# Interior-Based Topology in Fuzzy Class Theory

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## Abstract

Fuzzy topology based on interior operators is studied in the fully graded framework of Fuzzy Class Theory. Its relation to graded notions of fuzzy topology given by open sets and neighborhoods is shown.

**Keywords:** Fuzzy topology, Fuzzy Class Theory, interior operator, neighborhood.

## 1 Introduction

Fuzzy topology is a discipline of fuzzy mathematics developed since the beginning of the theory of fuzzy sets [13, 16, 21, 20, 22, 19]. Besides established approaches to fuzzy topology (categorial, lattice-valued, etc.), recent advances in metamathematics of fuzzy logic have enabled an approach to fuzzy topology based on formal fuzzy logic. The framework of higher-order fuzzy logic, also known as Fuzzy Class Theory [4], is especially suitable for fuzzy topology, as it easily accommodates fuzzy sets of fuzzy sets (of arbitrary orders), which are constantly encountered in fuzzy topology.

In classical mathematics, topology can be defined in several equivalent ways: by a system of open (closed) sets, by a system of neighborhoods, or by an interior (closure) operator. These definitions, however, are no longer equivalent in fuzzy logic. Notions of fuzzy topology given by open sets and neighborhoods have been investigated in the framework of Fuzzy Class Theory in [9]. In the present paper we focus on fuzzy topology given by interior operators. Unlike the authors of previous studies of interior and closure operators (e.g., [15, 10, 11]), we work in the fully graded and formal framework of Fuzzy Class Theory, following the methodology of [6]. This approach yields a specific kind of results [8], incomparable to those obtained in traditional fuzzy mathematics: they are on the one hand more general (namely fully graded, i.e., admitting partially valid assumptions), while on the other hand limited to the scope of applicability of deductive fuzzy logic [2].

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## 2 Preliminaries

Fuzzy Class Theory FCT, introduced in [4], is an axiomatization of Zadeh's notion of fuzzy set in formal fuzzy logic. We use its variant defined over $\mathrm{MTL}_\triangle$ [14], the logic of all left-continuous t-norms, which is arguably [2] the weakest fuzzy logic with good inferential properties for fully graded fuzzy mathematics in the framework of formal fuzzy logic [6].

We assume the reader's familiarity with $\mathrm{MTL}_\triangle$; for details on this logic see [14]. Here we only recapitulate its standard real-valued semantics:

| | | |
|---|---|---|
| $\&$ | ... | a left-continuous t-norm $*$ |
| $\to$ | ... | the residuum $\Rightarrow$ of $*$, defined as $x \Rightarrow y =_{\mathrm{df}} \sup\{z \mid z * x \leq y\}$ |
| $\wedge, \vee$ | ... | min, max |
| $\neg$ | ... | $x \Rightarrow 0$ |
| $\leftrightarrow$ | ... | the bi-residuum: $\min(x \Rightarrow y, y \Rightarrow x)$ |
| $\triangle$ | ... | $\triangle x = 1 - \mathrm{sgn}(1 - x)$ |
| $\forall, \exists$ | ... | inf, sup |

**Definition 2.1** Fuzzy Class Theory FCT *is a formal theory over multi-sorted first-order fuzzy logic (in this paper,* $\mathrm{MTL}_\triangle$*), with the sorts of variables for*

- *Atomic objects (lowercase letters $x, y, \ldots$)*
- *Fuzzy classes of atomic objects (uppercase letters $A, B, \ldots$)*
- *Fuzzy classes of fuzzy classes of atomic objects (Greek letters $\tau, \sigma, \ldots$)*

- *Fuzzy classes of the third order (in this paper denoted by sans serif letters* $\mathsf{A}, \mathsf{B}, \mathsf{a}, \mathsf{b}, \dots$*)*
- *Etc., in general for fuzzy classes of the $n$-th order ($X^{(n)}, Y^{(n)}, \dots$)*

*Besides the crisp identity predicate $=$, the language of* FCT *contains:*

- *The membership predicate $\in$ between objects of successive sorts*
- *Class terms $\{x \mid \varphi\}$ of order $n+1$, for any variable $x$ of any order $n$ and any formula $\varphi$*
- *Symbols $\langle x_1, \dots, x_k\rangle$ for $k$-tuples of individuals $x_1, \dots, x_k$ of any order*

FCT *has the following axioms (for all formulae $\varphi$ and variables of all orders):*

- *The logical axioms of multi-sorted first-order logic* MTL$_\triangle$
- *The axioms of crisp identity:*

$$x = x$$
$$x = y \rightarrow (\varphi(x) \rightarrow \varphi(y))$$
$$\langle x_1, \dots, x_k\rangle = \langle y_1, \dots, y_k\rangle \rightarrow x_i = y_i$$

- *The comprehension axioms:*

$$y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$$

- *The extensionality axioms:*

$$(\forall x)\triangle(x \in A \leftrightarrow x \in B) \rightarrow A = B$$

Note that in FCT, fuzzy sets are rendered as a *primitive notion* rather than modeled by membership functions. In order to capture this distinction, fuzzy sets are in FCT called *fuzzy classes;* the name *fuzzy set* is reserved for membership functions in the models of the theory.

The models of FCT are systems of fuzzy sets (and fuzzy relations) of all orders over a crisp universe of discourse, with truth degrees taking values in an MTL$_\triangle$-chain $\mathbf{L}$ (e.g., the interval $[0, 1]$ equipped with a left-continuous t-norm); thus all theorems on fuzzy classes provable in FCT are true statements about $\mathbf{L}$-valued fuzzy sets. Notice however that the theorems of FCT have to be derived from its axioms by the rules of the fuzzy logic MTL$_\triangle$ rather than classical Boolean logic. For details on proving theorems of FCT see [7] or [5].

In formulae of FCT we employ usual abbreviations known from classical mathematics or traditional fuzzy mathematics, including those listed in Table 1. Usual rules of precedence apply to the connectives of MTL$_\triangle$. Furthermore we define standard derived notions of FCT, summarized in Table 2, for all orders of fuzzy classes.

Table 1: Abbreviations used in the formulae of FCT

| | | |
|---|---|---|
| $Ax$ | $\equiv_{\mathrm{df}}$ | $x \in A$ |
| $x_1 \dots x_k$ | $=_{\mathrm{df}}$ | $\langle x_1, \dots, x_k\rangle$ |
| $x \notin A$ | $\equiv_{\mathrm{df}}$ | $\neg(x \in A)$ |
| $(\forall x \in A)\varphi$ | $\equiv_{\mathrm{df}}$ | $(\forall x)(x \in A \rightarrow \varphi)$ |
| $(\exists x \in A)\varphi$ | $\equiv_{\mathrm{df}}$ | $(\exists x)(x \in A \mathbin{\&} \varphi)$ |
| $(\forall x_1, \dots, x_k \in A)\varphi$ | $\equiv_{\mathrm{df}}$ | $(\forall x_1 \in A) \dots (\forall x_k \in A)\varphi$ |
| $(\exists x_1, \dots, x_k \in A)\varphi$ | $\equiv_{\mathrm{df}}$ | $(\exists x_1 \in A) \dots (\exists x_k \in A)\varphi$ |
| $\{x \in A \mid \varphi\}$ | $=_{\mathrm{df}}$ | $\{x \mid x \in A \mathbin{\&} \varphi\}$ |
| $\{t(x_1, \dots, x_k) \mid \varphi\}$ | $=_{\mathrm{df}}$ | $\{z \mid z = t(x_1, \dots, x_k) \mathbin{\&} \varphi\}$ |
| $y = F(x)$ | $\equiv_{\mathrm{df}}$ | $Fxy$, if $\triangle \operatorname{Fnc} F$ (see Tab. 2) is proved or assumed |
| $\varphi^n$ | $\equiv_{\mathrm{df}}$ | $\varphi \mathbin{\&} \dots \mathbin{\&} \varphi$ ($n$ times) |
| $\varphi^\triangle$ | $\equiv_{\mathrm{df}}$ | $\triangle\varphi$ |

Fuzzy counterparts of classical mathematical notions are in the present paper defined following the methodology sketched in [18, §5] and further elaborated in [4, §7], namely by choosing a suitable formula that expresses the classical definitions and re-interpreting it in fuzzy logic.

A distinguishing feature of FCT is that not only the membership predicate $\in$, but all defined notions are in general fuzzy (unless they are defined as provably crisp). FCT thus presents a fully graded approach to fuzzy mathematics. The importance of full gradedness in fuzzy mathematics is explained in [7, 3, 1]: its main merit lies in that it allows inferring relevant information even when a property of fuzzy sets is not fully satisfied. Fuzzy topology has a long tradition of attempting full gradedness, cf. graded definitions and theorems in [19, 22].

## 3 Open Fuzzy Topology

In classical mathematics, topology introduced by means of open sets is given by a crisp system $\tau$ of crisp subsets of a ground set V, where $\tau$ is required to satisfy certain conditions (closedness under $\bigcup, \cap, \emptyset, \mathrm{V}$, and possibly further properties, e.g., separation axioms). Generalization by admitting fuzzy subsets leads in FCT to regarding open fuzzy topology as a (possibly fuzzy) class of (possibly fuzzy) subclasses of the ground class V, i.e., a fuzzy class $\tau$ of the second order.[1]

[1] We keep the ground class crisp to avoid problems with quantification relativized to a fuzzy domain; generalization to fuzzy topological spaces with fuzzy universes is a topic

Table 2: Defined notions of FCT

| | | |
|---|---|---|
| $\emptyset$ | $=_{\mathrm{df}}$ | $\{x \mid 0\}$ ... empty class |
| $\mathrm{V}$ | $=_{\mathrm{df}}$ | $\{x \mid 1\}$ ... universal class |
| $\mathrm{Ker}\, A$ | $=_{\mathrm{df}}$ | $\{x \mid \triangle Ax\}$ ... kernel |
| $\alpha A$ | $=_{\mathrm{df}}$ | $\{x \mid \alpha \,\&\, Ax\}$ ... $\alpha$-resize |
| $-A$ | $=_{\mathrm{df}}$ | $\{x \mid \neg Ax\}$ ... complement |
| $A \cap B$ | $=_{\mathrm{df}}$ | $\{x \mid Ax \,\&\, Bx\}$ ... (strong) intersection |
| $A \sqcap B$ | $=_{\mathrm{df}}$ | $\{x \mid Ax \wedge Bx\}$ ... min-intersection |
| $A \cup B$ | $=_{\mathrm{df}}$ | $\{x \mid Ax \veebar Bx\}$ ... (strong) union |
| $A \times B$ | $=_{\mathrm{df}}$ | $\{xy \mid Ax \,\&\, By\}$ ... Cartesian product |
| $\mathrm{Rng}\, R$ | $=_{\mathrm{df}}$ | $\{y \mid (\exists x)Rxy\}$ ... range |
| $\bigcup \tau$ | $=_{\mathrm{df}}$ | $\{x \mid (\exists A \in \tau)(x \in A)\}$ ... class union |
| $\bigcap \tau$ | $=_{\mathrm{df}}$ | $\{x \mid (\forall A \in \tau)(x \in A)\}$ ... class intersec. |
| $\mathrm{Pow}\, A$ | $=_{\mathrm{df}}$ | $\{X \mid X \subseteq A\}$ ... power class |
| $\mathrm{Hgt}\, A$ | $\equiv_{\mathrm{df}}$ | $(\exists x)Ax$ ... height |
| $\mathrm{Plt}\, A$ | $\equiv_{\mathrm{df}}$ | $(\forall x)Ax$ ... plinth |
| $\mathrm{Crisp}\, A$ | $\equiv_{\mathrm{df}}$ | $(\forall x)\triangle(Ax \vee \neg Ax)$ ... crispness |
| $\mathrm{Refl}\, R$ | $\equiv_{\mathrm{df}}$ | $(\forall x)Rxx$ ... reflexivity |
| $\mathrm{Trans}\, R$ | $\equiv_{\mathrm{df}}$ | $(\forall x, y, z)(Rxy \,\&\, Ryz \to Rxz)$ ... transitivity |
| $\mathrm{Preord}\, R$ | $\equiv_{\mathrm{df}}$ | $\mathrm{Refl}\, R \,\&\, \mathrm{Trans}\, R$ ... preorder |
| $\mathrm{Fnc}\, R$ | $\equiv_{\mathrm{df}}$ | $(\forall x, y, y')(Rxy \,\&\, Rxy' \to y = y')$ ... functionality |
| $A \subseteq B$ | $\equiv_{\mathrm{df}}$ | $(\forall x)(Ax \to Bx)$ ... inclusion |
| $A \approx B$ | $\equiv_{\mathrm{df}}$ | $(A \subseteq B) \wedge (B \subseteq A)$ ... weak bi-incl. |
| $A \approxeq B$ | $\equiv_{\mathrm{df}}$ | $(A \subseteq B) \,\&\, (B \subseteq A)$ ... strong bi-incl. |

When investigating open fuzzy topologies, we are interested in such $\tau$ that satisfy analogous (but fuzzified) closure conditions as in classical topology. These are given by the following predicates that express the (degree of) closedness of $\tau$ under $\bigcup$ and $\cap$:

$$\mathsf{ic}(\tau) \equiv_{\mathrm{df}} (\forall A, B \in \tau)(A \cap B \in \tau)$$

$$\mathsf{Uc}(\tau) \equiv_{\mathrm{df}} (\forall \sigma \subseteq \tau)\big(\bigcup \sigma \in \tau\big)$$

These conditions (plus $\emptyset \in \tau$ and $\mathrm{V} \in \tau$) can be regarded as characteristic of open fuzzy topology. However, when studying open fuzzy topologies, we do *not* in general require that these axioms be satisfied as in classical topology. This is because they are (like all formulae of FCT) interpreted in many-valued logic; thus they need not be simply true or false, but are always true *to some degree.* By restricting our attention just to the systems that fully satisfy the above axioms, we would completely disregard systems that satisfy them to a degree of, e.g., 0.9999, even though graded theorems of FCT can provide us with useful information about such systems. Therefore we study *all* systems $\tau$, no matter to which degree they satisfy the above axioms. Similarly we proceed also in fuzzification of other definitions of fuzzy topology in the following sections.

It turns out [9] that besides the predicate $\mathsf{Uc}$, also predicates of the following forms are often met in the study of fuzzy topology (for $m, n \geq 1$):

$$\mathsf{Uc}_{m,n}(\tau) \equiv_{\mathrm{df}} (\forall \sigma)\big(\sigma \subseteq^m \tau \to \bigcup(\sigma \cap \overset{n}{\ldots} \cap \sigma) \in \tau\big)$$

Note that because $\varphi \,\&\, \varphi$ is not generally equivalent to $\varphi$ in $\mathrm{MTL}_\triangle$ (nor in stronger fuzzy logics except for Gödel fuzzy logic or stronger), $\sigma \cap \overset{n}{\ldots} \cap \sigma$ does not generally equal $\sigma$ (only $\sigma \cap \overset{n}{\ldots} \cap \sigma \subseteq \sigma$ holds for all $\sigma$). Similarly $(\sigma \subseteq \tau)^m$ is in general stronger than simple $\sigma \subseteq \tau$ if $m > 1$. Recall that the larger $m$, the stronger $\varphi^m$; informally $\varphi^m$ can be understood as $m$-times stressed $\varphi$ (consult, e.g., [7] for the role of multiple conjunction in formal proofs). Thus, like $\mathsf{Uc}$, the predicate $\mathsf{Uc}_{m,n}$ expresses the closedness of $\tau$ under a certain operation similar to the union of subsystems, only the condition of what counts as a subsystem is strengthened by $m$ and the union itself is strengthened by $n$.

By convention, we also admit the value "$\triangle$" for either $m$ or $n$ or both (cf. the last line of Table 1). Then, e.g., $\mathsf{Uc}_{\triangle,1}(\tau)$ expresses the closedness of $\tau$ under the unions of *crisp* subsystems of $\tau$, while $\mathsf{Uc}_{1,\triangle}(\tau)$ expresses the closedness of $\tau$ under the unions of *kernels* of subsystems of $\tau$ (i.e., only full members of the subsystem enter the union).

For convenience, we define a predicate that puts the properties monitored in open fuzzy topologies together. Since each of the properties can appear with varying multiplicity in theorems, we have to add further indices that parameterize the multiplicity of each of the conditions:

**Definition 3.1** *We define the predicate indicating the degree to which $\tau$ is an $(e, v, i, u, m, n)$*–open fuzzy topology *as*

$$\mathsf{OTop}^{e,v,i,u}_{m,n}(\tau) \equiv_{\mathrm{df}} (\emptyset \in \tau)^e \,\&\, (\mathrm{V} \in \tau)^v \,\&\, \mathsf{ic}^i(\tau) \,\&\, \mathsf{Uc}^u_{m,n}(\tau)$$

*For the sake of brevity, we drop the subscripts if both equal* 1*, and similarly for the superscripts.*

The properties of open fuzzy topologies have been studied in [9]. Since in this paper we are mainly interested in the interior operator, we repeat here the definition of the interior operator induced by an open fuzzy topology and list its basic properties.

**Definition 3.2** *Given a class of classes $\tau$, we define the* interior *of a class $A$ in $\tau$ as*

$$\mathrm{Int}_\tau(A) =_{\mathrm{df}} \bigcup \{B \in \tau \mid B \subseteq A\}$$

for future work. Since in this paper we always work within a single topological space, we can identify the ground class with the universal class V.

**Proposition 3.3** *It is provable in* FCT*:*

(i) $\mathrm{Int}_\tau(A) \subseteq A$

(ii) $A \in \tau \to \mathrm{Int}_\tau(A) \approxeq A$

(iii) $A \subseteq B \to \mathrm{Int}_\tau(A) \subseteq \mathrm{Int}_\tau(B)$

(iv) $\mathrm{Int}_\tau(A \sqcap B) \subseteq \mathrm{Int}_\tau(A) \sqcap \mathrm{Int}_\tau(B)$

**Proposition 3.4** *It is provable in* FCT*:*

(i) $\mathrm{V} \in \tau \to \mathrm{Int}_\tau(\mathrm{V}) \approxeq \mathrm{V}$

(ii) $\mathsf{Uc}(\tau) \to \mathrm{Int}_\tau(\mathrm{Int}_\tau(A)) \approxeq \mathrm{Int}_\tau(A)$

(iii) $\mathsf{ic}(\tau) \to \mathrm{Int}_\tau(A) \cap \mathrm{Int}_\tau(B) \subseteq \mathrm{Int}_\tau(A \cap B)$

Propositions 3.3 and 3.4 show that the interior operator generated by an open fuzzy topology $\tau$ satisfies properties expected from an interior operator—unconditionally in Proposition 3.3, and to a guaranteed degree (depending on the degree to which $\tau$ satisfies the conditions required from open fuzzy topologies) in Proposition 3.4.

If the antecedent conditions in Propositions 3.3 and 3.4 are fulfilled to the full degree, so are the conclusions. In particular, we have the following corollary:

**Corollary 3.5** FCT *proves:*

(i) $\triangle(A \in \tau) \to \mathrm{Int}_\tau(A) = A$

(ii) $\triangle\,\mathsf{Uc}(\tau) \to \mathrm{Int}_\tau(\mathrm{Int}_\tau(A)) = \mathrm{Int}_\tau(A)$

In words, whenever a fuzzy class $A$ is fully in $\tau$, it equals its interior (no matter what conditions $\tau$ does or does not satisfy to which degree). Similarly, if $\tau$ is fully closed under fuzzy unions, interiors are stable in $\tau$.

It will further be seen in Section 5 that an open fuzzy topology can vice versa be recovered from a primitive interior operator, under conditions similar to those above.

## 4 Neighborhood Fuzzy Topology

In classical mathematics, topology can also be introduced by assigning a system of neighborhoods to each point of a ground set V. Such a neighborhood system can be represented by a relation Nb between elements and subsets of V, where $\mathrm{Nb}(x, A)$ represents the fact that $A \subseteq \mathrm{V}$ is a neighborhood of $x \in \mathrm{V}$. The notion of neighborhood-based fuzzy topology, obtained by fuzzification of the classical notion in FCT, just allows the relation Nb and the class $A$ in $\mathrm{Nb}(x, A)$ to be fuzzy.[2] Thus in FCT, neighborhood fuzzy topologies will be second-order relations between atomic objects and first-order classes, i.e., classes Nb such that $\triangle(\mathrm{Nb} \subseteq \mathrm{V} \times \mathrm{Ker\,Pow\,V})$.

Neighborhood systems are in classical topology required to satisfy certain conditions. Fuzzified versions of these conditions will be of interest in neighborhood-based fuzzy topology, too:

**Definition 4.1** *Let* Nb *be a second-order class such that* $\triangle(\mathrm{Nb} \subseteq \mathrm{V} \times \mathrm{Ker\,Pow(V)})$*. Then we define the following predicates:*

$$
\begin{aligned}
\mathsf{N}_1(\mathrm{Nb}) &\equiv_{\mathrm{df}} (\forall x)\,\mathrm{Nb}(x, \mathrm{V})\\
\mathsf{N}_2(\mathrm{Nb}) &\equiv_{\mathrm{df}} (\forall x, A)(\mathrm{Nb}(x, A) \to x \in A)\\
\mathsf{N}_3(\mathrm{Nb}) &\equiv_{\mathrm{df}} (\forall x, A, B)(\mathrm{Nb}(x, A) \mathbin{\&} A \subseteq B \to\\
&\qquad \mathrm{Nb}(x, B))\\
\mathsf{N}_4(\mathrm{Nb}) &\equiv_{\mathrm{df}} (\forall x, A, B)(\mathrm{Nb}(x, A) \mathbin{\&} \mathrm{Nb}(x, B) \to\\
&\qquad \mathrm{Nb}(x, A \cap B))\\
\mathsf{N}_5(\mathrm{Nb}) &\equiv_{\mathrm{df}} (\forall x, A)(\mathrm{Nb}(x, A) \to (\exists B)(B \subseteq A \mathbin{\&}\\
&\qquad \mathrm{Nb}(x, B) \mathbin{\&} (\forall y \in B)\,\mathrm{Nb}(y, B))
\end{aligned}
$$

For convenience, we aggregate them in the following defined predicate:

**Definition 4.2** *We define the predicate indicating the degree to which* Nb *is a* $(k_1, \ldots, k_5)$–neighborhood fuzzy topology *as follows:*

$$
\mathsf{NTop}^{k_1,\ldots,k_5}(\mathrm{Nb}) \equiv_{\mathrm{df}} \mathrm{Nb} \subseteq^{\triangle} \mathrm{V} \times \mathrm{Ker\,Pow(V)} \mathbin{\&} \mathop{\&}_{i=1}^{5} \mathsf{N}_i^{k_i}(\mathrm{Nb})
$$

Basic properties of neighborhood fuzzy topologies and their relation to open fuzzy topologies have been summarized in [9]. Here we restrict our attention to their relationship to interior-based topologies. The following definition internalizes in FCT the classical definition of the interior of a class $A$:

**Definition 4.3** *Given a binary predicate* Nb *between elements and classes, we define*

$$
\mathrm{Int}_{\mathrm{Nb}}(A) =_{\mathrm{df}} \{x \mid \mathrm{Nb}(x, A)\}
$$

The behavior of $\mathrm{Int}_{\mathrm{Nb}}$ w.r.t. Kuratowski's (fuzzified) axioms of interior operators is studied in the following section.

## 5 Interior Fuzzy Topology

In classical topology, an interior operator is a function Int that assigns to each subset $A$ of a ground set V

[2] We again keep the ground set V crisp for simplicity and identify it with the universal class; see footnote 1.

a set $\mathrm{Int}(A) \subseteq \mathrm{V}$. In FCT we allow both the argument $A$ and the output $\mathrm{Int}(A)$ of the function to be fuzzy.[3] Fuzzy interior operators are thus construed as crisp second-order functions, i.e., classes Int such that $\mathrm{Int} \subseteq^{\triangle} \mathrm{Ker\,Pow(V)} \times \mathrm{Ker\,Pow(V)}\ \&\ \triangle\,\mathrm{Fnc(Int)}$.

The degrees to which Int satisfies (fuzzy versions of) Kuratowski's axioms for interior operators are given by the following predicates:

**Definition 5.1** *For second-order classes* Int *such that* $\mathrm{Int} \subseteq^{\triangle} \mathrm{Ker\,Pow(V)} \times \mathrm{Ker\,Pow(V)}\ \&\ \triangle\,\mathrm{Fnc(Int)}$ *we define the following predicates:*

$$\begin{aligned}
\mathsf{K}_1(\mathrm{Int}) &\equiv_{\mathrm{df}} \mathrm{Int(V)} \approxeq \mathrm{V}\\
\mathsf{K}_2(\mathrm{Int}) &\equiv_{\mathrm{df}} (\forall A)(\mathrm{Int}(A) \subseteq A)\\
\mathsf{K}_3(\mathrm{Int}) &\equiv_{\mathrm{df}} (\forall A)(\mathrm{Int}(\mathrm{Int}(A)) \approxeq \mathrm{Int}(A))\\
\mathsf{K}_4(\mathrm{Int}) &\equiv_{\mathrm{df}} (\forall A, B)(\mathrm{Int}(A) \cap \mathrm{Int}(B) \subseteq \mathrm{Int}(A \cap B))
\end{aligned}$$

Unlike in classical topology, in $\mathrm{MTL}_{\triangle}$ these conditions do not imply the monotonicity of Int. Therefore we define also the following predicates:

$$\begin{aligned}
\mathsf{Mon}(\mathrm{Int}) &\equiv_{\mathrm{df}} (\forall A, B)(A \subseteq B \to \mathrm{Int}(A) \subseteq \mathrm{Int}(B))\\
\mathsf{K}_5(\mathrm{Int}) &\equiv_{\mathrm{df}} (\forall A, B)(\mathrm{Int}(A \sqcap B) \subseteq \mathrm{Int}(A) \sqcap \mathrm{Int}(B))
\end{aligned}$$

Although $\mathsf{Mon}$ and $\mathsf{K}_5$ are not equivalent, the following relationships between them hold:

**Proposition 5.2** *It is provable in* FCT*:*

1. $\mathsf{K}_5(\mathrm{Int}) \to \mathsf{Mon}(\mathrm{Int})$
2. $\mathsf{Mon}^2(\mathrm{Int}) \to \mathsf{K}_5(\mathrm{Int})$
3. $\triangle\,\mathsf{K}_5(\mathrm{Int}) \leftrightarrow \triangle\,\mathsf{Mon}(\mathrm{Int})$

For convenience, we gather the conditions $\mathsf{K}_1$–$\mathsf{K}_5$ into one predicate $\mathsf{ITop}$:[4]

**Definition 5.3** *We define the notion of* $(k_1, \ldots, k_5)$–interior fuzzy topology *by the predicate*

$$\begin{aligned}
&\mathsf{ITop}^{k_1,\ldots,k_5}(\mathrm{Int}) \equiv_{\mathrm{df}}\\
&\mathrm{Int} \subseteq^{\triangle} \mathrm{Ker\,Pow(V)} \times \mathrm{Ker\,Pow(V)}\ \&\ \triangle\,\mathrm{Fnc(Int)}\\
&\qquad\qquad \&\ \mathop{\&}_{i=1}^{5} \mathsf{K}_i^{k_i}(\mathrm{Int})
\end{aligned}$$

[3]Again we keep V crisp and identify it with the universal class as in footnote 1. The function Int itself is conceived as crisp as well, to keep the correspondence to logical functions of [17]; if needed, it can be fuzzified by a similarity relation as in [1].

[4]It is not much important whether we take $\mathsf{K}_5$ or $\mathsf{Mon}$ in the definition of $\mathsf{ITop}$, as Proposition 5.2 "translates" between the two variants.

Open classes can be defined by means of the interior operator as usual:

$$\tau_{\mathrm{Int}} =_{\mathrm{df}} \{A \mid A \subseteq \mathrm{Int}(A)\}$$

The following graded theorem shows that if Int satisfies Kuratowski's axioms to a large degree, then $\tau_{\mathrm{Int}}$ satisfies the properties of open fuzzy topologies to a large degree, and the interior operator generated by $\tau_{\mathrm{Int}}$ equals Int to a large degree. Notice, however, that we have only got $\mathsf{OTop}_{2,1}(\tau_{\mathrm{Int}})$ rather than $\mathsf{OTop}(\tau_{\mathrm{Int}})$; in other words, we can only prove that the system of classes open w.r.t. a fuzzy Kuratowski interior operator is closed under unions of families "doubly included" in the system.

**Theorem 5.4** FCT *proves:*

$$\begin{aligned}
&\mathsf{ITop}^{1,1,1,1,2}(\mathrm{Int}) \to\\
&\qquad \mathsf{OTop}_{2,1}(\tau_{\mathrm{Int}})\ \&\ (\forall A)(\mathrm{Int}(A) \approxeq \mathrm{Int}_{\tau_{\mathrm{Int}}}(A))
\end{aligned}$$

**Corollary 5.5** FCT *proves:*

$$\triangle\,\mathsf{ITop}(\mathrm{Int}) \to \triangle\,\mathsf{OTop}_{2,1}(\tau_{\mathrm{Int}})\ \&\ \mathrm{Int} = \mathrm{Int}_{\tau_{\mathrm{Int}}}$$

Vice versa, interiors in well-behaved open fuzzy topologies are well-behaved fuzzy interior operators:

**Theorem 5.6** FCT *proves:*

$$\begin{aligned}
&\mathsf{OTop}^{0,1,1,1}(\tau) \to\\
&\qquad \mathsf{ITop}(\mathrm{Int}_\tau)\ \&\ (\forall A)(A \in \tau \leftrightarrow A \subseteq \mathrm{Int}_\tau(A))
\end{aligned}$$

**Corollary 5.7** FCT *proves:*

$$\triangle\,\mathsf{OTop}(\tau) \to \triangle\,\mathsf{ITop}(\mathrm{Int}_\tau)\ \&\ \tau = \tau_{\mathrm{Int}_\tau}$$

Neighborhoods can also be defined by means of the interior operator as usual:

$$\mathrm{Nb}_{\mathrm{Int}}(x, A) \equiv_{\mathrm{df}} x \in \mathrm{Int}(A)$$

It is immediate that $\mathrm{Nb}_{\mathrm{Int}}$ and $\mathrm{Int}_{\mathrm{Nb}}$ of Definition 4.3 are mutually inverse, i.e.,

$$\begin{aligned}
\mathrm{Int} &= \mathrm{Int}_{\mathrm{Nb}_{\mathrm{Int}}}\\
\mathrm{Nb} &= \mathrm{Nb}_{\mathrm{Int}_{\mathrm{Nb}}}
\end{aligned}$$

Moreover we have the following correspondence between the predicates $\mathsf{ITop}$ and $\mathsf{NTop}$:

**Theorem 5.8** FCT *proves:*

1. $\mathsf{ITop}^{1,2,2,1,1}(\mathrm{Int}) \to \mathsf{NTop}(\mathrm{Nb}_{\mathrm{Int}})$
2. $\mathsf{NTop}^{1,3,3,2,1}(\mathrm{Nb}) \to \mathsf{ITop}(\mathrm{Int}_{\mathrm{Nb}})$

As a corollary, we get the perfect match between the conditions $\mathsf{ITop}$ and $\mathsf{NTop}$ when true to degree 1:

**Corollary 5.9** FCT *proves:*

$$\begin{aligned} \triangle\, \mathsf{ITop}(\mathrm{Int}) &\leftrightarrow \triangle\, \mathsf{NTop}(\mathrm{Nb}_{\mathrm{Int}}), && \mathrm{Int} = \mathrm{Int}_{\mathrm{Nb}_{\mathrm{Int}}} \\ \triangle\, \mathsf{NTop}(\mathrm{Nb}) &\leftrightarrow \triangle\, \mathsf{ITop}(\mathrm{Int}_{\mathrm{Nb}}), && \mathrm{Nb} = \mathrm{Nb}_{\mathrm{Int}_{\mathrm{Nb}}} \end{aligned}$$

We conclude by giving three examples of interior-based fuzzy topology.

**Example 5.10** The operation sending a fuzzy class to its kernel is an interior operator that fully satisfies all of Kuratowski's axioms, as FCT proves

$$\begin{aligned} \operatorname{Ker} \mathrm{V} &= \mathrm{V} \\ \operatorname{Ker} A &\subseteq A \\ \operatorname{Ker} \operatorname{Ker} A &= \operatorname{Ker} A \\ \operatorname{Ker} A \cap \operatorname{Ker} B &= \operatorname{Ker}(A \cap B) \\ \operatorname{Ker}(A \sqcap B) &= \operatorname{Ker} A \sqcap \operatorname{Ker} B \end{aligned}$$

by [4, §3.4]. Thus $\triangle\, \mathsf{ITop}(\mathrm{Ker})$; we call it the *kernel fuzzy topology.*

In the kernel fuzzy topology, a class is fully open iff it is crisp: $\triangle(A \in \tau_{\mathrm{Ker}}) \leftrightarrow \mathrm{Crisp}\, A$. Partially open classes are those whose fuzzy elements only have low membership degrees. Since all crisp classes (including singletons) are open in the kernel fuzzy topology, it is a generalization of the notion of *discrete* crisp topology, with which it coincides in 2-valued models.

**Example 5.11** Define the interior of $A$ as $(\mathrm{Plt}\, A)\mathrm{V}$ (see Table 2 for the definitions of plinth and resize); i.e., $x \in \mathrm{Int}\, A \equiv_{\mathrm{df}} (\forall y) Ay$. In other words, the membership function of $\mathrm{Int}\, A$ is constant and all elements belong to $\mathrm{Int}\, A$ to the degree which is the infimum of the membership function of $A$. Then it is provable in FCT that $\triangle\, \mathsf{ITop}(\mathrm{Int})$; we call it the *plinth fuzzy topology.*

A class is fully open in the plinth topology iff it is a resize of the universal class. Thus, the plinth fuzzy topology is *stratified* (stratified topologies are defined as those in which all classes $\alpha\mathrm{V}$ are open [21, 19]). Partially open in the plinth topology are such classes whose membership functions have small amplitudes (i.e., the differences between their suprema and infima), as $\tau_{\mathrm{Int}} = \{A \mid \mathrm{Hgt}\, A \to \mathrm{Plt}\, A\}$. Since the only crisp open classes in the plinth topology are $\emptyset$ and V, it is a generalization of the notion of *anti-discrete* crisp topology (with which it coincides in 2-valued models).

**Example 5.12** In [12], an operation of the *opening* of a fuzzy set under a fuzzy relation has been introduced. In [3] the definition has been generalized to the graded framework of FCT and its graded properties have been investigated. The definition can be rephrased as follows:

$$\mathrm{Int}_R(A) =_{\mathrm{df}} \{y \mid (\exists x)(Rxy \,\&\, (\forall z)(Rxz \to Az))\}$$

From results proved in [3] it follows that for any relation $R$, the operator fully satisfies the conditions $\mathsf{K}_2$, $\mathsf{K}_3$, and $\mathsf{K}_5$. If $R$ is a crisp preorder, then furthermore $\mathrm{Int}_R$ fully satisfies $\mathsf{K}_4$. Since $\mathsf{K}_1(\mathrm{Int}_R)$ is equivalent to $\mathrm{V} \subseteq \mathrm{Rng}\, R$, we get

$$\triangle\, \mathrm{Preord}\, R \,\&\, \mathrm{Crisp}\, R \to \triangle\, \mathsf{ITop}(\mathrm{Int}_R)$$

This result can be generalized to a larger class of fuzzy relations: e.g., instead of crispness, $R = R \cap R$ is sufficient for $\triangle\, \mathsf{K}_4(\mathrm{Int}_R)$ if $\triangle\, \mathrm{Preord}\, R$; both conditions can further be relaxed if $\mathsf{ITop}$ is not required to degree 1. Furthermore it is shown in [3] that for any $R$ we have $\mathrm{Int}_R = \mathrm{Int}_{\tau_{\mathrm{Int}_R}}$.

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