Fuzzy Class Theory

Libor Běhounek [a,1] Petr Cintula [a,2,*]

[a] *Institute of Computer Science, Academy of Sciences of the Czech Republic*
Pod Vodárenskou věží 2, 182 07 Prague 8, Czech Republic
{behounek, cintula}@cs.cas.cz

Abstract

The paper introduces a simple, yet powerful axiomatization of Zadeh's notion of fuzzy set, based on formal fuzzy logic. The presented formalism is strong enough to serve as foundations of a large part of fuzzy mathematics. Its essence is elementary fuzzy set theory, cast as two-sorted first-order theory over fuzzy logic, which is generalized to simple type theory. We show a reduction of the elementary fuzzy set theory to fuzzy propositional calculus and a general method of fuzzification of classical mathematical theories within this formalism. In this paper we restrict ourselves to set relations and operations that are definable without any structure on the universe of objects presupposed; however, we also demonstrate how to add structure to the universe of discourse within our framework.

Key words: Formal fuzzy logic, fuzzy set, foundations of fuzzy mathematics, ŁΠ logic, higher-order fuzzy logic, fuzzy type theory, multi-sorted fuzzy logic
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1 Introduction

Fuzzy sets were introduced approximately 40 years ago by L.A. Zadeh [17]. During these years the notion of fuzziness spread to nearly all aspects of mathematics (fuzzy relations, fuzzy topology, fuzzy algebra etc.). There have been many (more or less successful) attempts to formalize or even axiomatize some areas of fuzzy mathematics. Very successful results were achieved especially in the area of fuzzy logic (in narrow sense). The work of Hájek, Gottwald,

* Coresponding author
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Mundici, and others established fuzzy logic as a formal theory. This success allows us to move further with the formalization of other parts of fuzzy mathematics.

Although fuzzy mathematics is nowadays very broad, the notion of fuzzy set is still a central concept. There have been several previous attempts at formalizing fuzzy sets in an axiomatic way. Early works, most notably [3] and [4], axiomatized the notion within classical logic by means of a *ternary* membership predicate, whose third argument represented the membership degree. Even though we do not follow this approach here, our motivation for the axiomatic method conforms with that of [3, 623–4]:

> "This unified theory in which sets, functions, etc. all are 'fuzzy' helps to obviate some of the [...] difficulties and to clarify the nature of the others. Further, it eliminates the necessity of having a predetermined theory of ordinary sets on top of which the 'fuzzy' sets are built as a superstructure by starting out axiomatically *ab initio,* as it were, assuming only elementary logic. Further, by developing the theory in a manner parallel to the usual development of other set theories, comparisons between this new theory and the more usual ones are facilitated."

The approach we adopt here consists in 'hiding' the third argument in the semantic meta-level of the theory and using formal fuzzy logic instead of classical logic for the background logic of the theory. The reasons for this design choice are explained in more detail in [1]. Here it suffices to say that it allows us to draw on the similarity with classical set theory even more extensively than the former approach, as the formulae of the theory become virtually the same as in classical mathematics, only governed by a weaker logic. (See also footnote 8 in Section 7 below.) Axiomatic fuzzy set theories construed in this way have already been explored by several predecessors; however, their agenda differs from ours in many respects. The papers [11] and [15] are mainly interested in metamathematical properties of fuzzified Zermelo-Fraenkel set theory, rather than developing fuzzy mathematics within its framework. The elegant theory of [16] is restricted to one particular t-norm logic, and so it cannot capture the general notion of fuzzy set. Inspecting these approaches we came to two conclusions: for the axiomatization of Zadeh's notion of fuzzy set, we *do not need* an analogue of full-fledged set theory, though we *do need* an expressively rich fuzzy logic as a logical background.

By an *analogue of full-fledged set theory* we mean a theory over fuzzy logic, which contains fuzzy counterparts of all concepts of classical set theory. We observed that real-world applications of fuzzy sets need only a small portion of set-theoretical concepts. The central notion in fuzzy sets is the membership of elements (rather than fuzzy sets) into a fuzzy set. In the classical setting, the theory of the membership of atomic objects into sets is called *elementary set* *theory,* or *class theory.* It is a theory with two sorts of individuals—objects and classes—and one binary predicate—the membership of objects into classes. In this paper we develop a *fuzzy class theory.* The classes in our theory correspond exactly to Zadeh's fuzzy sets.

By an *expressively rich logic* (which we need) we mean a logic of great expressive power, yet with a simple axiomatic system and good logical properties (deduction theorem, Skolem function introduction and eliminability, etc.). $\text{Ł}\Pi\forall$ seems to be the most suitable logic for our needs. In this paper we developed fuzzy class theory over the first-order logic ŁΠ, however if you examine the definitions and theorems you notice that nearly all of them will work in other fuzzy logics as well. We think that fixing the underlying logic will make important class-theoretical concepts clearer. Fuzzy class theory for a wider class of fuzzy logics can be a topic of some upcoming paper.

We show that the proposed theory is a simple, yet powerful formalism for working with elementary relations and operations on fuzzy sets (normality, equality, subsethood, union, intersection, kernel, support, etc.). By a small enhancement of our theory (adding tools to manage tuples of objects) we obtain a formalism powerful enough to capture the notion of fuzzy relation. Thus we can formally introduce the notions of T-transitivity, T-similarity, fuzzy ordering, and many other concepts defined in the literature. Finally, we extend our formalism to something which can be viewed as simple fuzzy type theory. Basically, we introduce individuals for classes of classes, classes of classes of classes etc. This allows us to formalize other parts of fuzzy mathematics (e.g., fuzzy topology). Our theory thus aspires to the status of *foundations of fuzzy mathematics* and a uniform formalism that can make interaction of various disciplines of fuzzy mathematics possible.

Of course, this paper cannot cover all the topics mentioned above. For the majority of them we only give the very basic definitions, and there is a lot of work to be done to show that the proposed formalism is suitable for them. We concentrate on the development of basic properties of fuzzy sets. In this area our formalism proved itself worthy, as it allows us to state several very general metatheorems that effectively reduce a wide range of theorems on fuzzy sets to fuzzy propositional calculus. This success is a promising sign for our formalism to be suitable for other parts of fuzzy mathematics as well.

As mentioned above, in this paper we restrict ourselves to notions that can be defined without adding a structure (similarity, metrics, etc.) to the universe of objects. Nevertheless, our formalism possesses means for adding a structure to the universe (usually by fixing a suitable class which satisfies certain axioms), which is necessary for the development of more advanced parts of fuzzy set theory. Such extensions of our theory will be elaborated in subsequent papers, for some hints see Section 6.

The proposed methodology of formal fuzzy mathematics is described in more details in our paper [1], in which we also make further references to related results. Let us just mention here that some of the roots of our approach (as well as some of the concepts we employ, like graded properties of fuzzy relations) can already be found in Gottwald's monograph [9]. The systematic way of defining fuzzy notions (see Section 7) is hinted at already in Höhle's 1987 paper [12].

2 Preliminaries

This section contains formal tools necessary for developing a theory over the multi-sorted first-order logic ŁΠ. Readers acquainted with classical multi-sorted calculi can go through this section quickly.

2.1 Propositional logic ŁΠ

Here we recall the definitions of the logic ŁΠ and some of its properties (the definition and theorems in this section are from [8] and [6]).

Definition 1 *The logic* ŁΠ *has the following* basic connectives *(they are listed together with their standard semantics in* $[0,1]$; *we use the same symbols for logical connectives and the corresponding algebraic operations):*

0	0	*truth constant falsum*
$\varphi \rightarrow_{\text{Ł}} \psi$	$x \rightarrow_{\text{Ł}} y = \min(1, 1 - x + y)$	*Łukasiewicz implication*
$\varphi \rightarrow_{\Pi} \psi$	$x \rightarrow_{\Pi} y = \min(1, \frac{x}{y})$	*product implication*
$\varphi \mathbin{\&_{\Pi}} \psi$	$x \mathbin{\&_{\Pi}} y = x \cdot y$	*product conjunction*

The logic $\text{Ł}\Pi\frac{1}{2}$ *has one additional truth constant* $\frac{1}{2}$ *with the standard semantics* $\frac{1}{2}$. *We define the following* derived connectives:

$\neg_{\text{L}}\varphi$	*is*	$\varphi \rightarrow_{\text{L}} 0$	$\neg_{\text{L}} x = 1 - x$
$\neg_{\Pi}\varphi$		$\varphi \rightarrow_{\Pi} 0$	$\neg_{\Pi} x = 1$ *if* $x = 0$, *otherwise* 0
1		$\neg_{\text{L}} 0$	1
$\Delta\varphi$		$\neg_{\Pi}\neg_{\text{L}}\varphi$	$\Delta x = 1$ *if* $x = 1$, *otherwise* 0
$\varphi \,\&_{\text{L}}\, \psi$		$\neg_{\text{L}}(\varphi \rightarrow_{\text{L}} \neg_{\text{L}}\psi)$	$x \,\&_{\text{L}}\, y = \max(0, x + y - 1)$
$\varphi \oplus \psi$		$\neg_{\text{L}}\varphi \rightarrow_{\text{L}} \psi$	$x \oplus y = \min(1, x + y)$
$\varphi \ominus \psi$		$\varphi \,\&_{\text{L}}\, \neg_{\text{L}}\psi$	$x \ominus y = \max(0, x - y)$
$\varphi \wedge \psi$		$\varphi \,\&_{\text{L}}\, (\varphi \rightarrow_{\text{L}} \psi)$	$x \wedge y = \min(x, y)$
$\varphi \vee \psi$		$(\varphi \rightarrow_{\text{L}} \psi) \rightarrow_{\text{L}} \psi$	$x \vee y = \max(x, y)$
$\varphi \rightarrow_{\text{G}} \psi$		$\Delta(\varphi \rightarrow_{\text{L}} \psi) \vee \psi$	$x \rightarrow_{\text{G}} y = 1$ *if* $x \leq y$, *otherwise* y

We assume the usual precedence of connectives. Occasionally we may write $\neg_{\text{G}}$ *and* $\&_{\text{G}}$ *as synonyms for* $\neg_{\Pi}$ *and* $\wedge$, *respectively. We further abbreviate* $(\varphi \rightarrow_{*} \psi) \,\&_{*}\, (\psi \rightarrow_{*} \varphi)$ *by* $\varphi \leftrightarrow_{*} \psi$ *for* $* \in \{\text{G}, \text{L}, \Pi\}$.

Definition 2 *An* ŁΠ-algebra *is a structure* $\mathbf{L} = (L, \oplus, \neg_{\text{L}}, \rightarrow_{\Pi}, \&_{\Pi}, 0, 1)$ *such that:*

- $(L, \oplus, \neg_{\text{L}}, 0)$ *is an MV-algebra*
- $(L, \vee, \wedge, \rightarrow_{\Pi}, \&_{\Pi}, 0, 1)$ *is a* Π*-algebra,*
- $x \,\&_{\Pi}\, (y \ominus z) = (x \,\&_{\Pi}\, y) \ominus (x \,\&_{\Pi}\, z)$.

Furthermore, a structure $\mathbf{L} = (L, \oplus, \neg_{\text{L}}, \rightarrow_{\Pi}, \&_{\Pi}, 0, 1, \frac{1}{2})$ *where the reduct* $\mathbf{L}' = (L, \oplus, \neg_{\text{L}}, \rightarrow_{\Pi}, \&_{\Pi}, 0, 1)$ *is an* ŁΠ*-algebra and the identity* $\frac{1}{2} = \neg_{\text{L}}\frac{1}{2}$ *holds is called an* ŁΠ$\frac{1}{2}$-algebra.

The standard ŁΠ-algebra $[\mathbf{0}, \mathbf{1}]$ *has the domain* $[0, 1]$ *and the operations as stated in Definition 1 above (analogously for the standard* ŁΠ$\frac{1}{2}$*-algebra).*

The two-valued ŁΠ *algebra is denoted by* $\{\mathbf{0}, \mathbf{1}\}$.

Definition 3 *The logic* ŁΠ *is given by the following axioms and deduction rules:*

(Ł) *The axioms of Łukasiewicz logic*

(Π) *The axioms of product logic*

(ŁΔ) $\Delta(\varphi \rightarrow_{\mathrm{L}} \psi) \rightarrow_{\mathrm{L}} (\varphi \rightarrow_{\Pi} \psi)$

(ΠΔ) $\Delta(\varphi \rightarrow_{\Pi} \psi) \rightarrow_{\mathrm{L}} (\varphi \rightarrow_{\mathrm{L}} \psi)$

(Dist) $\varphi \,\&_{\Pi}\, (\chi \ominus \psi) \leftrightarrow_{\mathrm{L}} (\varphi \,\&_{\Pi}\, \chi) \ominus (\varphi \,\&_{\Pi}\, \psi)$

The deduction rules are modus ponens and Δ*-necessitation (from* φ *infer* $\Delta\varphi$*).*

The logic ŁΠ$\frac{1}{2}$ *results from* ŁΠ *by adding the axiom* $\frac{1}{2} \leftrightarrow \neg_{\mathrm{L}}\frac{1}{2}$.

The notions of proof, derivability $\vdash$*, theorem, and theory over* ŁΠ *and* ŁΠ$\frac{1}{2}$ *are defined as usual.*

Theorem 4 (Completeness) *Let* φ *be a formula of* ŁΠ *(*ŁΠ$\frac{1}{2}$ *respectively). Then the following conditions are equivalent:*

- φ *is a theorem of* ŁΠ*(*ŁΠ$\frac{1}{2}$ *resp.)*
- φ *is an* **L***-tautology w.r.t. each* ŁΠ*-algebra(*ŁΠ$\frac{1}{2}$*-algebra resp.)* **L**
- φ *is a* $[\mathbf{0}, \mathbf{1}]$*-tautology.*

The following definitions and theorems demonstrate the expressive power of ŁΠ and ŁΠ$\frac{1}{2}$. Particularly, Corollary 8 shows that each propositional logic based on an arbitrary t-norm of a certain simple form is contained in ŁΠ$\frac{1}{2}$.

Definition 5 *A function* $f : [0, 1]^n \rightarrow [0, 1]$ *is called a* rational ŁΠ-function *iff there is a finite partition of* $[0, 1]^n$ *such that each block of the partition is a semi-algebraic set and* f *restricted to each block is a fraction of two polynomials with rational coefficients.*

Furthermore, a rational ŁΠ*-function* f *is* integral *iff all the coefficients are integer and* $f(\{0, 1\}^n) \subseteq \{0, 1\}$.

Definition 6 *Let* f *be a function* $f : [0, 1]^n \rightarrow [0, 1]$ *and* $\varphi(v_1, \ldots, v_n)$ *be a formula. We say that the function* f *is* represented by the formula φ *(*φ *is a representation of* f*) iff* $e(\varphi) = f(e(v_1), e(v_2), \ldots, e(v_m))$ *for each evaluation* e.

The following theorem was proved in [13]:

Theorem 7 (Functional representation) *A function* f *is an integral (rational respectively)* ŁΠ *function iff it is represented by some formula of* ŁΠ *(*ŁΠ$\frac{1}{2}$ *resp.).*

The following theorem was proved in [5], but it can be viewed as a corollary of the previous theorem.

Corollary 8 *Let $*$ be a continuous t-norm which is a finite ordinal sum of the three basic ones (i.e., of* G*,* L*, and* Π*), and $\Rightarrow$ be its residuum. Then there are derived connectives $\&_*$ and $\to_*$ of the* $\text{Ł}\Pi\frac{1}{2}$ *logic such that their standard* $[\mathbf{0},\mathbf{1}]$*-semantics are $*$ and $\Rightarrow$ respectively. The logic $PC(*)$ of the t-norm $*$ (see [10]) is contained in* $\text{Ł}\Pi\frac{1}{2}$ *if $\&$ and $\to$ of $PC(*)$ are interpreted as $\&_*$ and $\to_*$. Furthermore, if φ is provable in $PC(*)$ (and a fortiori, if it is provable in Hájek's logic* $\text{BL}\Delta$*, see [10]), then the formula φ_* obtained from φ by replacing the connectives $\&$ and $\to$ of $PC(*)$ (or* $\text{BL}\Delta$*) by $\&_*$ and $\to_*$ is provable in* $\text{Ł}\Pi\frac{1}{2}$.

Convention 9 *Further on, the signs $*$ and $\diamond$ will be reserved for t-norms definable in* $\text{Ł}\Pi\frac{1}{2}$ *(incl.* G*,* Ł *and* Π*), and the indexed connectives will always have the meaning introduced in the previous Corollary. However, we omit the indices of connectives whenever they are irrelevant, i.e., whenever all formulae obtained by subscripting any $*$ to such a connective are provably equivalent (for example, $\neg\neg_{\mathrm{G}}\varphi$, $\Delta(\varphi \to \psi)$, etc.), or equivalently provable (e.g., the principal implication in axioms and theorems).*

Corollary 10 *Let $r \in [0,1]$ be a rational number; then there is a formula φ of* $\text{Ł}\Pi\frac{1}{2}$ *such that $e(\varphi) = r$ for any* $[\mathbf{0},\mathbf{1}]$*-evaluation e.*

This corollary tells us that in $\text{Ł}\Pi\frac{1}{2}$ we have a truth constant $\bar{r}$ for each rational number $r \in [0,1]$. Using the completeness theorem we get the following corollary.

Corollary 11 *The following are theorems of the* $\text{Ł}\Pi\frac{1}{2}$ *logic:*

$$\overline{r \,\&_\Pi\, s} = \overline{r} \,\&_\Pi\, \overline{s}$$
$$\overline{r \to_\Pi s} = \overline{r} \to_\Pi \overline{s}$$
$$\overline{r \to_{\mathrm{L}} s} = \overline{r} \to_{\mathrm{L}} \overline{s}$$

where the symbols $\&_\Pi, \to_\Pi, \to_{\mathrm{L}}$ on the left-hand side are operations in $[\mathbf{0},\mathbf{1}]$ *and on the right-hand side they are logical connectives.*

2.2 Multi-sorted first-order logic $\text{Ł}\Pi\forall$

In this section we deal with first-order versions of the logics ŁΠ and $\text{Ł}\Pi\frac{1}{2}$. Since the difference between $\text{Ł}\Pi\forall$ and $\text{Ł}\Pi\frac{1}{2}\forall$ is purely "propositional", we focus on the logic $\text{Ł}\Pi\forall$; the definitions and theorems for the logic $\text{Ł}\Pi\frac{1}{2}\forall$ are analogous, for details see [5]. (For general first-order fuzzy logics see [10] and for multi-sorted first order fuzzy logic see [7].)

Definition 12 *A* multi-sorted predicate language $\mathbf{\Gamma}$ *for the logic* Ł$\Pi\forall$ *is a quintuple* $(\mathbf{S},\ \preceq,\ \mathbf{P},\ \mathbf{F},\ \mathbf{A})$*, where* $\mathbf{S}$ *is a non-empty set of sorts,* $\preceq$ *is an ordering on* $\mathbf{S}$ *(indicating the subsumption of sorts),* $\mathbf{P}$ *is a non-empty set of predicate symbols,* $\mathbf{F}$ *is a set of function symbols, and* $\mathbf{A}$ *is a function assigning to each predicate and function symbol a finite sequence of elements of* $\mathbf{S}$*.*

Let $|\mathbf{A}(P)|$ *denote the length of the sequence* $\mathbf{A}(P)$*. The number* $|\mathbf{A}(P)|$ *is called the* arity of the predicate symbol P*. The number* $|\mathbf{A}(f)|-1$ *is called the* arity of the function symbol f*. The functions* f *for which* $\mathbf{A}(f) = <s>$ *are called the* individual constants *of sort* s*. If* $s_1 \preceq s_2$ *holds for sorts* s_1, s_2 *we say that* s_2 subsumes s_1*.*

The logical symbols *of* Ł$\Pi\forall$ *are individual variables* $x^s, y^s, \dots$ *for each sort* s*, the logical connectives of* ŁΠ*, the quantifier* $\forall$ *and the identity sign* $=$*. For any variable* x^s*, we abbreviate* $\neg_{\mathrm{L}}(\forall x^s)\neg_{\mathrm{L}}$ *as* $(\exists x^s)$*.*

Definition 13 *Let* $\mathbf{\Gamma} = (\mathbf{S},\ \preceq,\ \mathbf{P},\ \mathbf{F},\ \mathbf{A})$ *be a multisorted predicate language. The notion of* $\mathbf{\Gamma}$-term *is defined inductively as follows:*

- *Each individual variable of sort* $s \in \mathbf{S}$ *is a* $\mathbf{\Gamma}$*-term of sort* s*.*
- *Let* $t_1, \dots, t_n$ *be* $\mathbf{\Gamma}$*-terms of respective sorts* $s_1, \dots, s_n \in \mathbf{S}$*, and* f *be a function symbol of* $\mathbf{\Gamma}$ *such that* $\mathbf{A}(f) = <w_1, \dots, w_n, w_{n+1}>$*, where* $s_i \preceq w_i$ *for* $i \le n$*. Then* $f(t_1, \dots, t_n)$ *is a* $\mathbf{\Gamma}$*-term of sort* w_{n+1}*.*
- *Nothing else is a* $\mathbf{\Gamma}$*-term.*

Definition 14 *Let* $\mathbf{\Gamma} = (\mathbf{S},\ \preceq,\ \mathbf{P},\ \mathbf{F},\ \mathbf{A})$ *be a multisorted predicate language. Let* $t_1, \dots, t_n$ *be* $\mathbf{\Gamma}$*-terms of respective sorts* $s_1, \dots, s_n \in \mathbf{S}$*, and* P *be a predicate symbol of* $\mathbf{\Gamma}$ *such that* $\mathbf{A}(P) = <w_1, \dots, w_n>$ *and* $s_i \preceq w_i$ *for* $i \le n$*. Then* $P(t_1, \dots, t_n)$ *is an* atomic $\mathbf{\Gamma}$-formula*. If* t_1 *and* t_2 *are* $\mathbf{\Gamma}$*-terms of arbitrary sorts, then* $t_1 = t_2$ *is also an atomic* $\mathbf{\Gamma}$*-formula.*

The notion of $\mathbf{\Gamma}$-formula *is defined inductively as follows:*

- *Each atomic* $\mathbf{\Gamma}$*-formula is a* $\mathbf{\Gamma}$*-formula.*
- *If* $\varphi_1, \dots, \varphi_n$ *are* $\mathbf{\Gamma}$*-formulae and* c *is an* n*-ary propositional connective of* ŁΠ*, then* $c(\varphi_1, \dots, \varphi_n)$ *is also a* $\mathbf{\Gamma}$*-formula.*
- *Let* φ *be a* $\mathbf{\Gamma}$*-formula and* x^s *a variable of sort* s*. Then* $(\forall x^s)\varphi$ *is also a* $\mathbf{\Gamma}$*-formula.*
- *Nothing else is a* $\mathbf{\Gamma}$*-formula.*

Bounded *and* free variables *in a formula are defined as usual. A formula is called a* sentence *iff it contains no free variables. A set of* $\mathbf{\Gamma}$*-formulae is called a* $\mathbf{\Gamma}$-theory.

Convention 15 *Instead of* $\xi_1, \dots, \xi_n$ *(where* ξ_i*'s are terms or formulae and* n *is arbitrary or fixed by the context) we shall sometimes write just* $\vec{\xi}$*.*

Unless stated otherwise, the expression $\varphi(x_1, \ldots, x_n)$ means that all free variables of φ are among $x_1, \ldots, x_n$. Similarly, in propositional logic the expression $\varphi(p_1, \ldots, p_n)$ will mean that all propositional variables occurring in φ are among $p_1, \ldots, p_n$.

If $\varphi(x_1, \ldots, x_n, \vec{z})$ is a formula and we substitute terms t_i for all x_i's in φ, we denote the resulting formula in the context simply by $\varphi(t_1, \ldots, t_n, \vec{z})$.

The expression $(\exists! x)_* \varphi(x, \vec{z})$ abbreviates the formula

$$(\exists x, \vec{z})[\varphi(x, \vec{z}) \; \&_* \; (\forall y)(\varphi(y, \vec{z}) \to_* y = x)]$$

Definition 16 *A term t of sort w is* substitutable *for the individual variable x^s in a formula $\varphi(x^s, \vec{z})$ iff $w \preceq s$ and no occurrence of any variable y occurring in t is bounded in $\varphi(t, \vec{z})$.*

Definition 17 *Let $\mathbf{L}$ be a linearly ordered ŁΠ-algebra. An* $\mathbf{L}$-structure $\mathbf{M}$ *for $\mathbf{\Gamma}$ has the following form: $\mathbf{M} = ((M_s)_{s \in \mathbf{S}}, (P_{\mathbf{M}})_{P \in \mathbf{P}}, (f_{\mathbf{M}})_{f \in \mathbf{F}})$, where M_s is a non-empty domain for each $s \in \mathbf{S}$ and $M_s \subseteq M_w$ iff $s \preceq w$; $P_{\mathbf{M}}$ is an n-ary fuzzy relation $\prod_{i=1}^{n} M_{s_i} \to \mathbf{L}$ for each predicate symbol $P \in \mathbf{P}$ such that $\mathbf{A}(P) = <s_1, \ldots, s_n>$; $f_{\mathbf{M}}$ is a function $\prod_{i=1}^{n} M_{s_i} \to M_{s_{n+1}}$ for each function symbol $f \in \mathbf{F}$ such that $\mathbf{A}(f) = <s_1, \ldots, s_n, s_{n+1}>$, and an element of M_s if f is a constant of sort s.*

Definition 18 *Let $\mathbf{L}$ be a linearly ordered ŁΠ-algebra and $\mathbf{M}$ be an $\mathbf{L}$-structure for $\mathbf{\Gamma}$. An* $\mathbf{M}$-evaluation *is a mapping e which assigns to each variable of sort s an element from M_s (for all sorts $s \in \mathbf{S}$).*

Let e be an $\mathbf{M}$-evaluation, x a variable of sort s, and $a \in M_s$. Then $e[x \to a]$ is an $\mathbf{M}$-evaluation such that $e[x \to a](x) = a$ and $e[x \to a](y) = e(y)$ for each individual variable y different from x.

Definition 19 *Let $\mathbf{L}$ be a linearly ordered ŁΠ-algebra. The* value *of a term and the* truth value *of a $\mathbf{\Gamma}$-formula in an $\mathbf{L}$-structure $\mathbf{M}$ for $\mathbf{\Gamma}$ and an $\mathbf{M}$-evaluation e are defined as follows:*

$$\begin{aligned}
\|x\|^{\mathbf{L}}_{\mathbf{M},e} &= e(x) \\
\|f(t_1, t_2, \ldots, t_n)\|^{\mathbf{L}}_{\mathbf{M},e} &= f_{\mathbf{M}}(\|t_1\|^{\mathbf{L}}_{\mathbf{M},e}, \|t_2\|^{\mathbf{L}}_{\mathbf{M},e}, \ldots, \|t_n\|^{\mathbf{L}}_{\mathbf{M},e}) \\
\|P(t_1, t_2, \ldots, t_n)\|^{\mathbf{L}}_{\mathbf{M},e} &= P_{\mathbf{M}}(\|t_1\|^{\mathbf{L}}_{\mathbf{M},e}, \|t_2\|^{\mathbf{L}}_{\mathbf{M},e}, \ldots, \|t_n\|^{\mathbf{L}}_{\mathbf{M},e}) \\
\|t_1 = t_2\|^{\mathbf{L}}_{\mathbf{M},e} &= 1 \textit{ if } \|t_1\|^{\mathbf{L}}_{\mathbf{M},e} = \|t_2\|^{\mathbf{L}}_{\mathbf{M},e} \textit{ and } 0 \textit{ otherwise} \\
\|0\|^{\mathbf{L}}_{\mathbf{M},e} &= 0 \\
\|\varphi_1 \circ \varphi_2\|^{\mathbf{L}}_{\mathbf{M},e} &= \|\varphi_1\|^{\mathbf{L}}_{\mathbf{M},e} \circ \|\varphi_2\|^{\mathbf{L}}_{\mathbf{M},e} \textit{ for } \circ \in \{\to_{\text{Ł}}, \to_{\Pi}, \&_{\Pi}\} \\
\|(\forall x^s)\varphi\|^{\mathbf{L}}_{\mathbf{M},e} &= \inf_{a \in M_s} \|\varphi\|^{\mathbf{L}}_{\mathbf{M},e[x^s \to a]}
\end{aligned}$$

If the infimum does not exist, we take its value as undefined. We say that an $\mathbf{L}$*-structure* $\mathbf{M}$ *for* $\mathbf{\Gamma}$ *is* safe *iff* $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},e}$ *is defined for each* $\mathbf{\Gamma}$*-formula* φ *and each* $\mathbf{M}$*-evaluation* e*.*

Definition 20 *Let* $\mathbf{L}$ *be a linearly ordered* ŁΠ*-algebra and* φ *a* $\mathbf{\Gamma}$*-formula. The* truth value *of the formula* φ *in an* $\mathbf{L}$*-structure* $\mathbf{M}$ *for* $\mathbf{\Gamma}$ *is defined as follows:*

$$\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = \inf\{\|\varphi\|^{\mathbf{L}}_{\mathbf{M},e} \mid e \textit{ is an } \mathbf{M}\textit{-evaluation}\}$$

We say that φ *is an* $\mathbf{L}$-tautology *iff* $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1$ *for each safe* $\mathbf{L}$*-structure* $\mathbf{M}$ *for* $\mathbf{\Gamma}$*. We say that an* $\mathbf{L}$*-structure* $\mathbf{M}$ *for* $\mathbf{\Gamma}$ *is an* $\mathbf{L}$-model *of a* $\mathbf{\Gamma}$*-theory* T *iff* $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1$ *for each* $\varphi \in T$*.*

Convention 21 *For a fixed* $\mathbf{L}$*-model* $\mathbf{M}$ *and an* $\mathbf{M}$*-evaluation* e *such that* $e(x_i) = a_i$ *(for all i's), we shall instead of* $\|\varphi(x_1, \ldots, x_n)\|^{\mathbf{L}}_{\mathbf{M},e}$ *write simply* $\|\varphi(a_1, \ldots, a_n)\|$ *and speak of the truth value of* $\varphi(a_1, \ldots, a_n)$*.*

Definition 22 *Let* $\varphi({x_1}^{s_1}, \ldots, {x_n}^{s_n})$ *be a formula of* ŁΠ∀ *and* $\mathbf{M}$ *be a safe structure for the language of* φ *over an* ŁΠ*-algebra* $\mathbf{L}$*. The function* $\chi_\varphi : \prod_{i=1}^{n} M_{s_i} \to \mathbf{L}$ *such that* $\chi_\varphi(a_1, \ldots, a_n) = \|\varphi(a_1, \ldots, a_n)\|^{\mathbf{L}}_{\mathbf{M}}$ *is called the* characteristic function *of* $\varphi(x_1, \ldots, x_n)$*.*

Definition 23 *The logic* ŁΠ∀ *is given by the following axioms and deduction rules:*

(P) *Substitution instances of the axioms of propositional* ŁΠ

(∀1) $(\forall x)\varphi(x, \vec{z}) \to \varphi(t, \vec{z})$*, where* t *is substitutable for* x *in* φ

(∀2) $(\forall x)(\chi \to_{\text{L}} \varphi) \to (\chi \to_{\text{L}} (\forall x)\varphi)$*, where* x *is not free in* χ

(=1) $x = x$

(=2) $(x = y) \to \Delta(\varphi(x, \vec{z}) \leftrightarrow \varphi(y, \vec{z}))$.

The deduction rules are modus ponens, Δ*-necessitation, and generalization.*

The notions of proof, theorem, and derivability $\vdash$ are defined as usual.

Instead of axiom (=2) we may use the usual axioms of congruence of identity w.r.t. all predicates and functions plus the axiom of crispness of identity, i.e. $(x = y) \vee \neg(x = y)$.

Lemma 24 *The following are theorems of* ŁΠ∀*:*

- $(x = y) \vee \neg(x = y)$
- $(x = y) \to (y = x)$

- $(x = y) \mathbin{\&_*} (y = z) \to_* (x = z)$
- $(x_1 = y_1) \mathbin{\&_*} \ldots \mathbin{\&_*} (x_n = y_n) \to_* (\varphi(x_1, \ldots, x_n, \vec{z}) \leftrightarrow_* \varphi(y_1, \ldots, y_n, \vec{z}))$.

The theorems of the next lemma will be needed in the following sections.

Lemma 25 *All formulae of the following forms are provable in* ŁΠ∀*:*

$$
\begin{aligned}
&(\forall x)(\varphi \to_* \psi) \;\to\; [(\forall x)\varphi \to_* (\forall x)\psi] && (1)\\
&(\forall x)(\varphi \to_* \psi) \;\to\; [(\exists x)\varphi \to_* (\exists x)\psi] && (2)\\
&(\forall x)(\varphi \wedge \psi) \;\to\; [(\forall x)\varphi \wedge (\forall x)\psi] && (3)\\
&(\exists x)(\varphi \vee \psi) \;\to\; [(\exists x)\varphi \vee (\exists x)\psi] && (4)\\
&(\forall x)(\varphi_1 \mathbin{\&_*} \ldots \mathbin{\&_*} \varphi_k \to_* \chi) \;\to\\
&\qquad \to\; [(\forall x)\varphi_1 \mathbin{\&_*} \ldots \mathbin{\&_*} (\forall x)\varphi_k \to_* (\forall x)\chi] && (5)\\
&(\forall x)(\varphi_1 \mathbin{\&_*} \ldots \mathbin{\&_*} \varphi_k \to_* \chi) \;\to\\
&\qquad \to\; [(\forall x)\varphi_1 \mathbin{\&_*} \ldots \mathbin{\&_*} (\forall x)\varphi_{k-1} \mathbin{\&_*} (\exists x)\varphi_k \to_* (\exists x)\chi] && (6)
\end{aligned}
$$

Proof. In the proof we use an easy generalization of Corollary 8 to the predicative case. Parts (1)–(4) are provable in BL∀ (see [10]). Part (5) is proved by a trivial inductive generalization of the following proof in BL∀:

$$
\begin{aligned}
&(\forall x)(\varphi \mathbin{\&} \psi \to \chi)\\
&\leftrightarrow (\forall x)(\varphi \to (\psi \to \chi))\\
&\to [(\forall x)\varphi \to (\forall x)(\psi \to \chi)]\\
&\to [(\forall x)\varphi \to ((\forall x)\psi \to (\forall x)\chi)]\\
&\leftrightarrow [(\forall x)\varphi \mathbin{\&} (\forall x)\psi \to (\forall x)\chi].
\end{aligned}
$$

Finally, part (6) is proved in the same way, only applying (2) instead of (1) when distributing $(\forall x)$ over $(\varphi_k \to \chi)$. Q.E.D.

Theorem 26 (Deduction) *Let T be a theory and φ be a sentence. Then $T \vdash \Delta\varphi \to \psi$ iff $T \cup \{\varphi\} \vdash \psi$.*

Theorem 27 (Strong Completeness) *Let φ be a $\mathbf{\Gamma}$-formula, T a $\mathbf{\Gamma}$-theory. Then the following are equivalent:*

- $T \vdash \varphi$
- $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1$ *for each* ŁΠ*-algebra* **L** *and each safe* **L***-model* **M** *of* T
- $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1$ *for each* linearly ordered ŁΠ*-algebra* **L** *and each safe* **L***-model* **M** *of* T

The following theorem of [7] vindicates the introduction and elimination of function symbols. Notice the connective Δ, which is provably indispensable for the validity of this theorem.

Theorem 28 *Let* $\varphi(x_1^{s_1}, \ldots, x_n^{s_n}, y^s)$ *be a* $\mathbf{\Gamma}$*-formula and* T *be a theory such that* $T \vdash (\forall x_1^{s_1}) \ldots (\forall x_n^{s_n})(\exists y^s)\Delta\varphi(x_1^{s_1}, \ldots, x_n^{s_n}, y^s)$*. Let* f *be a new function symbol such that* $\mathbf{A}(f) = <s_1, \ldots, s_n, s>$*. Then the* $\mathbf{\Gamma} \cup \{f\}$*-theory* $T' = T \cup \{(\forall x_1^{s_1}) \ldots (\forall x_n^{s_n})\Delta\varphi(x_1^{s_1}, \ldots, x_n^{s_n}, f(x_1^{s_1}, \ldots, x_n^{s_n}))\}$ *is a conservative extension of* T*.*

Furthermore, if $T \vdash (\forall x_1^{s_1}) \ldots (\forall x_n^{s_n})(\exists! y^s)\Delta\varphi(x_1^{s_1}, \ldots, x_n^{s_n}, y^s)$ *then for each* $\mathbf{\Gamma} \cup \{f\}$*-formula* φ *there is a* $\mathbf{\Gamma}$*-formula* φ' *such that* $T' \vdash \varphi \leftrightarrow \varphi'$*.*

3 Class theory over ŁΠ

3.1 Axioms

Fuzzy class theory FCT is a theory over ŁΠ∀ with two sorts of variables: *object variables,* denoted by lowercase letters $x, y, \ldots$, and *class variables,* denoted by uppercase letters $X, Y, \ldots$ None of the sorts is subsumed by the other.

The only primitive symbol of FCT is the binary membership predicate $\in$ between objects and classes (i.e., the first argument must be an object and the second a class; class theory takes into consideration neither the membership of classes in classes, nor of objects in objects).

The principal axioms of FCT are instances of the class comprehension scheme: for any formula φ not containing X (it may, however, contain any other object or class parameters),

$$(\exists X)\Delta(\forall x)(x \in X \leftrightarrow \varphi(x))$$

is an axiom of FCT. The strange Δ is neccessary for securing that the required class exists in the degree 1 (rather than being only approximated by classes satisfying the equivalence in degrees arbitrarily close to 1). The Δ is also necessary for the conservativeness of the introduction of comprehension terms[3] $\{x \mid \varphi(x)\}$ with axioms

$$y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$$

and their eliminability. In the standard recursive way one proves that φ in comprehension terms may be allowed to contain other comprehension terms.

[3] I.e., the Skolem functions of comprehension axioms, see Theorem 28.

The consistency of FCT is proved by constructing a model. Let M be an arbitrary set and $\mathbf{L}$ be a complete linear ŁΠ-algebra. The *Zadeh model* $\mathbf{M}$ *over the universe* M *and the algebra of truth-values* $\mathbf{L}$ is constructed as follows:

The range of object variables is M, the range of class variables is the set of all functions from M to $\mathbf{L}$. For any evaluation e we define $\|x \in X\|^{\mathbf{L}}_{\mathbf{M},e}$ as the value of the function $e(X)$ on $e(x)$. The value of the comprehension term $\{x \mid \varphi(x)\}$ is defined as the function taking an object a to $\|\varphi(a)\|^{\mathbf{L}}_{\mathbf{M},e}$ (in fact, the characteristic function of $\varphi(x)$ where e fixes the parameters). Then it is trivial that $\|y \in \{x \mid \varphi(x)\}\|^{\mathbf{L}}_{\mathbf{M},e} = \|\varphi(y)\|^{\mathbf{L}}_{\mathbf{M},e}$ which proves the comprehension axiom.

If $\mathbf{L} = [\mathbf{0}, \mathbf{1}]$, we call the described model *standard*.

Definition 29 *Let* $\mathbf{M}$ *be a model and* A *a class in* $\mathbf{M}$. *The characteristic function* $\chi_{x \in A}$ *is denoted briefly by* χ_A *and also called the* membership function *of* A. *(Instead of* $\chi_A(x)$ *or* $\|x \in A\|$ *many papers use just* Ax.*)*

It can be observed that the crisp formula $(\forall x)\Delta(x \in X \leftrightarrow x \in Y)$ expresses the identity of the membership functions of X and Y (as in all models $\|(\forall x)\Delta(x \in X \leftrightarrow x \in Y)\| = 1$ iff the membership functions of X and Y are identical, otherwise 0). Since our intended notion of fuzzy class is extensional, i.e., that fuzzy classes are determined by their membership functions, it is reasonable to require the *axiom of extensionality* which identifies classes with their membership functions:

$$(\forall x)\Delta(x \in X \leftrightarrow x \in Y) \to X = Y$$

(the converse implication follows from the axioms for identity). The consistency of this axiom is proved by its validity in Zadeh models.

The comprehension scheme of FCT still allows classical models, as the construction of Zadeh models works for the ŁΠ-algebra $\{\mathbf{0}, \mathbf{1}\}$. Sometimes it may be desirable to exclude classical models. This can be done either by taking ŁΠ$\frac{1}{2}$ instead of ŁΠ as the underlying logic, or equivalently by adding two constants C, c and the *axiom of fuzziness* $c \in C \leftrightarrow \neg_{\mathrm{L}} c \in C$ without changing the underlying logic. In both cases there is a sentence with the value $\frac{1}{2}$ in any model, and all rational truth constants are therefore definable. The consistency of this extension follows from the fact that it holds in standard Zadeh models.

General models of FCT correspond in the obvious way to Henkin's general models of classical second-order logic, while Zadeh models correspond to full second-order models. FCT with its axioms of comprehension and extensionality thus can be viewed as a notational variant of the second-order fuzzy logic ŁΠ (monadic, in the form presented in this section; for higher arities see Section 4). Following the axiomatic method, we prefer FCT formulated in the Henkin style (as a two-sorted first-order theory, rather than a second-order logic) because of its axiomatizability. For even though (standard) Zadeh models are the intended models of FCT, the theory of Zadeh models is not arithmetically definable, let alone recursively axiomatizable. This follows from the obvious fact that classical full second-order logic (which itself is non-arithmetical) can be interpreted in the theory of Zadeh models by inscribing Δ (or $\neg\neg_{\mathrm{G}}$) in front of every atomic formula.

3.2 Elementary class operations

Elementary class operations are defined by means of propositional combination of atomic formulae of FCT.

Convention 30 *Let $\varphi(p_1, \ldots, p_n)$ be a propositional formula and $\psi_1, \ldots, \psi_n$ be any formulae. By $\varphi(\psi_1, \ldots, \psi_n)$ we denote the formula φ in which all occurrences of p_i are replaced by ψ_i (for all $i \leq n$).*

Definition 31 *Let $\varphi(p_1, \ldots, p_n)$ be a propositional formula. We define the* n-ary class operation induced by φ *as*

$$\mathrm{Op}_\varphi(X_1, \ldots, X_n) =_{\mathrm{df}} \{x \mid \varphi(x \in X_1, \ldots, x \in X_n)\} .$$

Among elementary class operations we find the following important kinds:

- *Class constants.* We denote Op_0 by $\emptyset$ and call it the *empty class,* and Op_1 by V and call it the *universal class.*
- *α-Cuts.* Let α be a truth-constant. Then we call the class $\mathrm{Op}_{\Delta(\alpha \to p)}(X)$, i.e., $\{x \mid \Delta(\alpha \to (x \in X))\}$, the *$\alpha$-cut* of X and abbreviate it X_α. Similarly, $\mathrm{Op}_{\Delta(\alpha \leftrightarrow p)}(X)$ is called the *α-level* of X, denoted by $X_{=\alpha}$.
- *Iterated complements,* i.e., class operations Op_φ where φ is p prefixed with a chain of negations. In ŁΠ, there are only a few such formulae that are non-equivalent. They yield the following operations (their definitions are summarized in Table 1): *involutive* and *strict complements*, the *kernel* and *support*, and the complement of the kernel. Except for the involutive complement, all of them are crisp.
- *Simple binary operations.* Some of the class operations $\mathrm{Op}_{p \circ q}$ where $\circ$ is a (primitive or derived) binary connective have their traditional names and notation, listed in Table 1 (not exhaustively).

Table 1
Elementary class operations

φ	$\mathrm{Op}_\varphi(X_1,\ldots,X_n)$	Name
0	$\emptyset$	empty class
1	V	universal class
$\Delta(\alpha \to p)$	X_α	α-cut
$\Delta(\alpha \leftrightarrow p)$	$X_{=\alpha}$	α-level
$\neg_{\mathrm{G}} p$	$\backslash X$	strict complement
$\neg_{\mathrm{L}} p$	$-X$	involutive complement
$\neg_{\mathrm{G}}\neg_{\mathrm{L}} p$ (or Δp)	$\mathrm{Ker}(X)$	kernel
$\neg\neg_{\mathrm{G}} p$ (or $\neg\Delta\neg_{\mathrm{L}} p$)	$\mathrm{Supp}(X)$	support
$p \mathbin{\&_*} q$	$X \cap_* Y$	$*$-intersection
$p \vee q$	$X \cup Y$	union
$p \oplus q$	$X \uplus Y$	strong union
$p \mathbin{\&} \neg_{\mathrm{G}} q$	$X \setminus Y$	strict difference
$p \mathbin{\&_*} \neg_{\mathrm{L}} q$	$X -_* Y$	involutive $*$-difference

3.3 Elementary relations between classes

Most of important relations between classes have one of the two forms described in the following definition:

Definition 32 (Uniform and supremal relations) *Let $\varphi(p_1,\ldots,p_n)$ be a propositional formula. The n-ary* uniform relation *between $X_1,\ldots,X_n$ induced by φ is defined as*

$$\mathrm{Rel}^{\forall}_{\varphi}(X_1,\ldots,X_n) \equiv_{\mathrm{df}} (\forall x)\varphi(x \in X_1,\ldots,x \in X_n).$$

The n-ary supremal relation *between $X_1,\ldots,X_n$ induced by φ is defined as*

$$\mathrm{Rel}^{\exists}_{\varphi}(X_1,\ldots,X_n) \equiv_{\mathrm{df}} (\exists x)\varphi(x \in X_1,\ldots,x \in X_n).$$

Among elementary class relations we find the following important kinds (they are summarized in Table 2):

- *Equalities* $\mathrm{Rel}^{\forall}_{p \leftrightarrow_* q}$ denoted $\approx_*$. The value of $X \approx_{\mathrm{G}} Y$ is the maximal truth degree below which the membership functions of X and Y are identical. In standard $[\mathbf{0},\mathbf{1}]$-models, $1 - \|X \approx_{\mathrm{L}} Y\|$ is the maximal difference of the

Table 2
Class properties and relations

Relation	Notation	Name
$\mathrm{Rel}^{\exists}_{p}(X)$	$\mathrm{Hgt}(X)$	height
$\mathrm{Rel}^{\exists}_{\Delta p}(X)$	$\mathrm{Norm}(X)$	normality
$\mathrm{Rel}^{\forall}_{\Delta(p \vee \neg p)}(X)$	$\mathrm{Crisp}(X)$	crispness
$\mathrm{Rel}^{\exists}_{\neg\Delta(p \vee \neg p)}(X)$	$\mathrm{Fuzzy}(X)$	fuzziness
$\mathrm{Rel}^{\forall}_{p \to_* q}(X, Y)$	$X \subseteq_* Y$	$*$-inclusion
$\mathrm{Rel}^{\forall}_{p \leftrightarrow_* q}(X, Y)$	$X \approx_* Y$	$*$-equality
$\mathrm{Rel}^{\exists}_{p \&_* q}(X, Y)$	$X \Vert_* Y$	$*$-compatibility

(values of) the membership functions of X and Y, and $\|X \approx_\Pi Y\|$ is the infimum of their ratios. All $\approx_*$ get value 1 iff the membership functions are identical. For crisp classes, these notions of equality coincide with classical equality.

- *Inclusions* $\mathrm{Rel}^{\forall}_{p \to_* q}$, denoted $\subseteq_*$. Their semantics is analogous to that of equalities. They get the value 1 iff the membership function of X is majorized by that of Y.
- *Compatibilities* $\mathrm{Rel}^{\exists}_{p \&_* q}$. Their strict and involutive negations may respectively be called *strict* and *involutive $*$-disjointness.*
- *Unary properties* of height, normality, fuzziness, and crispness.

Notice that due to the axiom of extensionality, the relation $\mathrm{Rel}^{\forall}_{\Delta(p \leftrightarrow q)}$, which is obviously equivalent to $\Delta(X \approx_* Y)$, coincides with the identity of classes. Thus it is $\Delta(X \approx_* Y)$ that guarantees intersubstitutivity *salva veritate* in all formulae (equalities generally do not).

It can be noticed that Gödel equality $\approx_{\mathrm{G}}$ is highly true only if the membership functions are identical on low truth values; product equality $\approx_\Pi$ is also more restrictive on lower truth values. However, this does not conform with the intuition that the difference in the *high* values (on the "prototypes") should matter more than a negligible difference on objects that almost do not belong to the classes under consideration. Equality of involutive complements, $-X \approx_* -Y$, is therefore a better measure of similarity of classes. Similarly, $-Y \subseteq_* -X$ may give a better measure of containment of X in Y than $X \subseteq_* Y$.

3.4 Theorems on elementary class relations and operations

The following metatheorems show that a large part of elementary fuzzy set theory can be reduced to fuzzy propositional calculus.

Theorem 33 *Let $\varphi, \psi_1, \ldots, \psi_n$ be propositional formulae.*

Then $\vdash \varphi(\psi_1, \ldots, \psi_n)$

$$\textit{iff} \vdash \mathrm{Rel}^{\forall}_{\varphi}(\mathrm{Op}_{\psi_1}(X_{1,1}, \ldots, X_{1,k_1}), \ldots, \mathrm{Op}_{\psi_n}(X_{n,1}, \ldots, X_{n,k_n})) \tag{7}$$
$$\textit{iff} \vdash \mathrm{Rel}^{\exists}_{\varphi}(\mathrm{Op}_{\psi_1}(X_{1,1}, \ldots, X_{1,k_1}), \ldots, \mathrm{Op}_{\psi_n}(X_{n,1}, \ldots, X_{n,k_n})) \tag{8}$$

Proof. The substitution of the formulae $x \in X_{i,j}$ for $p_{i,j}$ into $\psi_i(p_{i,1}, \ldots, p_{i,k_i})$ everywhere in the (propositional) proof of $\varphi(\psi_1, \ldots, \psi_n)$ transforms it into the proof of

$$\varphi(x \in \mathrm{Op}_{\psi_1}(X_{1,1}, \ldots, X_{1,k_1}), \ldots, x \in \mathrm{Op}_{\psi_n}(X_{n,1}, \ldots, X_{n,k_n})).$$

Then use generalization on x to get $\mathrm{Rel}^{\forall}_{\varphi}$ and $\exists$-introduction to get $\mathrm{Rel}^{\exists}_{\varphi}$.

Conversely, given an evaluation e that refutes $\varphi(\psi_1, \ldots, \psi_n)$, we construct a Zadeh model $\mathbf{M}$ refuting (7) and (8) by assigning to the class variables $X_{i,j}$ the functions $A_{i,j}$ such that $A_{i,j}(a) = e(p_{i,j})$ for every a in the universe of $\mathbf{M}$. Applying Theorems 4 and 27, the proof is done. Q.E.D.

Corollary 34 *Let φ and ψ be propositional formulae.*

If $\vdash \varphi \to \psi$ *then* $\vdash \mathrm{Op}_{\varphi}(X_1, \ldots, X_n) \subseteq \mathrm{Op}_{\psi}(X_1, \ldots, X_n)$.

If $\vdash \varphi \leftrightarrow \psi$ *then* $\vdash \mathrm{Op}_{\varphi}(X_1, \ldots, X_n) = \mathrm{Op}_{\psi}(X_1, \ldots, X_n)$.

If $\vdash \varphi \vee \neg\varphi$ *then* $\vdash \mathrm{Crisp}(\mathrm{Op}_{\varphi}(X_1, \ldots, X_n))$.

By virtue of Theorem 33, the properties of propositional connectives directly translate to the properties of class relations and operations. For example:

$\vdash \Delta p \to p$	proves	$\vdash \mathrm{Ker}(X) \subseteq X$
$\vdash p \to p \vee q$	"	$\vdash X \subseteq X \cup Y$
$\vdash 0 \to p$	"	$\vdash \emptyset \subseteq X$
$\vdash p \mathbin{\&} q \to p \wedge q$	"	$\vdash X \cap_* Y \subseteq X \cap_{\mathrm{G}} Y$
$\vdash \neg_{\mathrm{G}} p \vee \neg\neg_{\mathrm{G}} p$	"	$\vdash \mathrm{Crisp}(\backslash X)$
$\vdash \Delta(\alpha \to p) \to \Delta(\beta \to p)$ for $\alpha \geq \beta$	"	$\vdash X_\alpha \subseteq X_\beta$ for $\alpha \geq \beta$, etc.

In order to translate monotonicity and congruence properties of propositional connectives to the same properties of class operations, we need another theorem:

Theorem 35 *Let $\varphi_i, \varphi'_i, \psi_{i,j}, \psi'_{i,j}$ be propositional formulae. Then*

$$\vdash \mathop{\&_*}_{i=1}^{k} \varphi_i(\psi_{i,1}, \ldots, \psi_{i,n_i}) \to \bigwedge_{i=1}^{k'} \varphi'_i(\psi'_{i,1}, \ldots, \psi'_{i,n'_i}) \tag{9}$$

iff

$$\vdash \mathop{\&_*}_{i=1}^{k} \mathrm{Rel}^{\forall}_{\varphi_i} \left(\mathrm{Op}_{\psi_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi_{i,n_i}}(\vec{X})\right) \to \\ \to \bigwedge_{i=1}^{k'} \mathrm{Rel}^{\forall}_{\varphi'_i} \left(\mathrm{Op}_{\psi'_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi'_{i,n'_i}}(\vec{X})\right) \tag{10}$$

Proof. Without loss of generality, the principal implications of (9) and (10) can be assumed to be $\to_*$. Replacing all propositional variables p_j in the proof of (9) by the atomic formulae $x \in X_j$ then yields the proof of

$$\mathop{\&_*}_{i=1}^{k} \varphi_i \left(\mathrm{Op}_{\psi_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi_{i,n_i}}(\vec{X})\right) \to_* \\ \to_* \bigwedge_{i=1}^{k'} \varphi'_i \left(\mathrm{Op}_{\psi'_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi'_{i,n'_i}}(\vec{X})\right).$$

Generalization on x and distribution of $\forall$ over all conjuncts using (1), (5), and (3) of Lemma 25 proves (10). The converse is proved as in Theorem 33. Q.E.D.

Examples of direct corollaries of the theorem:

Provability in BLΔ of	Proves in FCT
$(p \to q) \to ((p \,\&\, r) \to (q \,\&\, r))$	$X \subseteq_* Y \to X \cap_* Z \subseteq_* Y \cap_* Z$
$(p \to q) \to (p \to (p \wedge q))$	$X \subseteq_* Y \to X \subseteq_* X \cap_{\mathrm{G}} Y$
$[(p \to q) \,\&\, (q \to p)] \to (p \leftrightarrow q)$	$(X \subseteq_* Y \,\&\, Y \subseteq_* X) \to X \approx_* Y$
$(p \leftrightarrow q) \to [(p \to q) \wedge (q \to p)]$	$X \approx_* Y \to (X \subseteq_* Y \wedge Y \subseteq_* X)$
$[(p \to r) \,\&\, (q \to r)] \to (p \vee q \to r)$	$(X \subseteq_* Z \,\&\, Y \subseteq_* Z) \to X \cup Y \subseteq_* Z$
$\Delta(p \to q) \to [\Delta(\alpha \to p) \to \Delta(\alpha \to q)]$	$\Delta(X \subseteq Y) \to X_\alpha \subseteq Y_\alpha$
transitivity of $\to$, $\leftrightarrow$	transitivity of $\subseteq_*, \approx_*$, etc.

Similarly, $\mathrm{L} \vdash (\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow p)$ proves $-X \subseteq_{\mathrm{L}} -Y \leftrightarrow Y \subseteq_{\mathrm{L}} X$, etc.

To derive theorems about $\mathrm{Rel}^\exists$, we slightly modify Theorem 35:

Theorem 36 *Let $\varphi_i, \varphi'_i, \psi_{i,j}, \psi'_{i,j}$ be propositional formulae. Then*

$$\vdash \mathop{\&_*}_{i=1}^{k} \varphi_i(\psi_{i,1}, \dots, \psi_{i,n_i}) \to \bigvee_{i=1}^{k'} \varphi'_i(\psi'_{i,1}, \dots, \psi'_{i,n'_i}) \tag{11}$$

iff

$$\begin{aligned} \vdash \mathop{\&_*}_{i=1}^{k-1} \mathrm{Rel}^\forall_{\varphi_i}\left(\mathrm{Op}_{\psi_{i,1}}(\vec{X}), \dots, \mathrm{Op}_{\psi_{i,n_i}}(\vec{X})\right) \&_* \\ \&_* \mathrm{Rel}^\exists_{\varphi_k}\left(\mathrm{Op}_{\psi_{k,1}}(\vec{X}), \dots, \mathrm{Op}_{\psi_{k,n_k}}(\vec{X}\right) \to \\ \to \bigvee_{i=1}^{k'} \mathrm{Rel}^\exists_{\varphi'_i}\left(\mathrm{Op}_{\psi'_{i,1}}(\vec{X}), \dots, \mathrm{Op}_{\psi'_{i,n'_i}}(\vec{X})\right) \end{aligned} \tag{12}$$

Proof. Modify the proof of Theorem 35, using (6) of Lemma 25 instead of (5), and then (4) of the same Lemma to distribute $\exists$ over the disjuncts. Q.E.D.

Examples of direct corollaries:

Provability in BLΔ of	Proves in FCT
$p \mathbin{\&} (p \to q) \to q$	$\mathrm{Hgt}(X) \&_* (X \subseteq_* Y) \to \mathrm{Hgt}(Y)$
$\Delta(p \vee q) \to \Delta p \vee \Delta q$	$\mathrm{Norm}(X \cup Y) \to \mathrm{Norm}(X) \vee \mathrm{Norm}(Y)$
$(p \to r) \mathbin{\&} (p \mathbin{\&} q) \to (q \mathbin{\&} r)$	$X \subseteq_* Z \&_* X \|_* Y \to Y \|_* Z$, etc.

4 Tuples of objects

In order to be able to deal with fuzzy relations, we will further assume that the language of FCT contains an apparatus for forming tuples of objects and accessing their components. Such an extension can be achieved, e.g., by postulating variable sorts for any multiplicity of tuples (all of which are subsumed by the sort of objects), enriching the language with the functions for forming n-tuples of any combination of tuples and accessing its components, and adding axiom schemes expressing that tuples equal iff their respective constituents equal. The definition of Zadeh model then must be adjusted by partitioning the range of object variables and interpreting the tuples-handling functions. We omit elaborating this sort of syntactic sugar.

In what follows, the usual abbreviations of the form $\{\langle x_1,\dots,x_n\rangle \mid \varphi\}$ for $\{z \mid (\exists x_1)\dots(\exists x_n)(z = \langle x_1,\dots,x_n\rangle \mathbin{\&} \varphi)\}$ will be used.

FCT equipped with tuples of objects contains common operations for dealing with relations. We can define Cartesian products, domains, ranges and the relational operations as usual:[4]

$$
\begin{aligned}
X \times_* Y &=_{\mathrm{df}} \{\langle x,y\rangle \mid x \in X \mathbin{\&_*} y \in Y\}\\
\mathrm{Dom}(R) &=_{\mathrm{df}} \{x \mid \langle x,y\rangle \in R\}\\
\mathrm{Rng}(R) &=_{\mathrm{df}} \{y \mid \langle x,y\rangle \in R\}\\
R \circ_* S &=_{\mathrm{df}} \{\langle x,y\rangle \mid (\exists z)(\langle x,z\rangle \in R \mathbin{\&_*} \langle z,y\rangle \in S)\}\\
R^{-1} &=_{\mathrm{df}} \{\langle x,y\rangle \mid \langle y,x\rangle \in R\}\\
\mathrm{Id} &=_{\mathrm{df}} \{\langle x,y\rangle \mid x = y\}
\end{aligned}
$$

The introduction of tuples of objects also allows an axiomatic investigation of various kinds of fuzzy relations (e.g., similarities) and fuzzy structures (fuzzy preorderings, graphs, etc.). We can define the usual properties of relations, as summarized in Table 3 (for brevity's sake, we write just Rxy for $\langle x,y\rangle \in R$).[5]

Classical definitions of some properties of relations (e.g., antisymmetry) make use of the identity predicate on objects. One may be tempted to use the identity predicate $=$ of ŁΠ∀ in the rôle of the classical identity in these definitions. However, since $=$ is crisp, such definitions do not yield useful and genuine fuzzy notions. A fuzzy analogue of the crisp notion of identity is that of similarity or equality (see Table 3). We can therefore define these properties *relative to* a $*$-similarity or $*$-equality S. For details see the last section.

In this way, the properties of being a $*$-antisymmetric relation, a $*$-ordering, a linear $*$-ordering, a $*$-well-ordering, a $*$-function and a $*$-bijection (w.r.t. some fuzzy $*$-equality) can be introduced. By means of $*$-bijections, the notions of $*$-subvalence, $*$-equipotence and $*$-finitude of classes (again w.r.t. some fuzzy

[4] Obviously for crisp arguments these operations yield crisp classes; $X \times_* Y$ is crisp iff both X and Y are crisp. Unless X and Y are crisp, the property of being a relation from X to Y is double-indexed (a $*'$-subset of the Cartesian product $X \times_* Y$). Also the definitions of usual properties (e.g., reflexivity, $*$-symmetry, etc.) of a relation on a non-crisp Cartesian product have to be defined with relativized quantifiers which bring another index. It is doubtful that definitions combining various t-norms will have any real meaning. The situation is much easier if only relations on crisp classes are considered.

[5] Following the usual mathematical terminology, $*$-similarity may also be called $*$-*equivalence*; we respect the established fuzzy set terminology here. *Weak dichotomy* $(\forall x,y)(Rxy \oplus Ryx)$ could also be defined and weak versions of the properties that contain dichotomy, e.g. weakly linear $*$-ordering.

Table 3
Properties of relations

Notation	Definition	Name
$\mathrm{Refl}(R)$	$(\forall x)(Rxx)$	reflexive
$\mathrm{Sym}_*(R)$	$(\forall x, y)(Rxy \to_* Ryx)$	$*$-symmetric
$\mathrm{Trans}_*(R)$	$(\forall x, y, z)(Rxy \mathbin{\&_*} Ryz \to_* Rxz)$	$*$-transitive
$\mathrm{Dich}(R)$	$(\forall x, y)(Rxy \vee Ryx)$	dichotomic
$\mathrm{Quord}_*(R)$	$\mathrm{Refl}(R) \mathbin{\&_*} \mathrm{Trans}_*(R)$	$*$-quasiordering
$\mathrm{Linquord}_*(R)$	$\mathrm{Quord}_*(R) \mathbin{\&_*} \mathrm{Dich}(R)$	linear $*$-quasiordering
$\mathrm{Sim}_*(R)$	$\mathrm{Quord}_*(R) \mathbin{\&_*} \mathrm{Sym}_*(R)$	$*$-similarity
$\mathrm{Equ}_*(R)$	$\mathrm{Sim}_*(R) \mathbin{\&_*} (\forall x, y)(\Delta Rxy \to_* x = y)$	$*$-equality

$*$-equality) can be defined. A thorough investigation of these notions, however, exceeds the scope of this paper.

5 Higher types of classes

5.1 Second-level classes

Class theory does not contain an apparatus for dealing with families of classes. In many cases, a family of classes can be represented by a class of pairs or some other kind of 'encoding'. For instance, a relation R may be understood as representing the family of classes $X_i = \{x \mid \langle i, x \rangle \in R\}$ for all $i \in \mathrm{Dom}(R)$.

In other cases, however, no suitable class of indices can be found and such an 'encoding' is not possible. Then it is desirable to extend the apparatus of class theory by classes of the second level. This is done simply by repeating the same definitions one level higher. We introduce a new sort of variables for families of classes $\mathcal{X}, \mathcal{Y}, \ldots$, a new membership predicate between classes and families of classes $X \in \mathcal{X}$, and the comprehension scheme for families of classes

$$(\exists \mathcal{X})\Delta(\forall X)(X \in \mathcal{X} \leftrightarrow \varphi(X))$$

for all formulae φ (where φ may contain any parameters except for $\mathcal{X}$). The extensionality axiom for families of classes now reads

$$(\forall X)\Delta(X \in \mathcal{X} \leftrightarrow X \in \mathcal{Y}) \to \mathcal{X} = \mathcal{Y}.$$

Again it is possible to introduce second-level comprehension terms $\{X \mid \varphi(X)\}$, which introduction is conservative and eliminable by Theorem 28.

The consistency of this extension is proved by a construction of second-level Zadeh models over a linear ŁΠ-algebra $\mathbf{L}$, in which the object variables range over a universe U, the class variables over the set $\mathbf{L}^U$ of all functions from U to $\mathbf{L}$, and the second-level class variables range over the set $\mathbf{L}^{\mathbf{L}^U}$ of all functions from $\mathbf{L}^U$ to $\mathbf{L}$. The second-level class $\{X \mid \varphi(X)\}$ is again identified with the characteristic function of φ as in Section 3.1. Obviously, this construction makes both the second-level comprehension scheme and the axiom of extensionality satisfied in the model; the theory of second-level classes can thus be viewed as third-order fuzzy logic (we omit details).

All definitions of elementary class relations and operations and all theorems can directly be transferred from classes to second-level classes. Refining the language, axioms, and Zadeh models to tuples of classes is also straightforward.

It may be observed that the class operations and relations Op_φ, $\mathrm{Rel}^\forall_\varphi$, and $\mathrm{Rel}^\exists_\varphi$, which were introduced in Sections 3.2 and 3.3 as defined functors and predicates, are now individuals of the theory, viz. second-level classes.

5.2 Simple fuzzy type theory

If there be need for families of families of classes, it is straightforward to repeat the whole construction once again to get third-level classes. By iterating this process, we get a simple type theory over ŁΠ, for which the class theory described in Sections 3–4 is the induction step. The comprehension schemes and Zadeh models can easily be generalized to allow membership of elements of any type less then n in classes of the n-th level. [6]

A type theory over a particular fuzzy logic (viz. IMTLΔ, extended also to ŁΔ) has already been proposed by V. Novák in [14]. As mentioned in the Introduction, our theory can be built over various fuzzy logics with Δ; its variant over IMTLΔ and Novák's type theory seem to be equivalent (though radically different in notation, as Novák uses λ-terms).

Since almost all classical applied mathematics can be formalized within the first few levels of simple type theory, the formalism just described should be

[6] This is done simply by postulating that the n-th sort of variables is subsumed by the k-th sort if $n < k$. The sorts can further be refined to allow arbitrary tuples of individuals of lesser types with the appropriate tuple-forming, component-extracting, and tuple-identity axioms added. The generalization of Zadeh models is again quite straightforward.

sufficient for all applications of fuzzy sets based on t-norms or other functions definable in ŁΠ (see Theorem 7). To illustrate this, we show the formalization of Zadeh's extension principle.

Definition 37 (Extension by Zadeh's principle) *A (fuzzy) binary relation*[7] *R between objects is* extended by Zadeh's principle (based on a t-norm $*$) *to a relation $\mathcal{R}_*$ between (fuzzy) classes as follows:*

$$\mathcal{R}_*(X, Y) \equiv_{\mathrm{df}} (\exists x, y)(Rxy \mathbin{\&_*} x \in X \mathbin{\&_*} y \in Y)$$

Since relations between classes are classes of the second level in our simple type theory, Zadeh's extension principle in fact assigns to a first-level class R a second-level relation; such an assignment itself is an individual of the third level. Thus we can define Zadeh's principle as an *individual* of our theory—a special class $\mathcal{Z}_*$ of the third level:

Definition 38 (Zadeh's extension principle) Zadeh's extension principle based on $*$ *is a third-level function $\mathcal{Z}_*$ defined as follows (we adopt the usual functional notation for classes which are functions):*

$$\mathcal{Z}_*(R) =_{\mathrm{df}} \{\langle X, Y\rangle \mid (\exists x, y)(Rxy \mathbin{\&_*} x \in X \mathbin{\&_*} y \in Y)\}$$

Generally we can extend any fuzzy relation $R^{(n+1)}$ of type $n+1$ to one of type $n+2$ by Zadeh's principle of type $n+3$ (based on a t-norm $*$). All these 'principles' are in fact individuals of our theory, whose existence follows from the comprehension scheme.

Definition 39 (Zadeh's extension principle for higher types) Zadeh's extension principle for relations of type $n+1$ (for $n \geq 0$) based on $*$ *is the function of type $n+3$ defined as follows:*

$$\begin{aligned}\mathcal{Z}_*{}^{(n+3)}\left(R^{(n+1)}\right) =_{\mathrm{df}} \Big\{\Big\langle X_1^{(n+1)}, \ldots, X_k^{(n+1)}\Big\rangle \;\Big|\; & \\ (\exists W_1^{(n)}, \ldots, W_k^{(n)})\Big(\Big\langle W_1^{(n)}, \ldots, W_k^{(n)}\Big\rangle \in R^{(n+1)} \mathbin{\&_*} & \\ \mathop{\&_*}_{i=1}^{k} W_i^{(n)} \in X_i^{(n+1)}\Big)\Big\} &\end{aligned}$$

[7] The generalization to n-ary relations is trivial.

As we have shown, in FCT we can define many properties of individuals of our theory (objects or classes). Since our theory contains classical class theory (for classes which are crisp), we can introduce arbitrary relations and functions on the universe of objects which are definable in classical class theory. As they can be described by formulae, their existence is guaranteed by the comprehension axiom. So the only thing we need to add is a constant of the appropriate sort and the instance of the comprehension axiom. The following definition is the formalization of this approach for the first-order theories.

Definition 40 *Let* $\mathbf{\Gamma}$ *be a classical one-sorted predicate language and* T *be a* $\mathbf{\Gamma}$*-theory. For each n-ary predicate symbol* P *of* $\mathbf{\Gamma}$ *let us introduce a new constant* $\bar{P}$ *for a class of n-tuples, and for each n-ary function symbol* F *we take a new constant* $\bar{F}$ *for a class of (n+1)-tuples. We define the language* FCT($\mathbf{\Gamma}$) *as the language of* FCT *extended by the symbols* $\bar{Q}$ *for each symbol* $Q \in \mathbf{\Gamma}$*. The translation* $\bar{\varphi}$ *of a* $\mathbf{\Gamma}$*-formula* φ *to* FCT($\mathbf{\Gamma}$) *is obtained as the result of replacing all occurrences of all* $\mathbf{\Gamma}$*-symbols* Q *in* φ *by* $\bar{Q}$*.*

We define the theory FCT(T) *in the language* FCT($\mathbf{\Gamma}$) *as the theory with the following axioms:*

- *The axioms of* FCT
- *The translations* $\bar{\varphi}$ *of all axioms* φ *of* T
- Crisp($\bar{Q}$) *for each symbol* $Q \in \mathbf{\Gamma}$ *(for the definition of* Crisp*, see Table 2)*
- $\langle x_1, \ldots x_n, y\rangle \in \bar{F} \wedge \langle x_1, \ldots x_n, z\rangle \in \bar{F} \rightarrow y = z$ *for each n-ary function symbol* $F \in \mathbf{\Gamma}$*.*

Lemma 41 *Let* $\mathbf{\Gamma}$ *be a classical predicate language,* T *a* $\mathbf{\Gamma}$*-theory,* $\mathbf{L}$ *an* ŁΠ*-algebra. If* $\mathbf{M}$ *is an* $\mathbf{L}$*-model of* FCT(T)*, then* $\mathbf{M}^c = (M, (Q_{\mathbf{M}^c})_{Q\in\mathbf{\Gamma}})$*, where* $Q_{\mathbf{M}^c} = \bar{Q}_{\mathbf{M}}$ *for each* $Q \in \mathbf{\Gamma}$*, is a model (in the sense of classical logic) of the theory* T*.*

Vice versa, for each model $\mathbf{M}$ *of* T *there is an* $\mathbf{L}$*-model* $\mathbf{N}$ *of* FCT(T) *such that* $\mathbf{N}^c$ *is isomorphic to* $\mathbf{M}$*.*

Therefore (in virtue of Theorem 27), $T \vdash \varphi$ *iff* FCT(T) $\vdash \bar{\varphi}$*, for any* $\mathbf{\Gamma}$*-formula* φ*.*

Proof. If $\mathbf{M}$ is an $\mathbf{L}$-model of FCT, then for each $Q \in \mathbf{\Gamma}$, $\bar{Q}_{\mathbf{M}}$ is crisp due to the axiom Crisp($\bar{Q}$) of FCT(T). Setting the universe of $\mathbf{M}^c$ to that of $\mathbf{M}$, and for each symbol $Q \in \mathbf{\Gamma}$, setting $Q_{\mathbf{M}^c}$ to the set whose characteristic function is $\bar{Q}_{\mathbf{M}}$, we can see that $\mathbf{M}^c$ models T, because the axioms of T, which contain only crisp predicates, are evaluated classically in $\mathbf{M}^c$.

Conversely, we define $\mathbf{M}$ as the standard Zadeh model with the universe of $\mathbf{N}$, in which $\bar{F}_\mathbf{M} = F_\mathbf{N}$ for every function symbol $F \in \mathbf{\Gamma}$, and for every predicate $P \in \mathbf{\Gamma}$, $\bar{P}_\mathbf{M}$ is realized as the characteristic function of $P_\mathbf{N}$. Then $\mathbf{M}$ obviously satisfies all axioms of $\mathrm{FCT}(T)$; the axioms of T are again evaluated classically in $\mathbf{M}$, as the realizations of all predicates involved are crisp. Q.E.D.

Example 42 *Let R be a constant for a class of pairs. Then in each $\mathbf{L}$-model of the theory* $\mathrm{Crisp}(R), \mathrm{Refl}(R), \mathrm{Trans}(R), (\forall x, y)(Rxy \mathbin{\&} Ryx \to x = y)$*, the constant R is represented by a crisp ordering on the universe of objects. (For the definitions of* Refl *and* Trans*, see Table 3.)*

Example 43 *If T is a classical theory of the real closed field, then in each $\mathbf{L}$-model $\mathbf{M}$ of the theory $\mathrm{FCT}(T)$, the universe of objects with $\bar{\leq}_\mathbf{M}, \bar{+}_\mathbf{M}, \bar{-}_\mathbf{M}, \bar{\cdot}_\mathbf{M}, \bar{0}_\mathbf{M}, \bar{1}_\mathbf{M}$ is a real closed field.*

In Lemma 41 we speak of first-order theories only. Nevertheless, it can be extended to any theory formalizable in classical type theory. Here we present only one example.

Example 44 *Let τ be a constant for a class of classes and T the theory with the axioms:*

- $\mathrm{Crisp}(\tau)$
- $(\forall X)(X \in \tau \to \mathrm{Crisp}(X))$
- $(\forall \mathcal{X})(\mathrm{Crisp}(\mathcal{X}) \mathbin{\&} \mathcal{X} \subseteq \tau \to \{x \mid (\exists X \in \mathcal{X})(x \in X)\} \in \tau)$
- $(\forall X_1) \ldots (\forall X_n)(X_1 \in \tau \mathbin{\&} \ldots \mathbin{\&} X_n \in \tau \to X_1 \cap \ldots \cap X_n \in \tau)$ *for each* $n \in N$

Then in each $\mathbf{L}$-model of the theory T, the constant τ is represented by a classical topology on the universe of objects.

7 Fuzzy mathematics

If we examine the above definitions we see the crucial rôle of the predicate Crisp. If we remove this predicate from the above definitions we get the "natural" fuzzification of the above-mentioned concepts. [8]

[8] A sketch of this method can already be found in Höhle's 1987 paper [12, Section 5]:

> "It is the opinion of the author that from a mathematical viewpoint the important feature of fuzzy set theory is the replacement of the two-valued logic by a multiple-valued logic. [...I]t is now clear how we can find for every mathematical notion its 'fuzzy counterpart'. Since every mathematical notion can be written as a formula in a formal language, we have only to internalize, i.e. to interpret these expressions by the given multiple-valued logic."

In order to illustrate the methodology of fuzzification, let us concentrate on the concept of ordering. If we remove the predicate Crisp from the definition, then we have to distinguish which t-norm was used in the axioms of transitivity and antisymmetry. Thus we get the concept of $*$-fuzzy ordering. This is the way this concept was introduced by Zadeh. However, some carefulness is due here not to overlook some "hidden" crispness. There is crisp identity used in the antisymmetry axiom, and also in the reflexivity axiom which can be written as $(\forall x, y)(x = y \rightarrow Rxy)$. A more general definition is therefore parameterized also by a fuzzy equality in the following way:

Example 45 *Let E and R be two constants of classes of tuples. The following axioms define the concept of $(*, E)$-ordering R:*

- $\mathrm{Equ}_*(E)$
- $\mathrm{Trans}_*(R)$
- $(\forall x, y)(Exy \rightarrow Rxy)$
- $(\forall x, y)(Rxy \mathbin{\&_*} Ryx \rightarrow Exy)$

Observe that E is a $*$-equality, and the last two conditions can be written as $R \cap_* R^{-1} \subseteq E \subseteq R$. We thus get the notion of fuzzy ordering as defined by Bodenhoffer in [2].

In contemporary fuzzy mathematics the methodology of fuzzification of concepts is somewhat sketchy and non-consistent: usually only some features of a classical concept are fuzzified while other features are left crisp.

We would like to propose another "inductive" approach. We propose to follow the usual "inductive" development of mathematics (in some metamathematical setting—here in simple type theory) and fuzzify "along the way". In more words: develop a fuzzy generalization of basic classical concepts (the notion of class, relation, equality—as done in this paper); then define compound fuzzy notions by taking their classical definitions and consistently replacing classical sub-concepts in the definitions by their already fuzzified counterparts. The consistency of this approach promises that no crispness will be unintentionally "left behind".

This approach is formal and sometimes may lead to too complex notions. In such cases, some features of the complex notion may *intentionally* be left crisp by retaining some of the crispness axioms. The advantage of the proposed approach is that we always know *which* features are left crisp.

The framework presented in this paper provides a unified formalism for various disciplines of fuzzy mathematics. This may enable, i.a., an interchange of results and methods between distant disciplines of fuzzy mathematics, till now separated by differences in notation and incompatibilities in definitions. It can also bring new (proof-theoretic and model-theoretic) methods to traditional fuzzy disciplines and enable their further development in both theory and applications. Finally, the axiomatization of the whole fuzzy mathematics, independent of particular $[0,1]$-functions, can be an important step in understanding vague phenomena. Further elaboration of the proposed formalism and its application to various disciplines of fuzzy mathematics is thus a possible direction towards firm foundations of fuzzy mathematics.

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