Number-free Mathematics Based on T-norm Fuzzy Logic

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Abstract—*The paper presents a form of rendering classical mathematical notions by formal theories over suitable t-norm fuzzy logics in such a way that references to real numbers are eliminated from definitions and theorems, being removed to the standard semantics of fuzzy logic. Several examples demonstrate how this move conceptually simplifies the theory in exchange for non-classical reasoning, facilitates certain generalizations, and puts the concepts into a different perspective. The formal framework employed for the number-free formalization of mathematical concepts is that of higher-order fuzzy logic, also known as Fuzzy Class Theory.*

Keywords—Continuity, Fuzzy Class Theory, limit, metric, real-valued function, similarity.

1 Introduction

In the standard semantics of t-norm fuzzy logics [1], truth values are represented by real numbers from the unit interval $[0, 1]$; truth functions of n-ary propositional connectives are interpreted as certain real-valued functions on $[0, 1]^n$; and the quantifiers $\exists, \forall$ as the suprema and infima of sets of truth values. Read inversely, the logical apparatus of t-norm fuzzy logics expresses, by means of its standard semantics, certain first- and second-order constructions over real numbers. The axioms and rules of t-norm logics are designed to capture basic properties of such constructions and ensure the soundness of formally derivable theorems expressible in the language.

A suitable formal theory in first-order fuzzy logic thus can, by means of its standard semantics, express facts about classical mathematical notions and make them derivable by logical deductions in fuzzy logic. Since real numbers appear in the standard semantics of such a theory, they need no longer be explicitly mentioned by the theory itself. A notion of classical mathematics then becomes represented as the standard model of another notion of formal fuzzy mathematics that makes no explicit reference to real numbers: the reals are only implicitly present behind the logical axioms that govern reasoning in formal fuzzy mathematics.

This way of eliminating real numbers from the theory in favor of reasoning by means of t-norm fuzzy logic will here be called the *number-free* (or by analogy with "pointless topology", *numberless*) approach.

Number-free formalization of mathematical concepts is not new and has implicitly been around since the beginning of the theory of fuzzy sets: in fact, the notion of fuzzy set itself can be understood as a number-free rendering of the notion of real-valued function (see Section 2). However, it was only after the advancement of t-norm fuzzy logics, mostly in the past decade, that number-free notions could be treated rigorously in formal theories over first-order fuzzy logics. An early example of the number-free treatment of a classical notion is the formalization of finitely additive probability as a modality *Probable* in Łukasiewicz logic (see Section 3). Number-free rendering of more advanced mathematical notions, however, requires more complex concepts of formal fuzzy mathematics—esp. higher-order set-constructions and a formal theory of fuzzy relations. The latter prerequisites have only recently been developed in the framework of higher-order fuzzy logic [2, 3, 4], which made it possible to apply the number-free approach systematically to various classical mathematical notions.

The profit we gain under the number-free approach in exchange for having to use non-classical rules of reasoning is, in the first place, conceptual simplification (roughly speaking, we get 'a set for a function'). Secondly, the number-free rendering often reveals a new perspective upon the notion, exposing the gradual quality of the classical construct and rendering it as a primitive rather than derivative feature. Thirdly, many theorems of classical mathematics are under this approach detected as provable by simple (often, propositional) logical derivations in a suitable fuzzy logic, instead of complex classical proofs involving arithmetic, infima, functions, etc. Furthermore, adopting a non-standard semantics (e.g., taking Chang's MV-algebra instead of the standard real numbers) or a different interpretation of the logical symbols involved (e.g., taking another t-norm for conjunction) yields an effortless generalization that might be harder to find (and motivate) in the classical language of crisp mathematics. Finally, the many-valuedness of all formulae in fuzzy logic makes it possible to consider another kind of graded generalization, by admitting partial satisfaction of the axioms for the represented notion (e.g., a metric to degree .99, cf. [5, 6]).

The particular formal framework in which number-free formalization of classical mathematical concepts is carried out in this paper is that of higher-order fuzzy logic, also known as Fuzzy Class Theory (FCT), over a suitable propositional fuzzy logic at least as strong as $\mathrm{MTL}_\triangle$. A working knowledge of FCT will be assumed throughout; for an introduction to the theory and more information see [2, 5]. For reference, the definitions used in the present paper are repeated below in the Appendix.

The aim of this paper is only to introduce the number-free approach as a distinct paradigm of formalization, rather than to develop particular number-free theories in depth. Therefore it only gives definitions of and a few observations on several number-free notions and discusses the merits of such formalization; a detailed investigation of number-free theories is left for future work. A slightly more eloquent version of this paper has been made available as a research report [7].

2 Real-valued functions

The very first notion of fuzzy mathematics, namely Zadeh's notion of fuzzy set [8], can be regarded as a number-free representation of classical $[0, 1]$-valued functions by non-classical (namely, fuzzy) sets. Even though the formal apparatus of first-order fuzzy logic, which makes it possible to cast fuzzy sets as a primitive notion instead of representing them by classical real-valued functions, was developed years later, the tendency of regarding fuzzy sets and relations as a number-free rendition of real-valued functions has partly been present since the very beginning of the fuzzy set theory, as witnessed by the vocabulary and notation employed. E.g., the function $x \mapsto \min(A(x), B(x))$ is in the traditional fuzzy set theory denoted by $A \cap B$ and called the *intersection* of A and B: that is, the functions $A, B\colon X \to [0, 1]$ are regarded as (non-classical) *sets* rather than real-valued functions (as the intersection of real-valued functions is a different thing). Similarly, such notions as fuzzy relational composition or the image under a fuzzy relation would make little sense if the n-ary functions involved were not regarded as (a non-classical kind of) relations. The terminological shift towards the number-free discourse is expressed by the very term "fuzzy set" and its informal motivation of an unsharp collection of elements.

A certain part of the talk about *real-valued functions* and their properties was thus replaced by a talk about *sets* and *relations* that behave non-classically (e.g., do not follow the rule of excluded middle). This move eliminated references to numbers at least from the wording of some theorems, giving them compact forms and new conceptual meanings.[1] Clearly, this original number-free notion has proved immensely fruitful even in its semi-formal form of traditional fuzzy mathematics. The formal apparatus of logic-based fuzzy mathematics has provided means for accomplishing the long-present idea and developing a fully fledged number-free approach to fuzzy sets and fuzzy relations.[2]

3 Finitely additive probability measures

Another number-free representation, already based on formal fuzzy logic, was the axiomatization of finitely additive probability measures as models of a fuzzy modality *Probable* over propositional Łukasiewicz logic by Hájek, Godo, and Esteva [10]. Later it was elaborated in a series of papers by Flaminio, Marchioni, Montagna, and the authors of [10]. We shall briefly recapitulate the original axiomatization (adapted from [11]) as another illustration of the number-free approach.

Consider a classical probability space $(\Omega, \mathcal{B}, \pi)$, where Ω is a set of elementary events, $\mathcal{B}$ a Boolean algebra of subsets of Ω, and π a finitely additive probability measure on $\mathcal{B}$, i.e., a function $\pi\colon \mathcal{B} \to [0, 1]$ satisfying the following conditions:

$$\pi(\Omega) = 1$$
If $A \subseteq B$, then $\pi(A) \le \pi(B)$
If $A \cap B = \emptyset$, then $\pi(A \cup B) = \pi(A) + \pi(B)$

A number-free representation of π draws on the fact that a $[0, 1]$-valued function on a Boolean algebra can be understood as the standard model of a fuzzy modality P over an algebra of crisp propositions. The above conditions on π can be transformed into the axioms for P, which (due to the additivity) are expressible in Łukasiewicz logic:[3]

Definition 3.1. The axioms and rules of the logic FP(Ł) are those of Łukasiewicz propositional logic plus the following axioms and rules, for non-modal φ, ψ:

$$\varphi \vee \neg\varphi$$
From φ infer $P\varphi$
$P\varphi$, for all Boolean tautologies φ
$$P(\varphi \to \psi) \to (P\varphi \to P\psi)$$
$$P(\neg\varphi) \leftrightarrow \neg P\varphi$$
$$P(\varphi \vee \psi) \leftrightarrow ((P\varphi \to P(\varphi \wedge \psi)) \to P\psi)$$

The axioms and rules of FP(Ł) ensure the following representation theorem (adapted from [10]):

Theorem 3.2. *Any probability space* $(\Omega, \mathcal{B}, \pi)$ *is a standard model of* FP(Ł). *Vice versa, all standard models of* FP(Ł) *are probability spaces.*

The representation theorem shows that the number-free theory faithfully captures the original notion of finitely additive probability measure. Moreover, by the completeness theorem of FP(Ł) w.r.t. probability spaces proved in [11], all valid laws of finitely additive probability that are expressible in the language of FP(Ł) can in FP(Ł) be also (number-freely) proved.

In a given probability space $(\Omega, \mathcal{B}, \pi)$, i.e., a standard model of FP(Ł) with $\|P\| = \pi$, the truth value of $P\varphi$ is the probability of the event φ: $\|P\varphi\| = \pi(\|\varphi\|)$; the formula $P\varphi$ can therefore be understood as "φ is probable". Numerical calculations with probabilities are thus in FP(Ł) replaced by *logical derivations* with the modality "is probable". The key difference is that the latter represent inference *salva probabilitate* (i.e., salvo *probabilitatis gradu,* in the sense of P): e.g., it can be observed that numberless probability is transmitted by modus ponens, as $P\varphi \mathbin{\&} P(\varphi \to \psi) \to P\psi$, i.e., "if φ is probable and $\varphi \to \psi$ is probable, then ψ is probable", is a theorem of FP(Ł).

The number-free approach to probability facilitates several kinds of generalization. First, generalizations to models over non-standard algebras for Łukasiewicz logic: thus we can have, e.g., probability valued in Chang's MV-algebra, or in non-standard reals (as in [12]). Second, a generalization to measures with only partially satisfied additivity (by a many-valued interpretation of the axioms, see [6]). And third, a generalization to the probability of fuzzy events, where one discards the axiom $\varphi \vee \neg\varphi$ for events and adapts the finite additivity axiom to work well with fuzzy events (e.g., as $P(\varphi \oplus \psi) \leftrightarrow ((P\varphi \to P(\varphi \mathbin{\&} \psi)) \to P\psi)$).[4]

[1] The elimination of numbers also from proofs would have required a consistent use of first-order fuzzy logic. This approach was not embraced in the early works on fuzzy set theory, even though particular first-order fuzzy logics already existed, e.g., [9].

[2] Cf. [2, 3], where the formal theory of fuzzy sets and relations is developed without making any reference to real numbers in definitions, theorems, or proofs (only in explanatory semantic examples).

[3] Ł with rational truth constants is used in [10], but the truth constants are inessential for our account. The language is two-layered, admitting only non-modal formulae and propositional combinations of non-nested modal formulae.

[4] This approach has been taken in [13], though only over finitely-valued events, as the authors strove for the completeness of the logic; in [12] this was generalized to infinitely-valued events, with completeness w.r.t. non-standard reals.

4 Distribution functions

Classical distribution functions present a special way how to define a probabilistic measure on Borel sets, i.e., on the σ-algebra $\mathcal{B}$ of subsets of the real line generated by all intervals $(-\infty, x]$. A function $f\colon \mathbb{R} \to [0,1]$ defines a measure on $\mathcal{B}$ with $\mu(-\infty, x] = f(x)$ and $\mu(\mathbb{R}) = 1$ iff it satisfies the following conditions, which can thus be taken as the axioms for distribution functions:

1. *Monotony:* if $x \leq y$ then $f(x) \leq f(y)$, for all $x, y \in \mathbb{R}$
2. *Margin conditions:* $\lim\limits_{x\to-\infty} f(x) = 0, \ \lim\limits_{x\to+\infty} f(x) = 1$
3. *Right-continuity:* $\lim\limits_{x\to x_0^+} f(x) = f(x_0)$ for all $x_0 \in \mathbb{R}$

Let us translate these conditions into the number-free language. The function $f\colon \mathbb{R} \to [0,1]$ represents a standard fuzzy set of reals, i.e., in FCT over any expansion of MTL_Δ, the standard model of a predicate A on reals.[5] By the standard semantics of MTL_Δ, the above conditions translate into the following axioms on the predicate A:[6]

$$(\forall xy)(x \leq y \to (Ax \to Ay))$$
$$(\exists x)Ax, \ \neg(\forall x)Ax$$
$$(\forall x_0)[(\forall x > x_0)Ax \to Ax_0]$$

In FCT, these conditions express, respectively, the *upperness* of A in $\mathbb{R}$, the *full height* and *null plinth* of A, and the *left-closedness* of the fuzzy upper class A. Consequently, number-free distribution functions are left-closed upper sets in $\mathbb{R}$ with full height and null plinth, i.e., (weakly bounded) *fuzzy Dedekind cuts on* $\mathbb{R}$. The number-free rendering of distribution functions as fuzzy Dedekind cuts corresponds to the known fact that distribution functions represent Hutton fuzzy reals (cf., e.g., [15]). A use of fuzzy Dedekind cuts for the development of a logic-based theory of fuzzy intervals (or fuzzy numbers) is hinted at in [16].

5 Continuous functions on reals

In Section 4 we abused the presence of monotony for the number-free rendering of right-continuity. However, if we want to develop a graded theory with monotony satisfied to partial degrees, we need a different number-free characterization of continuity that does not rely on monotony. Again we shall work with functions $\mathbb{R} \to [0,1]$ only, even though various generalizations are easy to obtain.

For a number-free rendering of left-continuity, we shall use the following classical characterization. A function $f\colon \mathbb{R} \to [0,1]$ is left-continuous in x_0 iff

$$\limsup_{x\to x_0^-} f(x) = \liminf_{x\to x_0^-} f(x) = f(x_0), \text{ where}$$
$$\limsup_{x\to x_0^-} f(x) = \inf_{x_1<x_0} \sup_{x_1<x<x_0} f(x)$$
$$\liminf_{x\to x_0^-} f(x) = \sup_{x_1<x_0} \inf_{x_1<x<x_0} f(x).$$

This translates into the following number-free definitions in FCT over MTL_Δ:[7]

$$\mathrm{LimSup}^-(A, x_0) \equiv_{\mathrm{df}} (\forall x_1 < x_0)(\exists x \in (x_1, x_0))Ax$$
$$\mathrm{LimInf}^-(A, x_0) \equiv_{\mathrm{df}} (\exists x_1 < x_0)(\forall x \in (x_1, x_0))Ax$$
$$\Delta\,\mathrm{Cont}^-(A, x_0) \equiv_{\mathrm{df}} (Ax_0 = \mathrm{LimSup}^-(A, x_0))\ \&\ (Ax_0 = \mathrm{LimInf}^-(A, x_0))$$

The right-sided predicates LimSup^+ and LimInf^+ are defined dually (with $>$ for $<$), and the both-sided ones as

$$\mathrm{LimSup}(A, x_0) \equiv_{\mathrm{df}} \mathrm{LimSup}^-(A, x_0) \vee \mathrm{LimSup}^+(A, x_0)$$
$$\mathrm{LimInf}(A, x_0) \equiv_{\mathrm{df}} \mathrm{LimInf}^-(A, x_0) \wedge \mathrm{LimInf}^+(A, x_0).$$

These definitions reconstruct the classical notions in a number-free way in the framework of FCT; the representation theorems follow directly from the above considerations and the standard semantics of MTL_Δ. The following observation shows that many properties of the classical notions can be reconstructed in FCT as well.

Observation 5.1. By shifts of crisp relativized quantifiers valid in first-order MTL_Δ, the following theorems are easily provable in FCT:[8]

1. $\mathrm{LimInf}(A, x_0) \leq \mathrm{LimSup}(A, x_0)$
2. $A \subseteq B \to (\mathrm{LimSup}(A, x_0) \to \mathrm{LimSup}(B, x_0))$
   and analogously for LimInf.
3. $\mathrm{LimInf}(\backslash A, x_0) \leq \neg\,\mathrm{LimSup}(A, x_0)$
   $\mathrm{LimSup}(\backslash A, x_0) \geq \neg\,\mathrm{LimInf}(A, x_0)$
   (Equality holds in logics with involutive negation, but not generally in MTL_Δ.)
4. $\mathrm{LimInf}(A \sqcap B, x_0) = \mathrm{LimInf}(A, x_0) \wedge \mathrm{LimInf}(B, x_0)$
   $\mathrm{LimInf}(A \sqcup B, x_0) = \mathrm{LimInf}(A, x_0) \vee \mathrm{LimInf}(B, x_0)$
   and analogously for LimSup.

Since the definitions reconstruct classical notions, we have retained the classical terminology and notation referring to limits and continuity, even though these regard membership functions (i.e., semantic *models* of fuzzy classes), rather than fuzzy classes themselves. In FCT, the fuzzy predicate $\mathrm{LimSup}^-(A, x_0)$ actually expresses the condition that x_0 is a *left-limit point* of the fuzzy class A, and $\mathrm{LimInf}^-(A, x_0)$ that x_0 is an *interior point* of $A \cup \{x_0\}$ in the right half-open interval topology,[9] as these are the properties expressed by the defining formulae if all sets involved are crisp. Consequently, the formulae $(\forall x_0 \in A)\,\mathrm{LimInf}(A, x_0)$ and $(\forall x_0 \in A)\,\mathrm{LimSup}(A, x_0)$ express the notions of *openness* resp. *closedness* of A in a fuzzy interval topology on $\mathbb{R}$. The study of this fuzzy topology and its relationship to the fuzzy interval topologies of [17, 18] is left for future work.

Similarly as the definitions of LimInf and LimSup, also the theorems of Observation 5.1 have double meanings. On

[5] Recall that $\mathbb{R}$ as well as other crisp mathematical structures are available in FCT by means of the Δ-interpretation, see [2, §7] and [14, §4]. In this and the next section, we shall understand all first-order quantifications relativized to $\mathbb{R}$, unless specified otherwise.

[6] We use the fact that due to the monotony assumed, the margin conditions reduce to $\inf_x f(x) = 0$ and $\sup_x f(x) = 1$, and the right-continuity to $f(x_0) \geq \inf_{x>x_0} f(x)$. The representation theorem is then immediate by the standard semantics of MTL_Δ. Observe that as the axioms are required to degree 1, they are (due to the crispness of $\leq$) independent of the particular left-continuous t-norm used.

[7] Again it can be observed that the definitions are independent of a particular t-norm and are the same in all expansions of MTL_Δ.

[8] The theorems are stated for both-sided limits only, but hold equally well for one-sided limits.

[9] I.e., the topology with the open base of all half-open intervals $(a, b]$, also known as the upper-limit topology or the Sorgenfrey line.

the one hand they can be understood as number-free reconstructions and graded generalizations of the classical theorems on (membership) functions. On the other hand, they can be interpreted as fuzzy-mathematical theorems on (fuzzy) sets, under the above fuzzy interval topology on reals. In particular, 1. says that an interior point of a fuzzy set of reals is also its limit point; 2. that a limit point of a fuzzy set is also a limit point of a larger fuzzy set (and dually for interior points); 3. that an interior point of the complement of a fuzzy set A is not a limit point of A (and vice versa); and 4. that x_0 is a limit point of $A \sqcap B$ exactly to the degree it is a limit point of A *and* ($\wedge$) a limit point of B (and dually for $\sqcup$). (The theorems are graded, 'is' therefore represents fuzzy implication $\rightarrow$.)

6 Operations on reals

In the previous examples, only the codomain of real-valued functions of reals was rendered numberless. Obviously, the domain $\mathbb{R}$ (or more conveniently, $\mathbb{R} \cup \{\pm\infty\}$) can be re-scaled into $[0,1]$ and regarded as the standard set of truth degrees as well. The functions $\mathbb{R}^n \to \mathbb{R}$ then become n-ary functions from truth values to truth values, i.e., truth functions of fuzzy-logical connectives.

An apparatus for internalizing truth values and logical connectives in FCT was developed in [4, §3]. As shown there, the truth values can be internalized as the elements of the crisp class $\mathrm{L} = \mathrm{Ker}\,\mathrm{Pow}\{a\}$, i.e., subclasses of a fixed crisp singleton. The class L of internalized truth values is ordered by crisp inclusion $\subseteq^{\Delta}$, and the correspondence between internal and semantical truth values is given as follows:[10] $\alpha \in \mathrm{L}$ corresponds to the semantic truth value of $\emptyset \in \alpha$, and the semantic truth value of φ is represented by the class $\overline{\varphi} =_{\mathrm{df}} \{a \mid \varphi\}$; the correspondences $\varphi \leftrightarrow (a \in \overline{\varphi})$ and $\overline{\varphi} \subseteq \overline{\psi} \leftrightarrow (\varphi \rightarrow \psi)$ then hold.

Logical connectives are then internalized by crisp functions $\mathbf{c}\colon \mathrm{L}^n \to \mathrm{L}$ (which can be called *internal, inner,* or *formal* connectives). In particular, definable connectives c of the logic are represented by the corresponding class operations $\overline{c} = \{x \in \mathrm{L} \mid c(x \in X_1, \ldots, x \in X_n)\}$ on L (e.g., & by $\cap$, $\vee$ by $\sqcup$, etc.). Since the n-ary internal connectives are crisp functions valued in L, they can as well be regarded as fuzzy subsets of L^n, i.e., n-ary fuzzy relations on L. Usual fuzzy class operations then apply to them, making their theory graded: e.g., the graded inclusion

$$\mathbf{c} \subseteq \mathbf{d} \equiv_{\mathrm{df}} (\forall x_1 \ldots x_n)(\mathbf{c}(x_1, \ldots, x_n) \rightarrow \mathbf{d}(x_1, \ldots, x_n)).$$

The number-free theory of functions $\mathbb{R}^n \to \mathbb{R}$ is thus the fuzzy-logical theory of internal connectives, i.e., fuzzy relations on internal truth values. Usual generalizations are available (e.g., taking Chang's MV-algebra instead of reals), to which number-freely provable theorems on real functions automatically transfer.

An elaboration of the theory of unary and binary internal connectives has been sketched in [19, 20]. These preliminary papers focus on the defining properties of *t-norms* (i.e., monotony, commutativity, associativity, and the unit) and the relation of domination between internal connectives, making them graded by reinterpretation of their defining formulae in fuzzy logic (cf. [2, §7], [5, §2.3], or [6, §4]) and studying their graded properties. A full paper on the topic (by the authors of [20]) is currently under construction.

7 Metrics

Recall that a *pseudometric*[11] on a set X is a function $d\colon X^2 \to [0,+\infty]$ such that

$$d(x,x) = 0$$
$$d(x,y) = d(y,x)$$
$$d(x,z) \le d(x,y) + d(y,z)$$

The numberless reduction will first need to normalize the range of pseudometrics from $[0,+\infty]$ to $[0,1]$, e.g., setting

$$c(x,y) = 2^{-d(x,y)} \qquad (1)$$

The defining conditions on pseudometric then become the following equivalent conditions on c:[12]

$$c(x,x) = 1$$
$$c(x,y) = c(y,x)$$
$$c(x,z) \ge c(x,y) \cdot c(y,z)$$

These conditions are nothing else but the defining conditions of *fuzzy equivalences,* also known as *similarity relations* [21], in the standard semantics of product fuzzy logic [1]. We can thus equate number-free pseudometrics with *product similarities,* i.e., standard models of the following axioms in product fuzzy logic:

$$Cxx$$
$$Cxy \rightarrow Cyx$$
$$Cxy \,\&\, Cyz \rightarrow Cxz$$

The definition of a *metric* strengthens the first condition to $d(x,y) = 0$ iff $x = y$, which is equivalent to c being a *fuzzy equality* (also called *separated* similarity), i.e., $c(x,y) = 1$ iff $x = y$, thus replacing the first axiom by $Cxy \leftrightarrow x = y$.

Using a different left-continuous t-norm represents different ways of combining the distances $d(x,y)$ and $d(y,z)$ in the triangle inequality: e.g., with the minimum t-norm, c represents a *(pseudo)ultrametrics* under the same transformation (1), while with the Łukasiewicz t-norm, c represents a *bounded* pseudometric d under a different transformation $c(x,y) = (1 - d(x,y))/d_{\max}$, where $d_{\max} < +\infty$ is an upper bound on the distances. (The obvious both-way representation theorems are left to the reader.) Since furthermore many theorems on on number-free metrics hold generally over MTL_{Δ}, it is quite natural to generalize the notion of number-free metrics to any similarity, not only the product one.

Various notions based on such (generalized) number-free metrics can be defined and their properties investigated in the framework of FCT (over MTL_{Δ} or stronger). Only a few observations on number-free limits are given here as a further illustration of the numberless approach.

[10] However, see [4, Rem. 3.3] for certain metamathematical qualifications regarding this correspondence.

[11] For simplicity, we shall work with *extended* pseudometrics, allowing the value $+\infty$.

[12] Notice that since the function 2^{-x} reverses the order, the fuzzy relation $c\colon X^2 \to [0,1]$ expresses *closeness* rather than distance.

Fix a metric d rendered in the numberless way by a closeness predicate C under the transformation (1). The *limit* $\lim_{n\to\infty} x_n$ of a sequence $\{x_n\}_{n\in\mathbb{N}}$ (abbreviated $\vec{x}$) under C can be defined as follows:[13]

$$\Delta\mathrm{Lim}_C(\vec{x}, x) \equiv_{\mathrm{df}} \Delta(\exists n_0)(\forall n > n_0)Cxx_n \qquad (2)$$

Theorem 7.1. *Standard models over product logic validate* $\Delta\mathrm{Lim}_C(\vec{x}, x)$ *iff* $x = \lim_{n\to\infty} x_n$ *under* d*.*

Proof: $\lim x_n = x$ under d iff $\limsup d(x, x_n) = 0$, iff $\liminf 2^{-d(x,x_n)} = 1$, iff $\sup_{n_0} \inf_{n>n_0} c(x, x_n) = 1$, which is the semantics of $\Delta\mathrm{Lim}_C(\vec{x}, x)$. □

As noted above, the meaning of $\Delta\mathrm{Lim}_C$ is natural not only in product logic, but also in other t-norm logics, esp. if the relation C is interpreted as *indistinguishability* rather than mere closeness: then (2) expresses the condition that *from somewhere on,* x_n *is indistinguishable from* x. Similarly, Theorem 7.2 below expresses the fact that all limits of $\vec{x}$ are indistinguishable (to the degree the indistinguishability relation is symmetric and transitive).

Discarding the Δ in (2) yields a graded number-free notion of limit:[14]

$$\begin{aligned} \mathrm{Lim}_C(\vec{x}, x) &\equiv_{\mathrm{df}} (\exists n_0)(\forall n > n_0)Cxx_n \\ \lim_C \vec{x} &=_{\mathrm{df}} \{x \mid \mathrm{Lim}_C(\vec{x}, x)\} \\ \mathrm{Convg}_C(\vec{x}) &\equiv_{\mathrm{df}} (\exists x)\,\mathrm{Lim}_C(\vec{x}, x), \text{ i.e., } \mathrm{Hgt}(\lim_C \vec{x}) \end{aligned}$$

Interestingly, $\mathrm{Lim}_C(\vec{x}, x)$ coincides with G. Soylu's notion of *similarity-based fuzzy limit* [22]. Even without employing explicitly the formalism of t-norm fuzzy logic, the author was able to prove graded theorems such as [22, Prop. 3.5],

$$\mathrm{Lim}_C(\vec{x}, x) \ \&\ \mathrm{Lim}_C(\vec{y}, y) \to \mathrm{Lim}_C(\vec{x} + \vec{y}, x + y).$$

With the apparatus of FCT, the gradedness of Soylu's results can be extended even further by not requiring the full satisfaction of the defining properties of the similarity C (this conforms to the standard methodology of constructing graded theories [6, §7]). An example of such graded results is the following theorem on the fuzzy uniqueness of the limit:

Theorem 7.2. FCT *over* MTL_Δ *proves:*

$$\mathrm{Sym}\, C \ \&\ \mathrm{Trans}\, C \ \&\ \mathrm{Lim}_C(\vec{x}, x) \ \&\ \mathrm{Lim}_C(\vec{x}, x') \to Cxx'$$

Proof. By $\mathrm{Trans}\, C$ we obtain $Cxx_n \,\&\, Cx_nx' \to Cxx'$; thus $Cx_nx \,\&\, Cx_nx' \to Cxx'$ by $\mathrm{Sym}\, C$, whence

$$((n > n_0) \to Cx_nx) \ \&\ ((n > n_0) \to Cx_nx') \to Cxx'$$

follows propositionally. By generalization on n and distribution of the quantifier,

$$(\forall n > n_0)Cx_nx \ \&\ (\forall n > n_0)Cx_nx' \to Cxx'$$

is obtained (as in the consequent the quantification is void). Generalization on n_0 and the shift of the quantifier to the antecedent (as $\exists$) then yields the required formula. □

A more detailed investigation of convergence based on fuzzy indistinguishability in the formal framework of FCT exceeds the scope of the present paper, and is therefore left for future research.

[13] The Δ in $\Delta\mathrm{Lim}_C$ refers to the Δ in the defining formula, which will later be dropped.

[14] Observe that Lim_C is a Σ_2-formula: compare it with the classical Π_3-definition and the Π_1-definition in non-standard analysis.

Appendix: Fuzzy Class Theory

Fuzzy Class Theory (FCT) is a formal theory aimed at giving an axiomatic approximation of Zadeh's fuzzy sets of all orders over a fixed crisp domain. It can be characterized as Henkin-style higher-order fuzzy logic, or fuzzified Russell-style simple type theory. FCT can be regarded as a foundational theory for fuzzy mathematics [23], as other axiomatic mathematical theories over fuzzy logic can be formalized within its framework. For more details on FCT see [2, 5]; the relevant definitions of [5] are briefly repeated here for reference.

The reader's familiarity with the logic MTL_Δ and its main extensions is assumed; for details on these logics see [1, 24]. Here we only recapitulate its standard real-valued semantics, which is crucial for number-free mathematics:

$\&$	...	a left-continuous t-norm $*$
$\to$	...	the residuum $\Rightarrow$ of $*$, defined as $x \Rightarrow y =_{\mathrm{df}} \sup\{z \mid z * x \le y\}$
$\wedge, \vee$	...	min, max
$\neg$	...	$x \Rightarrow 0$
$\leftrightarrow$	...	the bi-residuum: $\min(x \Rightarrow y, y \Rightarrow x)$
Δ	...	$\Delta x = 1 - \mathrm{sgn}(1 - x)$
$\forall, \exists$	...	inf, sup

Łukasiewicz logic further specifies $x * y = (x + y - 1) \vee 0$, product logic sets $x * y = x \cdot y$, and Gödel logic sets $x * y = x \wedge y$.

Fuzzy Class Theory FCT is a formal theory over a given multi-sorted first-order fuzzy logic $\boldsymbol{L}$ (at least as strong as MTL_Δ), with sorts of variables for: atomic objects (lowercase letters $x, y, \dots$), fuzzy classes of atomic objects (uppercase letters $A, B, \dots$), fuzzy classes of fuzzy classes of atomic objects (calligraphic letters $\mathcal{A}, \mathcal{B}, \dots$), etc., in general for fuzzy classes of the n-th order ($X^{(n)}, Y^{(n)}, \dots$).

Besides the crisp identity predicate $=$, the language of FCT contains:

- The membership predicate $\in$ between objects of successive sorts
- The class terms $\{x \mid \varphi\}$ of order $n + 1$, for any variable x of any order n and any formula φ
- The symbols $\langle x_1, \dots, x_k\rangle$ for k-tuples of individuals $x_1, \dots, x_k$ of any order

In formulae of FCT we employ usual abbreviations and defined notions known from classical mathematics or traditional fuzzy mathematics, including those listed in Table 1, for all orders of fuzzy classes.

FCT has the following axioms, for all formulae φ and variables of all orders:

- The logical axioms of multi-sorted first-order logic $\boldsymbol{L}$
- The axioms of crisp identity: $x = x$; $x = y \ \&\ \varphi(x) \to \varphi(y)$; and $\langle x_1, \dots, x_k\rangle = \langle y_1, \dots, y_k\rangle \to x_i = y_i$
- The comprehension axioms: $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$
- The extensionality axioms: $(\forall x)(Ax = Bx) \to A = B$

Table 1: Abbreviations and defined notions of FCT

$\varphi = \psi$	$\equiv_{\mathrm{df}}$	$\Delta(\varphi \leftrightarrow \psi)$
$\varphi \leq \psi$	$\equiv_{\mathrm{df}}$	$\Delta(\varphi \rightarrow \psi)$
Ax	$\equiv_{\mathrm{df}}$	$x \in A$
$x_1 \dots x_k$	$=_{\mathrm{df}}$	$\langle x_1, \dots, x_k \rangle$
$(\forall x \in A)\varphi$	$\equiv_{\mathrm{df}}$	$(\forall x)(x \in A \rightarrow \varphi)$
$(\exists x \in A)\varphi$	$\equiv_{\mathrm{df}}$	$(\exists x)(x \in A \mathbin{\&} \varphi)$
$\{x \in A \mid \varphi\}$	$=_{\mathrm{df}}$	$\{x \mid x \in A \mathbin{\&} \varphi\}$
$\emptyset$	$=_{\mathrm{df}}$	$\{x \mid 0\}$
$\mathrm{Ker}\, A$	$=_{\mathrm{df}}$	$\{x \mid \Delta Ax\}$
$\backslash A$	$=_{\mathrm{df}}$	$\{x \mid \neg Ax\}$
$A \cap B$	$=_{\mathrm{df}}$	$\{x \mid Ax \mathbin{\&} Bx\}$
$A \sqcap B$	$=_{\mathrm{df}}$	$\{x \mid Ax \wedge Bx\}$
$A \sqcup B$	$=_{\mathrm{df}}$	$\{x \mid Ax \vee Bx\}$
$\mathrm{Pow}\, A$	$=_{\mathrm{df}}$	$\{X \mid X \subseteq A\}$
$\mathrm{Hgt}\, A$	$\equiv_{\mathrm{df}}$	$(\exists x)Ax$
$\mathrm{Plt}\, A$	$\equiv_{\mathrm{df}}$	$(\forall x)Ax$
$\mathrm{Crisp}\, A$	$\equiv_{\mathrm{df}}$	$(\forall x)\Delta(Ax \vee \neg Ax)$
$\mathrm{Sym}\, R$	$\equiv_{\mathrm{df}}$	$(\forall xy)(Rxy \rightarrow Rzx)$
$\mathrm{Trans}\, R$	$\equiv_{\mathrm{df}}$	$(\forall x, y, z)(Rxy \mathbin{\&} Ryz \rightarrow Rxz)$
$\mathrm{Fnc}\, R$	$\equiv_{\mathrm{df}}$	$(\forall x, y, y')(Rxy \mathbin{\&} Rxy' \rightarrow y = y')$
$A \subseteq B$	$\equiv_{\mathrm{df}}$	$(\forall x)(Ax \rightarrow Bx)$
$A \subseteq^{\Delta} B$	$\equiv_{\mathrm{df}}$	$(\forall x)(Ax \leq Bx)$

Models of FCT are systems of fuzzy sets (and fuzzy relations) of all orders over a crisp universe of discourse, with truth degrees taking values in an $\boldsymbol{L}$-chain $\mathbf{L}$ (e.g., the interval $[0, 1]$ equipped with a left-continuous t-norm); thus all theorems on fuzzy classes provable in FCT are true statements about $\mathbf{L}$-valued fuzzy sets. Note, however, that the theorems of FCT are derived from its axioms by the rules of the fuzzy logic $\boldsymbol{L}$ rather than classical Boolean logic. For details on proving theorems in FCT see [5] or [25].

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