Towards a formal theory of fuzzy Dedekind reals

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Abstract

In the framework of Henkin style higher-order fuzzy logic $\text{Ł}\Pi_\omega$ we construct fuzzy real numbers as fuzzy Dedekind cuts over crisp rationals, and show some of their properties provable in $\text{Ł}\Pi_\omega$. The definitions of algebraic operations and fuzzy intervals are sketched.

Keywords: Fuzzy Dedekind completion, fuzzy real numbers, higher-order fuzzy logic.

1 Introduction

In [3], Henkin-style higher-order fuzzy logic ŁΠ has been introduced and proposed as a unified foundational theory for fuzzy mathematics. This paper contributes to the programme of developing fuzzy mathematics within its framework by introducing the structure of fuzzy real numbers. A solid theory of fuzzy reals is indispensable for the more advanced disciplines of unified formal fuzzy mathematics, such as fuzzy measure theory or fuzzy probability.

The approach adopted in this paper conforms to the methodology of the Manifesto [2]. Real numbers and other concepts are therefore constructed in full analogy with classical mathematics, taking advantage of the similarity of both formalisms.

The method of construction of real numbers applied here is certainly not the only possible one, even within the framework of Henkin-style higher-order fuzzy logic. Another readily available method consists in implanting the first-order axioms of the real closed field in higher-order fuzzy logic. The systematic development of alternative notions of real number within higher-order fuzzy logic and their careful comparison, especially from the point of view of real-life applicability, is part of a broader long-term programme. Although the usability of the present notion for applications cannot yet be predicted, it nevertheless seems capable of capturing many features of fuzzy numbers already used in applied fuzzy mathematics, and furthermore shows many properties of independent mathematical interest. As sketched in Section 6, it can serve as a basis for a formal theory of fuzzy intervals, which is very close to applied practice.

We are going to construct fuzzy real numbers as fuzzy Dedekind cuts over crisp rationals. The reason why we use crisp rather than fuzzy rationals reflects the usual definitions of fuzzy numbers as fuzzy sets of (some kind of) common crisp numbers. However, unlike most definitions of fuzzy numbers, Dedekind cuts do not express the 'density' of the fuzzy number across the underlying crisp numbers, but rather its distribution (cumulative density), similar to the probabilistic distribution function. Intuitively, the membership $q \in A$ of a rational number q in a Dedekind fuzzy real number A expresses (the truth value of) the fact that q majorizes the fuzzy real.

In somewhat different settings, fuzzy Dedekind completion has already been studied in [7] and [1]. Dedekind reals in an axiomatic fuzzy set theory (over a slightly different logic) appear also in [8].

In [5], Dubois and Prade require of fuzzy reals that they be objects whose every α-cut is a (crisp) real. Interestingly, Dedekind fuzzy reals do meet this requirement, since every α-cut of a fuzzy

Dedekind cut is a crisp Dedekind cut, i.e., a crisp real (represented by the cut). We will see in Section 4 that (unlike the proposal of [5]) a monotonicity condition $\alpha \leq \beta \rightarrow A_\alpha \leq A_\beta$ is met here, which seems essential for some of the motivational aspects of fuzzy notions rendered horizontally (as sets of cuts); a thoroughful discussion of these requirements is yet to be carried out.

2 Preliminaries

For the ease of reference, we repeat here the definitions and axioms of Henkin-style higher-order fuzzy logic ŁΠ, which will be our framework in the rest of the paper. For details, see [3].

Definition 2.1 *The logic* ŁΠ *(introduced in [6]) has the following* primitive connectives *(listed here with their standard* $[0, 1]$*-semantics):*

The truth constant falsum $0 = 0$
Product conjunction $x \mathbin{\&_\Pi} y = x \cdot y$
Product implication $x \rightarrow_\Pi y = \min(1, x/y)$
Łukasiewicz implic. $x \rightarrow_\text{L} y = \min(1, 1 - x + y)$

We define various derived connectives *of* ŁΠ*:*

1	*is* $\neg_\text{L} 0$, *i.e.* 1
$\neg_\text{L} x$	*is* $x \rightarrow_\text{L} 0$, *i.e.* $1 - x$
$\neg_\Pi x$	*is* $x \rightarrow_\Pi 0$, *i.e.* $0/x$
Δx	*is* $\neg_\Pi \neg_\text{L} x$, *i.e.* $\Delta x = 1$ *if* $x = 1$, *else* 0
$x \mathbin{\&_\text{L}} y$	*is* $\neg_\text{L}(x \rightarrow_\text{L} \neg_\text{L} y)$, *i.e.* $\max(0, x + y - 1)$
$x \wedge y$	*is* $x \mathbin{\&_\text{L}} (x \rightarrow_\text{L} y)$, *i.e.* $\min(x, y)$
$x \vee y$	*is* $(x \rightarrow_\text{L} y) \rightarrow_\text{L} y$, *i.e.* $\max(x, y)$
$x \oplus y$	*is* $\neg_\text{L} x \rightarrow_\text{L} y$, *i.e.* $\min(1, x + y)$
$x \ominus y$	*is* $x \mathbin{\&_\text{L}} \neg_\text{L} y$, *i.e.* $\max(0, x - y)$
$x \rightarrow_\text{G} y$	*is* $\Delta(x \rightarrow_\text{L} y) \vee y$, *i.e.* 1 *if* $x \leq y$, *else* y

Bi-implications $\leftrightarrow_\text{L}$, $\leftrightarrow_\Pi$, *and* $\leftrightarrow_\text{G}$ *are defined as usual. Furthermore, for any t-norm* $*$ *representable in* ŁΠ, *the connectives* $\&_*$, $\rightarrow_*$, $\neg_*$, *and* $\leftrightarrow_*$ *can be defined. We employ the usual precedence of connectives.*

Convention 2.2 *We omit the t-norm indices of connectives and other defined symbols whenever they do not matter, i.e., whenever the substitution of any other t-norm index would yield a formula provably equivalent (or, in case of axioms and theorems, just equiprovable) to the original one. An index subscripted to a closing parenthesis distributes to all connectives and other indexed symbols within its scope that do not have their index explicitly marked.*

Definition 2.3 *The* propositional logic ŁΠ *has the following axioms:*

(Ł)	*The axioms of Łukasiewicz logic*
(Π)	*The axioms of Product logic*
(Ł_Π)	$\Delta(\varphi \rightarrow_\text{L} \psi) \rightarrow_\text{L} (\varphi \rightarrow_\Pi \psi)$
(Π_L)	$\Delta(\varphi \rightarrow_\Pi \psi) \rightarrow_\text{L} (\varphi \rightarrow_\text{L} \psi)$
(D)	$(\varphi \mathbin{\&_\Pi} (\chi \ominus \psi)) \leftrightarrow_\text{L} ((\varphi \mathbin{\&_\Pi} \chi) \ominus (\varphi \mathbin{\&_\Pi} \psi))$

The deduction rules of ŁΠ *are modus ponens and* Δ*-necessitation (from* φ *infer* $\Delta\varphi$*).*

Definition 2.4 *The* first-order logic ŁΠ *[4] adds the deduction rule of generalization and the following axioms for quantifiers and (crisp) identity:*

(∀1)	$(\forall x)\varphi(x) \rightarrow \varphi(t)$ *if* t *is substitutable for* x *in* ψ
(∀2)	$(\forall x)(\chi \rightarrow_\text{L} \varphi) \rightarrow (\chi \rightarrow_\text{L} (\forall x)\varphi)$ *if* x *is not free in* χ
(=1)	$x = x$
(=2)	$x = y \rightarrow \Delta(\varphi(x) \leftrightarrow \varphi(y))$

The symbol $(\exists x)$ *is an abbreviation for* $\neg_\text{L}(\forall x)\neg_\text{L}$.

Definition 2.5 *The* Henkin-style second-order logic ŁΠ *is a theory in the multi-sorted first-order logic* ŁΠ, *with sorts for objects (lowercase variables) and classes (uppercase variables). Both of the sorts subsume subsorts of* n*-tuples, for all* $n \geq 1$. *Apart from the obvious necessary function symbols and axioms for tuples (tuples equal iff their respective constituents equal), the only primitive symbol is the membership predicate* $\in$ *between objects and classes. The axioms for* $\in$ *are (i) the comprehension axioms*

$$(\exists X)\Delta(\forall x)(x \in X \leftrightarrow \varphi),$$

for all φ *not containing* X, *which enable the (eliminable) introduction of comprehension terms* $\{x \mid \varphi\}$ *with axioms* $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$ *(where* φ *may be allowed to contain other comprehension terms); and (ii) the extensionality axiom*

$$(\forall x)\Delta(x \in X \leftrightarrow x \in Y) \rightarrow X = Y.$$

Convention 2.6 *Formulae* $(\forall x)(x \in X \rightarrow_* \varphi)$, $(\exists x)(x \in X \mathbin{\&_*} \varphi)$ *are abbreviated* $(\forall x \in X)_* \varphi$

and $(\exists x \in X)_*\varphi$, resp.; $x \notin_* X$ stands for $\neg_*(x \in X)$; alternatively we write Ax and $Rx_1 \dots x_n$ for $x \in A$ and $\langle x_1, \dots, x_n \rangle \in R$, resp.

Definition 2.7 *The* Henkin-style logics ŁΠ of higher orders *are obtained by repeating the previous definition on each level of the type hierarchy. Obviously, all defined symbols of any type can then be shifted to all higher types as well. (Consequently, all theorems are preserved by uniform upward type-shifts.) Types may be allowed to subsume all lower types.*

Henkin-style ŁΠ of order n will be denoted by $\text{Ł}\Pi_n$, the whole hierarchy by $\text{Ł}\Pi_\omega$. The types of terms are either denoted by a superscripted parenthesized number (e.g., $X^{(3)}$), or understood from the context.

Definition 2.8 *In $\text{Ł}\Pi_2$, we define the following relations and operations:*

$$\begin{array}{rcl}
\emptyset & =_{\text{df}} & \{x \mid 0\} \\
\text{Ker}(X) & =_{\text{df}} & \{x \mid \Delta(x \in X)\} \\
X_\alpha & =_{\text{df}} & \{x \mid \Delta(\alpha \to x \in X)\} \\
\backslash_* X & =_{\text{df}} & \{x \mid x \notin_* X\} \\
X \cap_* Y & =_{\text{df}} & \{x \mid x \in X \,\&_*\, x \in Y\} \\
X \cup Y & =_{\text{df}} & \{x \mid x \in X \vee x \in Y\} \\
\text{Crisp}(X) & \equiv_{\text{df}} & (\forall x)\Delta(x \in X \vee x \notin X) \\
\text{Fuzzy}(X) & \equiv_{\text{df}} & \neg\text{Crisp}(X) \\
X \subseteq_* Y & \equiv_{\text{df}} & (\forall x)(x \in X \to_* x \in Y) \\
X \approx_* Y & \equiv_{\text{df}} & (\forall x)(x \in X \leftrightarrow_* x \in Y) \\
X \times_* Y & =_{\text{df}} & \{\langle x, y \rangle \mid x \in X \,\&_*\, y \in Y\} \\
R^{-1} & =_{\text{df}} & \{\langle x, y \rangle \mid Ryx\} \\
\text{Id} & =_{\text{df}} & \{\langle x, y \rangle \mid x = y\}
\end{array}$$

We shall freely use all elementary theorems on these notions which follow from the metatheorems proved in [3], and thus can be checked by simple propositional calculations.

Definition 2.9 *In $\text{Ł}\Pi_2$, we can also define the usual properties of relations:*

Reflexivity $\text{Refl}(R) \equiv_{\text{df}} (\forall x)(Rxx)$
Dichotomy $\text{Dich}(R) \equiv_{\text{df}} (\forall x, y)(Rxy \vee Ryx)$
$*$-Symmetry $\text{Sym}_*(R) \equiv_{\text{df}} (\forall x, y)(Rxy \to_* Ryx)$
$*$-Transitivity $\text{Trans}_*(R)$
$\equiv_{\text{df}} (\forall x, y, z)(Rxy \,\&\, Ryz \to Rxz)_*$
$*$-Antisymmetry $\text{Asym}_{E,*}(R)$ *(w.r.t. E)*
$\equiv_{\text{df}} (\forall x, y)(Rxy \,\&\, Ryx \to Exy)_*$
$*$-Quasi-ordering $\text{QOrd}_*(R)$
$\equiv_{\text{df}} (\text{Refl}(R) \,\&\, \text{Trans}(R))_*$
$*$-Ordering $\text{Ord}_{E,*}(R)$ *(w.r.t. E)*
$\equiv_{\text{df}} (\text{QOrd}(R) \,\&\, \text{Asym}_E(R))_*$
$*$-Linear ordering $\text{LOrd}_{E,*}(R)$ *(w.r.t. E)*
$\equiv_{\text{df}} (\text{Ord}_E(R) \,\&\, \text{Dich}(R))_*$
$*$-Function $\text{Fnc}_{E,*}(R)$ *(w.r.t. E)*
$\equiv_{\text{df}} (\forall x, y, z)(Rxy \,\&\, Rxz \to Eyz)_*$

We adopt the convention that the index E can be dropped if $\Delta(E = \text{Id})$. If $\Delta\text{Fnc}_*(F)$, we can write $y = F(x)$ instead of ΔFxy.

Definition 2.10 *The class union and class intersection are the functions $\bigcup_*^{(n+3)}$ and $\bigcap_*^{(n+3)}$, respectively, assigning a class $A^{(n+1)}$ to a class of classes $\mathcal{A}^{(n+2)}$ and defined as follows:*

$$\begin{array}{rcl}
\bigcup_* \mathcal{A} & =_{\text{df}} & \{x \mid (\exists A \in \mathcal{A})_*(x \in A)\} \\
\bigcap_* \mathcal{A} & =_{\text{df}} & \{x \mid (\forall A \in \mathcal{A})_*(x \in A)\}
\end{array}$$

3 Formal theory of suprema and infima

The notions defined in this section are most meaningful for (quasi)orderings. Nevertheless, the definitions can be formulated for just any relation and most of the results hold regardless of any properties of the relation involved.

Definition 3.1 *The* upper *and* lower $*$-cone *of a class A w.r.t. $\leq$ is defined as follows:*

$$\begin{array}{rcl}
A^{\uparrow_*} & =_{\text{df}} & \{x \mid (\forall a \in A)_*(a \leq x)\} \\
A^{\downarrow_*} & =_{\text{df}} & \{x \mid (\forall a \in A)_*(x \leq a)\}
\end{array}$$

Let us fix some relation $\leq$ and denote its converse as usual by $\geq$. The usual definition of suprema and infima as least upper bounds and greatest lower bounds can then be formulated as follows (notice that they are *fuzzy classes,* since the property of being a supremum is graded):

Definition 3.2 *The classes of* $*$-suprema *and* $*$-infima *of a class A w.r.t. $\leq$ are defined as*

$$\begin{array}{rcl}
\leq\text{-}\text{Sup}_* A & =_{\text{df}} & A^{\uparrow_*} \cap_* A^{\uparrow_* \downarrow_*} \\
\leq\text{-}\text{Inf}_* A & =_{\text{df}} & A^{\downarrow_*} \cap_* A^{\downarrow_* \uparrow_*}
\end{array}$$

Example 3.3 *$\bigcup_* \mathcal{A}$ is a $*$-supremum of $\mathcal{A}$ w.r.t. $\subseteq_*$. Similarly, $\bigcap_* \mathcal{A} \in \subseteq_*\text{-}\text{Inf}_* \mathcal{A}$.*

The following lemmata on suprema and infima, needed for the formal theory of Dedekind reals, are mostly known in the algebraic setting (see e.g. [1]); here we reconstruct them in the formal theory $Ł\Pi_\omega$. In the rest of this section we drop the $\leq$ sign in $\leq$-Sup_* and $\leq$-Inf_*, and assume all formulae indexed by $*$. We formulate the lemmata only for suprema, omitting their dual versions.

Lemma 3.4 $\mathrm{Sup}\, A = \mathrm{Inf}\, A^{\uparrow}$

Lemma 3.5
$(x \in \mathrm{Sup}\, A \;\&\; y \in \mathrm{Sup}\, A) \to (x \leq y \;\&\; y \leq x)$

Corollary 3.6 *The $*$-suprema w.r.t. $\subseteq_*$ are $\approx_*$-unique. By the extensionality axiom, the element of the kernel of $\subseteq_*$-$\mathrm{Sup}_*\, \mathcal{A}$ is unique w.r.t. identity. (Generally,* 1*-true suprema w.r.t. R are E-unique if R is antisymmetric w.r.t. E.)$_*$*

Lemma 3.7
$(A \subseteq B \;\&\; x \in \mathrm{Sup}\, A \;\&\; y \in \mathrm{Sup}\, B) \to x \leq y$

4 Fuzzy Dedekind reals

In [3] it is shown that any classical nth-order theory can be interpreted in $Ł\Pi_n$ by adding the axioms of crispness of all predicates and functions in the language of the theory. Thus we may assume that in $Ł\Pi_\omega$ we have at our disposal a theory of crisp natural numbers (obtained e.g. by the interpretation of 1st- or 2nd-order Peano arithmetic or any sufficiently strong theory of natural numbers in $Ł\Pi_\omega$). By the standard construction we get integers and rationals as certain pairs of natural numbers, with the usual crisp ordering and operations. Further on we shall therefore presuppose the existence of the class Q of crisp rational numbers, equipped with all usual relations and operations. We shall freely use any classical theorem of the classical theory of rational numbers, as they are provable in $Ł\Pi_\omega$ due to Lemma 41 of [3].

We require the following axioms of Dedekind cuts $A \subseteq \mathrm{Q}$ (which will represent Dedekind reals):

1. $(\forall p, q \in \mathrm{Q})[(p \leq q \to (p \in A \to q \in A)]$
2. $(\forall p \in \mathrm{Q})[(\forall p \in \mathrm{Q})(q > p \to q \in A) \to p \in A]$

The first axiom (which says that A is an upper set) reflects the intuitive motivation (see Section 1) that the membership $p \in A$ of a rational p in the Dedekind fuzzy real A expresses (the truth value of) the fact that p *majorizes* the fuzzy real: thus if $q \geq p$, then *a fortiori* q majorizes A at least in the degree p does.

The second axiom (the right-continuity of the membership function of A) is aimed at excluding the "left-continuous" doppelgangers of cuts with discontinuous membership functions. The reason for this requirement is the same as in classical mathematics, where the set of all cuts must similarly be pruned. Keeping the *left*-closed cuts corresponds to the choice of the informal meaning of $q \in A$ as "$A \leq q$" (rather than "$A < q$").

Definition 4.1 *The (second order) class* R *of* fuzzy Dedekind reals *is the class of all $A \subseteq Q$ that satisfy both axioms 1 and 2 above. (It exists by the comprehension axiom of* $Ł\Pi_3$*.)*

Crisp cuts in R correspond to (all and only) classical real numbers. A crisp cut with the least element q can be identified with the rational number q itself; if the distinction is necessary, we denote the cut by $\overline{q}$. Crisp cuts lacking the least element represent classical irrational numbers; those which are definable can be given the same name as in classical mathematics, e.g. $\sqrt{2} =_{\mathrm{df}} \{q \in \mathrm{Q} \mid q^2 > 2\}$. We denote the empty cut $\emptyset$ by $+\infty$ and the whole Q by $-\infty$.

Zadeh's extension principle does not yield a useful notion of ordering for cumulative distributions (e.g., we would have $A \leq B$ for any crisp $A, B \neq +\infty$, as surely $(\exists p, q \in \mathrm{Q})(Ap \;\&\; Bq \;\&\; p \leq q)$). On the other hand, the usual definition of ordering as inclusion (reversed, as we chose the upper cuts) used in classical Dedekind completions is well-motivated and works well:

Definition 4.2 *Let $A, B \in \mathrm{R}$, then*

$$A \leq_* B \;\equiv_{\mathrm{df}}\; B \subseteq_* A$$

Obviously, $\leq_*$ extends the order on Q, i.e., $(\forall p, q \in \mathrm{Q})(\overline{p} \leq \overline{q} \leftrightarrow p \leq q)$. Moreover, it embodies our original motivation of interpreting $q \in A$ as "$A \leq q$", since it can be proved that for $q \in \mathrm{Q}$ and $A \in \mathrm{R}$,

$$q \in A \;\leftrightarrow\; A \leq_* \overline{q}. \qquad (1)$$

It follows immediately from the properties of inclusion that $\leq_*$ is an $(\approx_*, *)$-ordering, though not linear. Like in classical mathematics, $+\infty$ is the greatest and $-\infty$ the least real.

There are several candidates for the definition of strict ordering $<$ on R. Here we only give one of the strongest $<$-like notions, which is analogous to the intuitionistic relation of apartness:

Definition 4.3 *For $A, B \in \mathrm{R}$,*

$$A \ll B \equiv_{\mathrm{df}} (\exists q)(\Delta Aq \ \&\ \Delta\neg Bq)$$

Reals A such that $-\infty \ll A \ll +\infty$ are bounded, and thus can be called proper *reals.*

Like in classical mathematics, the chief merit of the Dedekind completion is the existence of all suprema and infima:

Theorem 4.4 $\mathcal{A} \subseteq \mathrm{R} \rightarrow \bigcap_* \mathcal{A} \in \mathrm{R}$

From Example 3.3 and Corollary 3.6 it follows that $\bigcap_* \mathcal{A}$ is the unique 1-true $*$-supremum w.r.t. $\leq_*$. On the contrary, $\bigcup_* \mathcal{A}$ need not be in R (it is an upper subset of Q, but not necessarily left-closed). Nevertheless, due to Lemma 3.4, all infima exist in R as well. We shall denote the unique element of $\mathrm{Ker}(\leq_*\text{-}\mathrm{Sup}_* \mathcal{A})$ by $\sup_* \mathcal{A}$ (and similarly for $\inf_* \mathcal{A}$). The suprema and infima that already existed in Q are obviously (since all sets involved are crisp) preserved.

5 Algebraic operations

We only sketch the definitions of addition and multiplication of fuzzy reals.

Since the addition of rationals is monotonous w.r.t. $\leq$, Zadeh's principle yields a well-motivated extension of $+$ to fuzzy reals: if defined as

$$q \in A +_* B \equiv_{\mathrm{df}} (\exists a \in A)_* (\exists b \in B)_* (q = a + b)$$

then $q \in A +_* B$ (i.e. $A +_* B \leq_* q$) is true just as much as Aq and Bq (i.e. $A, B \leq_* q$) guarantee. It can be proved that addition of fuzzy reals is commutative and associative, $\overline{0}$ is the neutral element, and it extends addition of crisp reals.

A similarly straightforward application of Zadeh's principle to multiplication on Q (which is not monotonous w.r.t. $\leq$) would yield a counter-intuitive results. Like in classical Dedekind reals, one must restrict Zadeh's extension to subdomains of rationals where multiplication is monotonous (i.e., positive and negative rationals) and take the union of Zadeh's extensions on these pieces (I omit the details here for space reasons).

A task yet to be done is to define further operations on reals (subtraction, division, exponentiation, etc.) with suitable properties. Preliminary results (to be presented in a subsequent paper) suggest that these tasks are viable.

6 Fuzzy intervals

The formal theory presented in the previous sections can be extended to a theory of fuzzy intervals (often called just 'fuzzy numbers'), of which we give a brief sketch here.

Observe that since no special property of Q has been used, the results of the previous sections hold for the fuzzy Dedekind completion of any crisp poset (in particular, it always yields a fuzzy complete lattice). From the applicational point of view, probably the most useful are fuzzy intervals over crisp reals; further on we shall therefore assume that the crisp numbers (denoted by lowercase variables) are crisp reals instead of rationals (the results, however, again hold for any crisp ordered domain).

By (1), an upper Dedekind cut A is in fact an upper interval $\{q \mid A \leq q\}$. Obviously, the results for upper Dedekind cuts can be dualized for lower cuts as well; thus in the same way, a lower cut B is a lower interval $\{q \mid B \geq q\}$. A fuzzy interval

$$[A, B]_* =_{\mathrm{df}} \{q \mid A \leq q \ \&_*\ q \leq B\}$$

is therefore an intersection of an upper cut A and a lower cut B. In other words, the upper cut A represents the *left endpoint* of an upper interval $[A, +\infty)$; similarly B represents the *right endpoint* of $(-\infty, B]$, and $[A, B]_* = [A, +\infty) \cap_* (-\infty, B]$.

The operations of Section 5 have been motivated by (1); thus they are subject to this 'interval interpretation'. We thus get an algebra of intervals with natural operations induced by the cut operations on the endpoints, e.g.

$$[A, B]_* +_* [C, D]_* =_{\text{df}} [A +_* C, B +_* D]_*$$

The crisp points where the kernel of an interval ends play an important role. In virtue of the lattice completeness of the system of cuts we can define them within the theory:

Definition 6.1 *Let A be an upper cut and B a lower cut. Then we define the upper cut $A^{\leftarrow}$ and the lower cut $B^{\rightarrow}$ as follows:*

$$\begin{aligned} A^{\leftarrow} &=_{\text{df}} \inf\{q \mid \Delta Aq\} \\ B^{\rightarrow} &=_{\text{df}} \sup\{q \mid \Delta Bq\} \end{aligned}$$

(If the system of underlying crisp numbers is a complete lattice, as in the case of crisp reals, the cuts $A^{\leftarrow}$ and $B^{\rightarrow}$ can be identified with the corresponding crisp numbers.)

Observe that in virtue of axiom 2 for Dedekind cuts (Section 4), $A^{\leftarrow}$ is in fact a *minimum* of the kernel of the cut (and dually). These crisp endpoints are preserved by arithmetical operations on cuts (since kernels behave classically in good definitions); thus, e.g., $(X +_* Y)^{\leftarrow} = X^{\leftarrow} + Y^{\leftarrow}$. (On the other hand, one can easily find counterexamples to $X \leq_* Y \rightarrow X^{\leftarrow} \leq_* Y^{\leftarrow}$ or the converse; only $\Delta(X \leq Y) \rightarrow X^{\leftarrow} \leq Y^{\leftarrow}$ holds.)

It can be observed that a fuzzy interval is normal iff $A^{\leftarrow} \leq B^{\rightarrow}$. In such a case the membership function of $[A, B]$ is that of A on $(-\infty, A^{\leftarrow}]$, that of B on $[B^{\rightarrow}, +\infty)$, and 1 on $[A^{\leftarrow}, B^{\rightarrow}]$.

If $A^{\leftarrow} = B^{\rightarrow}$, then there is exactly one element in the kernel of $[A, B]$. We will call such degenerate intervals *fuzzy points*. Due to the axioms for Dedekind cuts, fuzzy points satisfy the most usual requirements on 'fuzzy real numbers' (singleton kernel, convexity of cuts, monotony of membership function towards the central point). Conforming to the tradition of fuzzy mathematics, we can therefore (ambiguously, but intelligibly) denote representatives of the (crisp) equivalence class $\{[A, B] \mid A^{\leftarrow} = B^{\rightarrow} = r\}$ by $\widetilde{r}$.

The set of all fuzzy points is closed under usual arithmetical operations (since, as stated above, they preserve the crisp endpoints of cuts). Furthermore, their arithmetics (sketched above) extends the arithmetics of crisp numbers (thus, e.g., $\widetilde{1} + \widetilde{1} = \widetilde{2}$). However, the arithmetics of fuzzy points differs somewhat from the traditional arithmetics of fuzzy intervals, as our operations are defined separately for the upper and lower endpoints of fuzzy intervals. It is beyond the scope of this short paper to argue why this is well-motivated; it will be elaborated in more details in a separate paper. At present we only propose this new formal theory of fuzzy intervals and fuzzy points (or, "fuzzy numbers") for further study and for trying it in applications.

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