzzy inference correctly, we can abstract from the model that helped us to find them.

Fuzzy predicates are essentially not different from crisp predicates: the only difference is the graded boundary of fuzzy sets, due to which some of the laws of classical reasoning about them fail. The laws of inference valid for fuzzy predicates form fuzzy logic; classical logic is its limit case, applicable if by chance all predicates involved are crisp. Reasoning about fuzzy predicates therefore follows the laws of fuzzy logic in the same manner as reasoning about crisp predicates follows the laws of classical logic. In mathematics, such reasoning can be formalized into formal theories, in which the deduction follows the rules of classical logic if all predicates are crisp, or fuzzy logic if any of them are fuzzy. The mathematics of structures involving fuzziness can thus assume the form of formal theories over fuzzy logic, rather than the study of membership functions which uses classical logic. The former way is to be preferred as a genuinely fuzzy approach, while the latter is only a secondary classical model of fuzziness.

Admittedly, a formal theory over fuzzy logic is just a notational abbreviation of classical reasoning about the class of all models of the theory. Nonetheless, the axiomatic method is the general paradigm of mathematics; one of its main advantages is that the appropriate choice of the language of the formal theory screens off irrelevant features of the models. An axiomatic system is thus not only the means of generalization over all models, but rather an abstraction to their constitutive features.

Obviously, the formalistic imperative applies mainly to the development of mathematical fuzzy logic and various branches of fuzzy mathematics, not to particular applications of fuzzy sets. In an application, we are modeling particular phenomena and thus we naturally work with a particular model. For instance, some real-life problems (e.g., processing of a questionnaire with five grades between absolute yes and absolute no) may invite a definite algebra of truth values. However, having a general theory may of course help even in particular cases, since it will describe the general features of the problem. The programme of developing fuzzy mathematics in a theoretical manner stresses the priority of general theories over immediate applicational needs.

The idea that fuzzy inference cannot be reduced to a particular model able to account for its rules entails that in the investigation of fuzzy inference we should not limit ourselves to one particular fuzzy logic (e.g., Łukasiewicz). The model which underlies it—e.g., a specific t-norm—is particular, while fuzzy reasoning in general is broader. There are examples of fuzzy reasoning that follow variant inference rules, all of which are suitable for different respective contexts of real-life situations and invite explanations in terms of various individual t-norms or other semantics. The multitude of existing fuzzy logics varying both in expressive power and inference rules is not only explicable by the need of capturing of all aspects of fuzzy inference in diverse contexts, but even indispensable for this enterprise.

Similar considerations are related to Hájek's preference for fuzzy logics without truth constants in the language (except for those which are definable). First, the truth constants have little support in natural language. Second, by incorporating the truth values into the syntax, we force the logic to follow too closely a particular model of vague inference, viz. that using truth values. Of course, we cannot be too dogmatic about rejecting truth constants: it turns out that in sufficiently strong theories, at least rational truth constants are definable. Sometimes, the truth degrees are useful for a particular application. However, we should be cautious of deliberately introducing them into logic and thus restricting the possible models of vague inference.

Thus, even though liberal in both the expressive power and inference rules, we—following Hájek—believe a certain style of logical systems to be a most suitable formalism for representing fuzzy inference. For brevity's sake, in what follows we shall call them *Hájek-style fuzzy logics*. Put in a nutshell, they are fuzzy logics retaining the syntax of classical logic (preferably without truth constants), defined as axiomatic systems (rather than non-axiomatizable sets of tautologies). A prototypical example is Hájek's Basic Logic BL, propositional or predicate. This certainly does not mean that other systems (for instance, with some kind of labelled formulae) have no merits of their own; only they are not preferred for the development of fuzzy mathematics by the Prague school. In the following paragraphs we give some reasons for such preferences.

There is a pragmatic motive for retaining as much of classical syntax as possible. The way of working in theories over Hájek-style fuzzy logics resembles closely the way of working in classical logic: Hájek-style fuzzy logics are often just weaker variants of Boolean logic—syntactically fully analogous, just lacking some of its laws. Therefore, many theoretical and metatheoretical methods developed for classical logic can be mimicked and employed, resulting in a quick and sound development of the theory. This feature has already been utilized in the metamathematics of fuzzy logic—the proofs of the completeness, deduction, and other metatheorems have often been obtained by adjustments of classical proofs.

To illustrate the utility of this guideline, we allege that an axiomatic theory of fuzzy sets can more easily be developed as a formal theory of binary membership predicate over some fuzzy logic than if the graded membership is rendered, e.g., as a ternary predicate between elements, sets, and truth values in the framework of classical logic. Many constructions and even proofs of the classical theory will work in the former case and need not be rediscovered (nor even reformulated). Even though both theories may turn out equivalent, the resemblance of fuzzy concepts to classical ones becomes more visible in Hájek's approach: cf. the many 'breakthrough' definitions of fuzzy set inclusion which, if put down in Hájek-style fuzzy logic have exactly the form of the classical definition of set inclusion. This is another reason for preferring the classical syntax in fuzzy logic, over non-standard logical systems.

The imperative to work deductively in a formal theory explains also our preference for axiomatic systems over non-axiomatizable sets of tautologies. The infeasibility of algorithmical recognition of valid inferences in the latter is a strong reason supporting the preference. Thus, e.g., predicate fuzzy logics are better conceived as the systems of axioms and rules for quantifiers than the sets of valid $[0, 1]$-tautologies, even though the former usually admit non-intended models.

The respect for the priority of formal theories to models can partly be seen as emphasizing the syntax against the semantics of fuzzy logic. Hájek's approach thus can be viewed as a *syntactic turn* in fuzzy logic. The accent on syntax is of course not meant to contest the fundamental rôle of semantics in logic, nor the heuristic value of the models. Nevertheless, playing up the importance of formal deduction in fuzzy logic corresponds to its motivation as a description of the rules of correct reasoning under vagueness.

Such, then, is a reconstruction of the methodological background we adopt. It has already proved worthy in the area of metamathematics of fuzzy logic. Thus it seems reasonable to apply its doctrines to other branches of fuzzy mathematics as well.

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The need for axiomatization of further areas of fuzzy mathematics besides fuzzy logic is beyond doubt. Axiomatization has always aided the development of mathematical theories. There have been many—more or less successful—attempts to formalize or even axiomatize some areas of fuzzy mathematics. However, these axiomatics are usually designed ad hoc: some concepts in a classical theory are turned fuzzy, however their selection is based on non-systematic intuitions or intended applications; seldom all is fuzzified that could be. (To fuzzify as much as possible is desirable for generality's sake; if an application requires some features to be crisp, they can be 'defuzzified' by an additional assumption of the crispness of these particular features.) Many of these axiomatics are in fact semi-classical, being founded upon the notions of truth degrees and membership functions, which are merely a classical rendering of fuzzy sets.

Further problems of contemporary fuzzy mathematics lie in its fragmentation. Even if some axiomatic theories of various parts of fuzzy mathematics exist, they use completely different sets of primitive concepts and incompatible formalisms. This makes it virtually impossible to combine any two of them into one broader theory. It would certainly be better if fuzzy mathematics as a whole could employ a unified methodology in building its axiomatic theories, because it would facilitate the exchange of results between its branches. Applying the doctrines sketched above, we propose such a unified methodology for the axiomatization of fuzzy mathematics. Obviously, in our approach it assumes the form of constructing formal theories over Hájek-style fuzzy logics.

In the axiomatic construction of classical mathematic, a three-layer architecture has proved useful, with the layers of logic, foundations, and only then individual mathematical disciplines. Individual disciplines are thus developed within the framework of a unifying formal theory, be it some variant of set theory, type theory, category theory, or another sufficiently rich and general kind of theory. In fuzzy mathematics, the level of logic seems to be developed far enough so as to support sufficiently strong formal theories. The search for a suitable foundational theory is thus the task of the day. As hinted above, the close analogy between Hájek-style fuzzy logics and classical logic gives rise to a hope that fuzzy analogues of classical foundational theories will be able to harbour all (or at least nearly all) parts of existing fuzzy mathematics.

As conceivable candidates for a foundational theory, several ZF-style fuzzy set theories have already arisen. Many of them are certainly capable of doing the job. Nevertheless, they seem to be more complex than necessary for the task. Largely this is induced by the fact that such theories have to deal with a specific set-theoretical agenda and take into account the structure of the whole set universe (expressed, e.g., by the axiom of well-foundedness). Moreover, for many of them it is not clear whether they can straightforwardly be generalized to other fuzzy logics than the one in which they have been developed; thus they are only capable of providing the foundation for a limited part of fuzzy mathematics. Besides the repertoire of ZF-style set theories, fuzzy logic also offers set theories based on naïve comprehension. Although their axiomatic system is very elegant, their consistency is limited to (certain) fuzzy logics where no bivalent operator is definable (roughly speaking, to infinite-valued Łukasiewicz logic or weaker).

If nevertheless a universal foundational theory is successfully found, the development of individual concepts of fuzzy mathematics has to proceed in a systematic way, taking into the account the dependencies between them as in classical mathematics. For example, the notion of cardinality should only be defined after the introduction and investigation of the notion of function, upon which it is based (and which in turn is based upon the concept of fuzzy equality, i.e., similarity). When more than one counterpart of a classical definition suggest themselves, the choice between them should be made according to their fruitfulness, applicability, and the practice of traditional fuzzy mathematics; in many cases more than one analogue of the classical notion will have to be introduced and studied within the theory. Defined notions should also be checked against conformity with category theory; for instance, a proposed definition of Cartesian product should accord with that of mapping (one must, however, take into account that many natural notions of morphism become fuzzy under fuzzy logic).

Only this kind of systematic approach can avoid giving ad hoc definitions of fuzzy concepts, which often suffer from arbitrariness and hidden crispness, or even references to particular crisp models of fuzziness (e.g. membership functions) which are not objects of the formal theory.

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As a concrete implementation of the general programme sketched above we propose a specific foundational theory described below. We do not claim it to be the only possible way either of doing the foundations of fuzzy mathematics, or of fulfilling our foundational programme. The methodology itself is independent of this particular solution we propose. Nevertheless, we think that our theory embodies its guidelines very well and is a viable foundation for fuzzy mathematics of the present day. Moreover, because of the simplicity of its apparatus, the work done within its framework can be of use for other possible systems via a formal interpretation.

By inspecting the existing approaches and having in mind the need for generality and simplicity, it becomes obvious that a fully fledged set theory is not necessary for the foundations of fuzzy mathematics. What is necessary is only the ability to perform within the theory the basic constructions of fuzzy mathematics. On the other hand, a great variability of the backround fuzzy logic is required in order to encompass the whole of fuzzy mathematics.

Most notions of classical mathematics can be defined within the first few levels of a simple type theory. The similarity between Hájek-style and classical theories hints that this could be true of fuzzy concepts defined in a fuzzified simple type theory as well. Indeed, many important notions can be defined already at the first level, which is in fact second-order predicate fuzzy logic. Most notably, elementary fuzzy set theory, or the axiomatization of *Zadeh's notion of fuzzy set,* is contained in second-order fuzzy logic (second-order models are exactly Zadeh's universes of fuzzy sets). Some theories (e.g., topology), however, need more levels of type hierarchy, thus we employ higher-order fuzzy logic (in the limit, logic of order ω).

Unfortunately, fuzzy higher-order logic is not recursively axiomatizable. Since we prefer axiomatic deductive theories over non-axiomatizable sets of tautologies, we choose its Henkin-style variant, even though it admits non-intended models. We thus get a *first-order theory,* axiomatized very naturally by the extensionality and comprehension axioms for each order. Moreover, the construction works for virtually all imaginable fuzzy logics (and many non-fuzzy logics as well). The bunch of foundational theories we propose thus can be called *Henkin-style higher-order fuzzy logic* (for an individual fuzzy logic of one's choice; expressively rich logics like ŁΠ seem to be sufficient for all practical purposes; nevertheless, the investigation of the fragments over weaker logics has also its own importace). Equipping the theory with the obvious axioms of tuples yields an apparatus which seems to be of enough expressive power for a great part of fuzzy mathematics, since a structure on the universe of discourse (metric, measure, etc.) can then be introduced by means of relations and higher-order predicates. Furthermore, if the background logic is sufficiently strong, there is a general method of embedding any classical theory, and even of its natural fuzzification (as well as conscious and controlled 'defuzzification' of its concepts if some of their features are to be left crisp). The details of this formalism can be found in [1].

As indicated above, elementary fuzzy set theory and some parts of the theory of fuzzy relations are already formalized within our foundational theory. Several other parts of fuzzy mathematics are currently (re-)developed in our formalism. However, the reconstruction (and expected further advance) of the whole of fuzzy mathematics is an infinite task. Everybody is therefore invited to participate in this research programme of systematic formal development of fuzzy mathematics, as well as to continue the discussion of its best foundation.

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Acknowledgements. As the reader could easily observe, this methodological programme has close links to the works of many predecessors, and in fact only applies their accomplishments to the area of fuzzy mathematics.

Our formalistic approach to mathematics is close to that of Hilbert's [6]. Our aspiration to lay down the logical foundations for fuzzy mathematics is only a derivative of the admirable enterprise of Russell and Whitehead [8]. In some (and only some) respects our programme is similar to that of Bourbaki [2], though we hope not only to reconstruct and codify, but also advance the field of our interest. The link to Vienna circle [3] which results from the circumstances of the first presentation of this manifesto is rather incidental (though in some aspects one could perhaps find distant parallels).

We cannot mention all the outstanding works of fuzzy logic upon which our contribution is based. Here we mention only the most important works relevant to our approach; further citations can be found in [1]. Apparently the first monograph close in spirit to our programme was Gottwald's [4]. Hájek's [5] gave firm foundations to the kind of formal fuzzy logic we use. A great influence in the propagation of rigorous fuzzy logic in the Czech mathematical community had Novák's book [7]. And, needless to say, the whole field of fuzzy mathematics we try to formalize originated with Zadeh's [9].

References

[1] L. Běhounek and P. Cintula, Fuzzy class theory, Fuzzy Sets and Systems 154 (2005) 34–55.

[2] N. Bourbaki, Théorie des Ensembles, volume 1 of Éléments de Mathématique, Hermann, Paris, 1966.

[3] R. Carnap, H. Hahn, and O. Neurath, Wissenschaftliche Weltauffassung. Der Wiener Kreis, in: R. Hegselmann, editor, Wissenschaftliche Weltauffassung,

Sozialismus und Logischer Empirismus, Frankfurt, 1929, 81–101.

[4] S. Gottwald, Fuzzy Sets and Fuzzy Logic: Foundations of Application—from a Mathematical Point of View, Vieweg, Wiesbaden, 1993.

[5] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer, Dordercht, 1998.

[6] D. Hilbert, Grundlagen der Geometrie, in: Festschrift zur Feier der Enthüllung des Gauss-Weber Denkmals, B.G. Teubner, Leipzig, 1899, 1–92.

[7] V. Novák, Fuzzy množiny a jejich aplikace, SNTL, Prague, 1986. (English version: V. Novák, Fuzzy Sets and Their Applications, Adam Hilger, Bristol, 1989.)

[8] A.N. Whitehead and B. Russell, Principia Mathematica, Cambridge University Press, Cambridge, 1910, 1911, 1913.

[9] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.