Relational compositions in Fuzzy Class Theory

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Abstract

We present a method for mass proofs of theorems of certain forms in a formal theory of fuzzy relations and classes. The method is based on formal identification of fuzzy classes and inner truth values with certain fuzzy relations, which allows transferring basic properties of sup-T and inf-R compositions to a family of more than 30 composition-related operations, including sup-T and inf-R images, pre-images, Cartesian products, domains, ranges, resizes, inclusion, height, plinth, etc. Besides yielding a large number of theorems on fuzzy relations as simple corollaries of a few basic principles, the method provides a systematization of the family of relational notions and generates a simple equational calculus for proving elementary identities between them, thus trivializing a large part of the theory of fuzzy relations.

Keywords: Fuzzy relation, sup-T-composition, inf-R-composition, BK-product, fuzzy class theory, formal truth value. MSC 2000: 03E72, 03E70.

1 Introduction

The theory of fuzzy relations is a prerequisite to any other discipline of fuzzy mathematics. In this paper we show a method for mass proofs of theorems of certain forms in a formal theory of fuzzy relations. The method is based on transferring the properties of sup-T and inf-R relational compositions [42, 2] to a family of related notions in the theory of fuzzy sets and relations. We work in the formal framework of higher-order fuzzy logic, also known as Fuzzy Class Theory (FCT), introduced in [12]; we follow the methodology of [15].

Some part of the method we employ has already been briefly and informally sketched in Bělohlávek's book [18, Remark 6.16]. Our formal setting allows us to extend it to a larger family of notions and exploit the analogies between composition-related notions systematically, thus obtaining a large number of theorems on fuzzy relations for free. Furthermore, the syntactical apparatus of FCT makes it possible to show the soundness of this method by means of formal interpretations [9].

In consequence of methodological assumptions of deductive fuzzy logic explained in [10], our framework is constrained by certain requirements. First, our fuzzy sets can only take membership degrees in $\mathrm{MTL}_\triangle$-algebras [23] (possibly expanded by additional operators). In particular, if the system of membership degrees is the real unit interval $[0, 1]$, then our conjunction is bound to be a left-continuous t-norm $*$ and implication its residuum $\Rightarrow$. Thus we do not deal with more general conjunctive or implicational operators, such as mean conjunctions, Kleene–Dienes or early Zadeh implication, etc., which have also been considered for relational products [2]. Secondly, we always assume that we work over a fixed crisp ground set V. That is, our atomic objects (urelements) x are always elements of V (so *for all* x means actually *for all* $x \in V$); our fuzzy sets are always

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elements of the system $\mathcal{F}(V)$ of all fuzzy subsets of V (so their membership functions are defined for all $x \in V$); our n-ary fuzzy relations are elements of $\mathcal{F}(V^n)$; our second-order fuzzy sets are elements of $\mathcal{F}(\mathcal{F}(V))$, or $\mathcal{F}\big(\bigcup_{n=1}^{\infty} \mathcal{F}(V^n)\big)$ if fuzzy sets of fuzzy relations are considered; etc.

For simplicity, our exposition only deals with homogeneous binary fuzzy relations. Nevertheless, the results can easily be extended to heterogeneous binary fuzzy relations (see Remarks 4.10 and 5.6); a further extension to fuzzy relations of larger arities is hinted at in Remark 5.23.

The paper presents an application of formal methods of FCT to fuzzy relational notions; hence it is inevitably loaded with heavy formalism. Its details may therefore be hard to follow for readers unfamiliar with the apparatus of FCT or formal fuzzy logic. Nevertheless, some of the results and the general picture of interrelations between the composition-based notions might still be of interest to readers who are not interested in formalistic details. Therefore we shall first give an informal account of the methods presented in the paper and indicate which parts of the paper could be relevant for a broader audience.

The basic idea of the paper is to systematically exploit the similarity of the definitions of several fuzzy relational concepts. For instance, the definition of sup-T-composition of fuzzy relations, which in the traditional style of fuzzy mathematics reads

$$(R \circ S)xy = \bigvee_z Rxz * Szy, \tag{1}$$

is very similar to the definitions of the preimage and image of a fuzzy set under a fuzzy relation, which read, respectively,

$$(R^{\leftarrow} A)x = \bigvee_z Rxz * Az \tag{2}$$

$$(S^{\rightarrow} A)y = \bigvee_z Az * Szy \tag{3}$$

(where $*$ is a left-continuous t-norm). As observed in [18, Remark 6.16], the similarity extends to the point that many properties of sup-T-composition transfer to the properties of images and preimages. By formalization of the definitions in a suitable formal framework (viz, that of FCT), we are able to delimit a class of relational notions (listed in Tables 1–5 below) and a class of their properties that transfer automatically, without the need of separate proofs.

Obviously the reason why many properties of sup-T-compositions transfer to images and preimages is the same form of the definitions (1)–(3), the only difference being the absence of one of the variables occurring in (1) from the formulae (2) and (3). The definitions (2) and (3) can actually be reduced to instances of (1), by substituting a dummy object $\underline{0}$ for the variable to be eliminated from (1). By this trick, the fuzzy set A in the definition of preimage becomes identified with a suitable fuzzy relation S, namely such S that $Sz\underline{0} = Az$ and $Szy = 0$ for $y \neq \underline{0}$, where $\underline{0}$ is an arbitrarily chosen (but fixed) element.

It turns out to be useful to employ this representation of a fuzzy set by a suitable fuzzy relation systematically, as it will enable us to reduce several more notions to relational compositions. Thus in general we identify a fuzzy set A with the fuzzy relation $\mathbf{R}_A$ such that

$$\mathbf{R}_A xy = \begin{cases} Ax & \text{if } y = \underline{0} \\ 0 & \text{otherwise} \end{cases}$$

(in the following sections, the relation $\mathbf{R}_A$ is denoted simply by $\boldsymbol{A}$ or even just A). The operation of preimage then satisfies $\mathbf{R}_{R^{\leftarrow} A} = R \circ \mathbf{R}_A$, i.e., is represented as a special case of $R \circ S$ (for $S = \mathbf{R}_A$). Simplifying the notation, we may write simply $R^{\leftarrow} A = R \circ A$. Similarly, the operation of image satisfies $\mathbf{R}_{R^{\rightarrow} A} = R^{\mathrm{T}} \circ \mathbf{R}_A$ (or simply $R^{\rightarrow} A = R^{\mathrm{T}} \circ A$), where R^{T} is the transposition of R, i.e., $R^{\mathrm{T}} xy = Ryx$ (the transposition is needed for substituting $\underline{0}$ for the first rather than second variable in the definition of $R \circ S$, to match with the definition of $R^{\rightarrow} A$).

With the identification of A and $\mathbf{R}_A$, we can extend the compositional representation to further notions, for instance the Cartesian product of two fuzzy sets,

$$(A \times B)xy = Ax * By.$$

This is done by substituting the dummy object $\underline{0}$ for the variable z in the definition (1) of $R \circ S$ (notice that $\bigvee_z$ becomes void if z is fixed to the single element $\underline{0}$), which yields $A \times B = \mathbf{R}_A \circ \mathbf{R}_B^{\mathrm{T}}$, or simply $A \times B = A \circ B^{\mathrm{T}}$. The properties of sup-T-compositions (e.g., associativity) thus automatically transfer to Cartesian products as well.

Besides fuzzy sets, single membership degrees can also be represented by suitable fuzzy relations: namely, if both arguments in Rxy are replaced by $\underline{0}$, then the expression $R\underline{00}$ denotes the particular membership degree of the single pair $\underline{00}$ in R. Conversely, a membership degree α can be represented by a fuzzy relation $\mathbf{R}_\alpha$ defined as

$$\mathbf{R}_\alpha xy = \begin{cases} \alpha & \text{for } x = y = \underline{0} \\ 0 & \text{otherwise.} \end{cases}$$

(Again, in the following sections we simply write $\boldsymbol{\alpha}$ or just α instead of $\mathbf{R}_\alpha$.) This representation of membership degrees by fuzzy relations yields further composition-based relational notions, obtained by replacing more than one variable by the dummy object $\underline{0}$ in (1).

For instance, replacing all three variables x, y, z in (1) by $\underline{0}$ will yield the conjunction $*$ of truth degrees, as clearly $\mathbf{R}_{\alpha * \beta} = \mathbf{R}_\alpha \circ \mathbf{R}_\beta$ (notice that $\bigvee_z$ is again void as z is fixed to the single value $\underline{0}$). Similarly, by setting $x = z = \underline{0}$ (or $y = z = \underline{0}$) we obtain the operation of α-resize αA, defined as $(\alpha A)x = \alpha * Ax$ for all x, satisfying, as again the supremum over $z = \underline{0}$ is void, $\mathbf{R}_{\alpha A} = \mathbf{R}_\alpha \circ \mathbf{R}_A$, or simply $\alpha A = \alpha \circ A$. Finally, by setting $x = y = \underline{0}$ in (1) we obtain the graded relation of compatibility (or the height of intersection) of two fuzzy sets,

$$(A \parallel B) = \bigvee_z (Az * Bz),$$

with $\mathbf{R}_{A \parallel B} = \mathbf{R}_A^{\mathrm{T}} \circ \mathbf{R}_B$.

Further useful notions can be obtained, e.g., by substituting the maximal fuzzy set, i.e., the fuzzy set V such that $Vx = 1$ for all x, for some of the arguments in the above definitions. Thus, e.g., the graded property of height of a fuzzy set,

$$\operatorname{Hgt} A = \bigvee_z Az,$$

arises as $V \parallel A$, i.e., $\mathbf{R}_{\operatorname{Hgt} A} = \mathbf{R}_V^{\mathrm{T}} \circ \mathbf{R}_A$, or $\operatorname{Hgt} A = V^{\mathrm{T}} \circ A$. Similarly the domain and range of a fuzzy relation R are defined as $R^{\leftarrow} V$ and $R^{\rightarrow} V$, respectively, i.e., $\operatorname{Dom} R = R \circ V$ and $\operatorname{Rng} R = R^{\mathrm{T}} \circ V$.

Such properties of sup-T-composition that are preserved under restricting its arguments to relations of the form $\mathbf{R}_A$ or $\mathbf{R}_\alpha$ then automatically transfer to all members of the above family of notions. Among such properties are, e.g., the associativity of $\circ$, its monotony with respect to fuzzy inclusion, its invariance or monotony under unions and intersections, etc. Representing the family of notions as special cases of composition thus yields a mass proof method for their properties, as it is only necessary to prove such properties for the single notion of sup-T-composition $\circ$; their validity for the whole family of derived notions then follows automatically.

Furthermore, the associativity and transposition properties of sup-T-composition

$$(R \circ S) \circ T = R \circ (S \circ T) \tag{4}$$

$$(R \circ S)^{\mathrm{T}} = S^{\mathrm{T}} \circ R^{\mathrm{T}} \tag{5}$$

allow us to derive interrelations between the composition-based notions by simple equational calculations. For instance, $R^{\rightarrow}(S^{\rightarrow} A) = (S \circ R)^{\rightarrow} A$ is proved by the following identities, which just apply (4) and (5) to the derived notions:

$$R^{\rightarrow}(S^{\rightarrow} A) = R^{\mathrm{T}} \circ (S^{\mathrm{T}} \circ A) = (R^{\mathrm{T}} \circ S^{\mathrm{T}}) \circ A = (S \circ R)^{\mathrm{T}} \circ A = (S \circ R)^{\rightarrow} A.$$

The application of the simple rules (4), (5) to nested composition-based notions thus yields an infinite number of easily derivable corollaries.

The method just described for sup-T-compositions can also be applied to other kinds of fuzzy relational products—for instance, the BK-products, i.e., the inf-R-composition $\lhd$ and the related products $\rhd$ and $\square$, introduced by Bandler and Kohout in [2] and defined as

$$(R \lhd S)xy = \bigwedge_z (Rxz \Rightarrow Szy) \tag{6}$$

$$(R \rhd S)xy = \bigwedge_z (Szy \Rightarrow Rxz) \tag{7}$$

$$(R \square S)xy = \bigwedge_z ((Rxz \Rightarrow Szy) \wedge (Szy \Rightarrow Rxz)) \tag{8}$$

(where $\Rightarrow$ is the residuum of the left-continuous t-norm $*$). The elimination of some variables from (6)–(8), formally achieved by the same trick of substituting the dummy object $\underline{0}$, produces a family of notions analogous to those based on sup-T-composition. The family includes further well-known operations, such as:

- The graded inclusion of fuzzy sets $(A \subseteq B) = \bigwedge_z (Az \Rightarrow Bz)$, which can be represented as the BK-product $\mathbf{R}_A^{\mathrm{T}} \lhd \mathbf{R}_B$, or simply $A^{\mathrm{T}} \lhd B$, and thus is the BK-analogue of compatibility $(A \parallel B) = A^{\mathrm{T}} \circ B$
- The operation of plinth, $\operatorname{Plt} A = \bigwedge_z Az = V^{\mathrm{T}} \lhd A$, which is the BK-analogue of height $\operatorname{Hgt} A = V^{\mathrm{T}} \circ A$
- The implication $\Rightarrow$ itself, as $\mathbf{R}_{\alpha \Rightarrow \beta} = \mathbf{R}_\alpha \lhd \mathbf{R}_\beta$: thus by our conventions, $\alpha \Rightarrow \beta$ can also be written as $\alpha \lhd \beta$; it is the BK-analogue of the conjunction $*$.

The BK-analogues of the operations of image, preimage, Cartesian product, and α-resize are also important and appear frequently in fuzzy mathematics (see Examples 5.9–5.14 below). The present approach systematizes these notions and suggests their systematic names (e.g., $\lhd$-image, $\rhd$-preimage, etc.).

Again, the well-known properties of BK-products, such as their monotony with respect to inclusion, their invariance or monotony under unions and intersections, etc., are transferred to the whole family of BK-based notions. Furthermore, (4) and (5) jointly with the identities valid for BK-products

$$(R \lhd S)^{\mathrm{T}} = S^{\mathrm{T}} \rhd R^{\mathrm{T}}$$
$$R \lhd (S \lhd T) = (R \circ S) \lhd T$$
$$R \lhd (S \rhd T) = (R \lhd S) \rhd T$$

enable us to derive interrelations between all sup-T and BK-based notions by easy equational calculations. The resulting simple equational calculus contains more than thirty notions from both sup-T and BK families and covers a large part of the theory of fuzzy sets and fuzzy relations. The calculus thus may serve as a basis for an automated generation of a broad class of valid theorems on fuzzy sets and fuzzy relations.

The present paper carries out the above ideas in a rigorous manner within the formal framework of Fuzzy Class Theory:

Section 2 briefly introduces the apparatus of FCT over the logic $\mathrm{MTL}_\triangle$ and gives definitions of the standard notions employed in the paper. It also contains several lemmata needed later for proofs of some theorems; readers who are not interested in formal proofs can safely skip them.

Section 3 gives a formal account of the representation of fuzzy sets A and membership degrees α by the fuzzy relations $\mathbf{R}_A$ and $\mathbf{R}_\alpha$ (denoted there just $\boldsymbol{A}$ and $\boldsymbol{\alpha}$ for the sake of simplicity) and illustrates it on the matrix representation of fuzzy relations, under which fuzzy sets correspond to (file) vectors and membership degrees to scalars. For the representation of truth degrees, however, it is necessary first to internalize semantic truth values within the theory: recall that FCT has no variables for truth degrees, so a model that represents them by some FCT-defined fuzzy sets has to be constructed first. The construction of inner truth values is important for many parts of fuzzy mathematics formalized in FCT (cf. Remark 3.5). Nevertheless, the readers who are not interested in metamathematical issues can safely skip the part on the internalization and simply assume that we have the lattice L of truth values α at our disposal within the theory.

The formal definition of the family of notions based on sup-T-compositions is given in Section 4, where the reduction to compositions is also illustrated by showing how they work under the matrix and graph representations of fuzzy relations. The notions based on BK-products are then treated in Section 5; their importance for fuzzy mathematics is exemplified by Examples 5.9–5.14. The basic properties of sup-T-compositions are given in Theorem 4.2 and Corollary 4.3, and those of BK-compositions in Theorem 5.3 and Corollary 5.4. Their automatic consequents for the derived notions are listed in Corollaries 4.7–4.14 and 5.15–5.19. Independently of the formalism employed in their derivation, these corollaries may be of interest for a broader fuzzy community as a reference table listing a number of properties of fuzzy relational notions.

2 Preliminaries

Fuzzy Class Theory FCT, introduced in [12], is an axiomatization of Zadeh's notion of fuzzy set in formal fuzzy logic. Here we use its variant defined over $\mathrm{MTL}_\triangle$ [23], the logic of all left-continuous t-norms enriched with the connective $\triangle$, since it is arguably [10] the weakest fuzzy logic with good inferential properties for fully graded fuzzy mathematics and its expressive power is sufficient for our needs. The results of the present paper are readily transferable to any well-behaved extension of $\mathrm{MTL}_\triangle$(formally, to any deductive fuzzy logic in the sense of [10]), e.g., Łukasiewicz, product, or Gödel logic, Hájek's basic logic BL, etc. [30, 23].

We assume the reader's familiarity with first-order $\mathrm{MTL}_\triangle$; for details on this logic see [23, 32]. We only recapitulate its standard $[0, 1]$ semantics here:

$\&$	...	any left-continuous t-norm $*$
$\rightarrow$	...	the residuum $\Rightarrow$ of $*$, defined as $x \Rightarrow y =_{\mathrm{df}} \sup\{z \mid z * x \leq y\}$
$\wedge$, $\vee$	...	min, max
$\neg$	...	$\neg x =_{\mathrm{df}} x \Rightarrow 0$
$\leftrightarrow$	...	bi-residuum: $\min(x \Rightarrow y, y \Rightarrow x)$
$\triangle$	...	$\triangle x =_{\mathrm{df}} 1 - \mathrm{sgn}(1 - x)$
$\forall$, $\exists$	...	inf, sup

For reference, the following definition lists the axioms of multi-sorted first-order $\mathrm{MTL}_\triangle$ with crisp identity.

Definition 2.1 The language of *multi-sorted first-order logic* $\mathrm{MTL}_\triangle$ *with identity* consists of the binary connectives $\rightarrow$, $\&$, and $\wedge$, unary connective $\triangle$, propositional constant 0, quantifiers $\forall$ and $\exists$, binary predicate $=$, an arbitrary fixed set of predicate and function symbols of arbitrary arities, a pre-ordered set of sorts of variables, and countably many variables of each sort. There are the following defined connectives:

$$\begin{aligned}
\varphi \vee \psi &\equiv_{\mathrm{df}} ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi) \\
\neg\varphi &\equiv_{\mathrm{df}} \varphi \rightarrow 0 \\
\varphi \leftrightarrow \psi &\equiv_{\mathrm{df}} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) \\
1 &\equiv_{\mathrm{df}} \neg 0
\end{aligned}$$

The deduction rules of first-order $\mathrm{MTL}_\triangle$ are the *modus ponens* (from φ and $\varphi \rightarrow \psi$ infer ψ), $\triangle$-*necessitation* (from φ infer $\triangle\varphi$), and *generalization* (from φ infer $(\forall x)\varphi$), for all well-formed formulae φ and ψ of the given language.

The axioms of first-order $\mathrm{MTL}_\triangle$ with crisp identity are the following, for all well-formed formulae φ, ψ, χ of the given language:

(MTL1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(MTL2)	$(\varphi \mathbin{\&} \psi) \to \varphi$
(MTL3)	$(\varphi \mathbin{\&} \psi) \to (\psi \mathbin{\&} \varphi)$
(MTL4a)	$(\varphi \mathbin{\&} (\varphi \to \psi)) \to (\varphi \wedge \psi)$
(MTL4b)	$(\varphi \wedge \psi) \to \varphi$
(MTL4c)	$(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
(MTL5a)	$(\varphi \to (\psi \to \chi)) \to ((\varphi \mathbin{\&} \psi) \to \chi)$
(MTL5b)	$((\varphi \mathbin{\&} \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(MTL6)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(MTL7)	$0 \to \varphi$

($\triangle$1)	$\triangle\varphi \vee \neg\triangle\varphi$
($\triangle$2)	$\triangle(\varphi \vee \psi) \to (\triangle\varphi \vee \triangle\psi)$
($\triangle$3)	$\triangle\varphi \to \varphi$
($\triangle$4)	$\triangle\varphi \to \triangle\triangle\varphi$
($\triangle$5)	$\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$

($\forall$1)	$(\forall x)\varphi(x) \to \varphi(t)$	if t is substitutable for x in $\varphi(x)$
($\exists$1)	$\varphi(t) \to (\exists x)\varphi(x)$	if t is substitutable for x in $\varphi(x)$
($\forall$2)	$(\forall x)(\chi \to \varphi(x)) \to (\chi \to (\forall x)\varphi(x))$	if x is not free in χ
($\exists$2)	$(\forall x)(\varphi(x) \to \chi) \to ((\exists x)\varphi(x) \to \chi)$	if x is not free in χ
($\forall$3)	$(\forall x)(\chi \vee \varphi(x)) \to (\chi \vee (\forall x)\varphi(x))$	if x is not free in χ

(=1)	$x = x$	
(=2)	$x = y \to (\varphi(x) \leftrightarrow \varphi(y))$	if y is substitutable for x in $\varphi(x)$

In ($\forall$1)–(=2), x and y can be of any sort of variables in the given language. (Recall that in multi-sorted logics, the definition of substitutability requires the compatibility of sorts besides the usual conditions.)

By appropriate restrictions of language we get the propositional logics $\mathrm{MTL}_\triangle$ or MTL (without $\triangle$) and the first-order logics $\mathrm{MTL}_\triangle$ or MTL, with or without crisp identity.

Convention 2.2 In order to save some parentheses, we apply usual rules of precedence to propositional connectives of $\mathrm{MTL}_\triangle$, namely, $\to$ and $\leftrightarrow$ have lower priority than other binary connectives, and unary connectives have the highest priority. We use the sign $\equiv$ for equivalence-by-definition. A chain of implications $\varphi_1 \to \varphi_2,\ \varphi_2 \to \varphi_3,\ \ldots,\ \varphi_{n-1} \to \varphi_n$ can be written as $\varphi_1 \longrightarrow \varphi_2 \longrightarrow \cdots \longrightarrow \varphi_n$ (and similarly for the equivalence connective).

Besides the axioms, we shall use the theorems of first-order $\mathrm{MTL}_\triangle$ listed in [23, 16] without mention, as they are standard instruments for proving in $\mathrm{MTL}_\triangle$ (for more details on proof techniques in $\mathrm{MTL}_\triangle$, see [13, 16]). Furthermore we shall need the following lemmata:

Lemma 2.3 $\mathrm{MTL}_\triangle$ *proves:*

1. $\triangle\neg\varphi \leftrightarrow \triangle(\varphi \leftrightarrow 0)$
2. $\triangle\neg\varphi \mathbin{\&} \triangle\neg\psi \to \triangle(\varphi \leftrightarrow \psi)$
3. $\varphi \mathbin{\&} (\chi \to \psi) \to (\chi \to \varphi \mathbin{\&} \psi)$
4. $\varphi \mathbin{\&} (\psi \to \chi) \to ((\varphi \to \psi) \to \chi)$
5. $(\exists y)(\forall x)\varphi \to (\forall x)(\exists y)\varphi$
6. $\chi \mathbin{\&} (\forall x)\varphi \to (\forall x)(\chi \mathbin{\&} \varphi))$*, if x is not free in χ.*

Proof: 1. By (MTL7) and $\triangle$-necessitation, $\triangle(0 \to \varphi)$ is a theorem; thus $\triangle\neg\varphi \longleftrightarrow (\triangle(\varphi \to 0) \wedge \triangle(0 \to \varphi)) \longleftrightarrow \triangle(\varphi \leftrightarrow 0)$.

2. By 1., $\triangle\neg\varphi \to \triangle(\varphi \leftrightarrow 0)$ and $\triangle\neg\psi \to \triangle(0 \leftrightarrow \psi)$, whence the statement follows by the ($\triangle$-necessitated) transitivity of $\leftrightarrow$.

3. follows from the MTL-theorems $\zeta \longleftrightarrow (1 \to \zeta) \longleftrightarrow (1 \mathbin{\&} \zeta)$ and $(\vartheta \to \varphi) \mathbin{\&} (\psi \to \chi) \to (\vartheta \mathbin{\&} \psi \to \varphi \mathbin{\&} \chi)$ with 1 for ϑ.

4. is proved by the following chain of equivalences:
$[(\varphi \to \psi) \mathbin{\&} (\psi \to \chi) \to (\varphi \to \chi)] \longleftrightarrow [\varphi \to ((\varphi \to \psi) \mathbin{\&} (\psi \to \chi) \to \chi)] \longleftrightarrow$
$[\varphi \to ((\psi \to \chi) \to ((\varphi \to \psi) \to \chi))] \longleftrightarrow [(\varphi \mathbin{\&} (\psi \to \chi)) \to ((\varphi \to \psi) \to \chi)]$.

5. From the instance $(\forall x)\varphi \to \varphi$ of $(\forall 1)$ we get $(\forall y)((\forall x)\varphi \to \varphi)$ by generalization, whence $(\exists y)(\forall x)\varphi \to (\exists y)\varphi$ follows by quantifier distribution. Generalization over x and a quantifier shift completes the proof.

6. is proved by the following chain of equivalences and implications:
$(\forall x)(\chi \mathbin{\&} \varphi \to \chi \mathbin{\&} \varphi) \longleftrightarrow (\forall x)(\chi \to (\varphi \to \chi \mathbin{\&} \varphi)) \longleftrightarrow [\chi \to (\forall x)(\varphi \to \chi \mathbin{\&} \varphi)] \longrightarrow$
$[\chi \to ((\forall x)\varphi \to (\forall x)(\chi \mathbin{\&} \varphi))] \longleftrightarrow [\chi \mathbin{\&} (\forall x)\varphi \to (\forall x)(\chi \mathbin{\&} \varphi)]$
by (MTL5a,b), $(\forall 2)$, and quantifier distribution. QED

Lemma 2.4 *The following shifts of relativized quantifiers (cf. Convention 2.6 below) are provable in first-order* MTL *(with or without $\triangle$), if x is not free in χ and y is not free in ϑ:*

1. $(\exists y)(\chi \mathbin{\&} (\forall x)(\vartheta \to \varphi)) \to (\forall x)(\vartheta \to (\exists y)(\chi \mathbin{\&} \varphi))$
2. $(\forall x)(\varphi \to (\chi \to \psi)) \leftrightarrow (\chi \to (\forall x)(\varphi \to \psi))$
3. $(\forall x)(\varphi \to (\psi \to \chi)) \leftrightarrow ((\exists x)(\varphi \mathbin{\&} \psi) \to \chi)$
4. $(\exists x)(\varphi \mathbin{\&} (\chi \to \psi)) \to (\chi \to (\exists x)(\varphi \mathbin{\&} \psi))$
5. $(\exists x)(\varphi \mathbin{\&} (\psi \to \chi)) \to ((\forall x)(\varphi \to \psi) \to \chi)$

Proof: 1. is proved by the following chain of implications based respectively on Lemma 2.3(6,5,3), and the shift of $\exists$ over implication:
$(\exists y)(\chi \mathbin{\&} (\forall x)(\vartheta \to \varphi)) \longrightarrow (\exists y)(\forall x)(\chi \mathbin{\&} (\vartheta \to \varphi)) \longrightarrow (\forall x)(\exists y)(\chi \mathbin{\&} (\vartheta \to \varphi)) \longrightarrow$
$(\forall x)(\exists y)(\vartheta \to \chi \mathbin{\&} \varphi) \longrightarrow (\forall x)(\vartheta \to (\exists y)(\chi \mathbin{\&} \varphi))$.

2. follows from the following chain of equivalences:

$$(\forall x)(\varphi \to (\chi \to \psi)) \longleftrightarrow (\forall x)(\chi \to (\varphi \to \psi)) \longleftrightarrow (\chi \to (\forall x)(\varphi \to \psi))$$

3.–5. follow in a similar way from (MTL5a,b), Lemma 2.3(3) and Lemma 2.3(4), respectively, by usual quantifier shifts. QED

We now proceed to the definition of the apparatus of Fuzzy Class Theory (i.e., Henkin-style higher-order fuzzy logic) over $\mathrm{MTL}_\triangle$.

Definition 2.5 *Fuzzy Class Theory* FCT is a formal theory over a multi-sorted first-order deductive fuzzy logic (in this paper, $\mathrm{MTL}_\triangle$), with the sorts of variables for

- *Atomic objects* ('urelements'), denoted by lowercase letters $x, y, \ldots$
- *Fuzzy classes of atomic objects* (uppercase letters $A, B, \ldots$)
- *Fuzzy classes of fuzzy classes of atomic objects* (calligraphic letters $\mathcal{A}, \mathcal{B}, \ldots$)
- Etc., in general for *fuzzy classes of the n-th order* $(X^{(n)}, Y^{(n)}, \ldots)$

Besides the crisp *identity predicate* $=$, the language of FCT contains:

- The *membership predicate* $\in$ between objects of successive sorts
- *Class terms* $\{x \mid \varphi\}$ of order $n+1$, for any formula φ and any variable x of any order n
- Symbols $\langle x_1, \ldots, x_k \rangle$ for *k-tuples* of individuals $x_1, \ldots, x_k$ of any order

FCT has the following axioms (for all formulae φ and variables of any order):

- The logical axioms of multi-sorted first-order logic $\mathrm{MTL}_\triangle$ with crisp identity
- The *tuple-identity axioms* (for all k): $\langle x_1, \ldots, x_k \rangle = \langle y_1, \ldots, y_k \rangle \to x_1 = y_1 \mathbin{\&} \ldots \mathbin{\&} x_k = y_k$
- The *comprehension axioms*: $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$
- The *extensionality axioms*: $(\forall x)\triangle(x \in A \leftrightarrow x \in B) \to A = B$

The models of FCT are systems (closed under definable operations) of fuzzy sets of all orders over a fixed crisp universe of discourse, with truth degrees taking values in an $\mathrm{MTL}_\triangle$-chain $\mathbf{L}$ (e.g., the interval $[0, 1]$ equipped with a left-continuous t-norm). Thus all theorems on fuzzy classes provable in FCT are true statements about $\mathbf{L}$-valued fuzzy sets, for any $\mathrm{MTL}_\triangle$-chain $\mathbf{L}$.

For details on the apparatus of FCT we refer the reader to [12, 14] or a freely available primer [16]. Peculiar properties of fuzzy mathematics axiomatized over formal fuzzy logic are described in [17]. The following features of FCT are worth mentioning here:

- In FCT, fuzzy sets are rendered as a *primitive notion,* rather than modeled by membership functions. In order to capture this distinction, the objects of FCT are called *fuzzy classes* rather than fuzzy sets; the name *fuzzy set* is reserved for membership functions in the models of the theory.[1] Nevertheless, since FCT is sound w.r.t. models formed of *all* fuzzy subsets, the reader can always safely substitute fuzzy sets for our classes.

- Not only the membership predicate $\in$, but all defined notions of FCT are in general fuzzy (unless they are defined as provably crisp). FCT thus presents a fully graded approach to fuzzy mathematics. The importance of full gradedness in fuzzy mathematics is explained in [16, 11, 8]: its main merit is in that it allows inferring relevant information even when a property of fuzzy sets is not fully satisfied.

- Since FCT is a formal theory over the fuzzy logic $\mathrm{MTL}_\triangle$, its theorems have to be derived by the rules of $\mathrm{MTL}_\triangle$ rather than classical Boolean logic which is used in usual mathematical theories. For details on proving theorems in FCT see [16] or [13].

- Since the language and axioms of FCT have the same form for all orders of fuzzy classes, it is sufficient to formulate conventions, definitions, and theorems only for the lowest order, as they can be propagated to all higher orders automatically.

Convention 2.6 In formulae of FCT, we employ usual abbreviations known from classical and fuzzy mathematics, including the following ones:

$$
\begin{array}{rcl}
Ax & \equiv_{\mathrm{df}} & x \in A \\
x_1 \ldots x_k & =_{\mathrm{df}} & \langle x_1, \ldots, x_k \rangle \\
x \notin A & \equiv_{\mathrm{df}} & \neg(x \in A), \text{ and similarly for } \neq \\
(\forall x \in A)\varphi & \equiv_{\mathrm{df}} & (\forall x)(x \in A \to \varphi) \\
(\exists x \in A)\varphi & \equiv_{\mathrm{df}} & (\exists x)(x \in A \mathbin{\&} \varphi) \\
\{x \in A \mid \varphi\} & =_{\mathrm{df}} & \{x \mid x \in A \mathbin{\&} \varphi\} \\
(\forall \tau)\varphi & \equiv_{\mathrm{df}} & (\forall z)(z = \tau \to \varphi), \text{ for any term } \tau \text{ of the same sort as } z, \text{ and } z \text{ not free in } \varphi \\
(\exists \tau)\varphi & \equiv_{\mathrm{df}} & (\exists z)(z = \tau \mathbin{\&} \varphi), \text{ for any term } \tau \text{ of the same sort as } z, \text{ and } z \text{ not free in } \varphi \\
\{\tau \mid \varphi\} & =_{\mathrm{df}} & \{z \mid z = \tau \mathbin{\&} \varphi\}, \text{ for any term } \tau \text{ of the same sort as } z, \text{ and } z \text{ not free in } \varphi \\
\{x_1, \ldots, x_n\} & =_{\mathrm{df}} & \{z \mid z = x_1 \vee \ldots \vee z = x_n\} \\
t_1 = \cdots = t_n & \equiv_{\mathrm{df}} & (t_1 = t_2) \mathbin{\&} \ldots \mathbin{\&} (t_{n-1} = t_n) \\
\varphi^n & \equiv_{\mathrm{df}} & \varphi \mathbin{\&} \ldots \mathbin{\&} \varphi \ (n \text{ times}) \\
y = F(x) & \equiv_{\mathrm{df}} & Fxy, \text{ if } \triangle(\forall x y y')(Fxy \mathbin{\&} Fxy' \to y = y') \text{ is proved or assumed} \\
\bigcup_\varphi \tau & =_{\mathrm{df}} & \bigcup\{\tau \mid \varphi\} \text{ for any term } \tau, \text{ and similarly for } \bigcap \text{ (see Definition 2.12 for } \bigcup, \bigcap)
\end{array}
$$

[1] The difference between fuzzy sets and classes is not just terminological: due to the first-order axiomatization, some fuzzy subsets may be missing from a model of FCT. An extreme example is provided by models consisting only of *crisp* subsets: it can be observed that they satisfy all axioms of FCT over $\mathrm{MTL}_\triangle$. Such non-intended models can be excluded by additional axioms ensuring the existence of non-crisp classes.

Convention 2.7 Let φ be a propositional formula and let all propositional variables that occur in φ be among $p_1, \dots, p_k$. The result of substitution of first-order formulae $\psi_1, \dots, \psi_k$ respectively for the variables $p_1, \dots, p_k$ in $\varphi(p_1, \dots, p_k)$ will be symbolized by $\varphi(\psi_1, \dots, \psi_k)$.

Definition 2.8 In FCT, we define the following class constants and operations:

$$\begin{array}{rcll}
\emptyset & =_{\mathrm{df}} & \{x \mid 0\} & \textit{empty class} \\
\mathrm{V} & =_{\mathrm{df}} & \{x \mid 1\} & \textit{universal class} \\
\mathrm{Ker}\, A & =_{\mathrm{df}} & \{x \mid \triangle Ax\} & \textit{kernel} \\
-A & =_{\mathrm{df}} & \{x \mid \neg Ax\} & \textit{complement} \\
A - B & =_{\mathrm{df}} & \{x \mid Ax \,\&\, \neg Bx\} & \textit{difference} \\
A \cap B & =_{\mathrm{df}} & \{x \mid Ax \,\&\, Bx\} & \textit{(strong) intersection} \\
A \cap_{\wedge} B & =_{\mathrm{df}} & \{x \mid Ax \wedge Bx\} & \textit{min-intersection} \\
A \cup_{\vee} B & =_{\mathrm{df}} & \{x \mid Ax \vee Bx\} & \textit{max-union}
\end{array}$$

Generally for any propositional formula $\varphi(p_1, \dots, p_k)$ of $\mathrm{MTL}_\triangle$ we define the corresponding class operation

$$\mathrm{Op}_\varphi(A_1, \dots, A_k) =_{\mathrm{df}} \{x \mid \varphi(A_1 x, \dots, A_k x)\}$$

Example 2.9 $A \cap B = \mathrm{Op}_{p \& q}(A, B)$, $-A = \mathrm{Op}_{\neg p}(A)$, $\mathrm{Ker}\, A = \mathrm{Op}_{\triangle p}(A)$, $\emptyset = \mathrm{Op}_0$, etc.

Definition 2.10 In FCT, we define the following elementary relations between fuzzy classes:

$$\begin{array}{rcll}
A \subseteq B & \equiv_{\mathrm{df}} & (\forall x)(Ax \to Bx) & \textit{inclusion} \\
A \approx B & \equiv_{\mathrm{df}} & (\forall x)(Ax \leftrightarrow Bx) & \textit{weak bi-inclusion} \\
A \subseteq^{\triangle} B & \equiv_{\mathrm{df}} & (\forall x)\triangle(Ax \to Bx) & \textit{crisp inclusion} \\
A \parallel B & \equiv_{\mathrm{df}} & (\exists x)(Ax \,\&\, Bx) & \textit{compatibility} \\
\mathrm{Hgt}(A) & \equiv_{\mathrm{df}} & (\exists x)Ax & \textit{height} \\
\mathrm{Crisp}(A) & \equiv_{\mathrm{df}} & (\forall x)\triangle(Ax \vee \neg Ax) & \textit{crispness}
\end{array}$$

Generally for any propositional formula $\varphi(p_1, \dots, p_k)$ of $\mathrm{MTL}_\triangle$ we define two induced elementary relations between fuzzy classes

$$\begin{array}{rcl}
\mathrm{Rel}^{\forall}_{\varphi}(A_1, \dots, A_k) & \equiv_{\mathrm{df}} & (\forall x)\varphi(A_1 x, \dots, A_k x) \\
\mathrm{Rel}^{\exists}_{\varphi}(A_1, \dots, A_k) & \equiv_{\mathrm{df}} & (\exists x)\varphi(A_1 x, \dots, A_k x)
\end{array}$$

Example 2.11 $(A \subseteq B) \equiv \mathrm{Rel}^{\forall}_{p \to q}(A, B)$ and $\mathrm{Hgt}(A) \equiv \mathrm{Rel}^{\exists}_{p}(A)$ by definition, and $(A = B) \leftrightarrow \mathrm{Rel}^{\forall}_{\triangle(p \leftrightarrow q)}(A, B)$ by the axiom of extensionality.

Metatheorems of [12, §3.4] reduce proofs of a broad class of theorems on elementary operations and relations between fuzzy classes to simple propositional calculations. In the present paper we shall freely use corollaries of these metatheorems (like $A \cap B \subseteq A$, $\mathrm{Ker}\, A \subseteq A$, etc.), as their direct proofs in FCT are easy anyway.

Definition 2.12 In FCT, we define the following higher-order fuzzy class operations:

$$\begin{array}{rcll}
\bigcup \mathcal{A} & =_{\mathrm{df}} & \{x \mid (\exists A \in \mathcal{A})(x \in A)\} & \textit{class union} \\
\bigcap \mathcal{A} & =_{\mathrm{df}} & \{x \mid (\forall A \in \mathcal{A})(x \in A)\} & \textit{class intersection} \\
\mathrm{Pow}\, A & =_{\mathrm{df}} & \{X \mid X \subseteq A\} & \textit{power class}
\end{array}$$

Definition 2.13 In FCT, we define the following relational operations:

$$\begin{array}{rcll}
A \times B & =_{\mathrm{df}} & \{xy \mid Ax \,\&\, By\} & \textit{Cartesian product} \\
\mathrm{Dom}(R) & =_{\mathrm{df}} & \{x \mid Rxy\} & \textit{domain} \\
\mathrm{Rng}(R) & =_{\mathrm{df}} & \{y \mid Rxy\} & \textit{range} \\
R^{\rightarrow} A & =_{\mathrm{df}} & \{y \mid (\exists x)(Ax \,\&\, Rxy)\} & \textit{image} \\
R^{\leftarrow} B & =_{\mathrm{df}} & \{x \mid (\exists y)(By \,\&\, Rxy)\} & \textit{pre-image} \\
R^{\mathrm{T}} & =_{\mathrm{df}} & \{xy \mid Ryx\} & \textit{transposition} \\
\mathrm{Id} & =_{\mathrm{df}} & \{xy \mid x = y\} & \textit{identity relation} \\
A^n & =_{\mathrm{df}} & \{x_1 \dots x_n \mid Ax_1 \,\&\, \dots \,\&\, Ax_n\} & \textit{Cartesian power}
\end{array}$$

In particular, V^n is the class of all n-tuples of atomic objects. Subclasses of V^n are called *n-ary fuzzy relations*; the condition that a class R is an n-ary relation is expressed by the formula $R \subseteq^{\triangle} \mathrm{V}^n$. Instead of "unary relations" we usually speak simply of fuzzy classes, unless we want to stress the distinction from the general meaning of the term "class", which includes relations of arities larger than one.[2]

Since all classes in FCT are in principle fuzzy, we often omit the word "fuzzy" and speak simply of *classes* and *relations*, meaning "fuzzy (including possibly crisp) classes or relations". Since crisp classes are just a special kind of fuzzy classes, we do not distinguish operations on crisp relations from their counterparts operating on fuzzy relations (unlike certain traditions in the theory of fuzzy relations), and use the same symbols for both kinds of arguments; if necessary, the crispness of arguments can explicitly be expressed in the formula by means of the predicate Crisp introduced in Definition 2.10.

The operation of transposition (see Definition 2.13) applied to R yields its *converse* relation R^{T}. The following simple properties of transposition will be needed in subsequent sections:

Proposition 2.14 FCT *proves:*

1. $R^{\mathrm{T}\mathrm{T}} = R$
2. $R \subseteq^{\triangle} \mathrm{Id} \to R^{\mathrm{T}} = R$
3. *For any propositional formula* $\varphi(p_1, \ldots, p_n)$,
$$\begin{array}{rcl} \mathrm{Rel}^{\forall}_{\varphi}(R_1^{\mathrm{T}}, \ldots, R_n^{\mathrm{T}}) & \leftrightarrow & \mathrm{Rel}^{\forall}_{\varphi}(R_1, \ldots, R_n) \\ \mathrm{Rel}^{\exists}_{\varphi}(R_1^{\mathrm{T}}, \ldots, R_n^{\mathrm{T}}) & \leftrightarrow & \mathrm{Rel}^{\exists}_{\varphi}(R_1, \ldots, R_n) \end{array}$$
In particular, $R \subseteq S \leftrightarrow R^{\mathrm{T}} \subseteq S^{\mathrm{T}}$ *and* $R = S \leftrightarrow R^{\mathrm{T}} = S^{\mathrm{T}}$.
4. $(\mathrm{Op}_{\varphi}(R_1, \ldots, R_n))^{\mathrm{T}} = \mathrm{Op}_{\varphi}(R_1^{\mathrm{T}}, \ldots, R_n^{\mathrm{T}})$ *for any propositional formula* $\varphi(p_1, \ldots, p_n)$.
In particular, $(R \cap S)^{\mathrm{T}} = R^{\mathrm{T}} \cap S^{\mathrm{T}}$, $(-R)^{\mathrm{T}} = -(R^{\mathrm{T}})$, $\emptyset^{\mathrm{T}} = \emptyset$, *etc.*
5. $\displaystyle\bigcup_{R\in\mathcal{A}} R^{\mathrm{T}} = \Big(\bigcup_{R\in\mathcal{A}} R\Big)^{\mathrm{T}}$, $\displaystyle\bigcap_{R\in\mathcal{A}} R^{\mathrm{T}} = \Big(\bigcap_{R\in\mathcal{A}} R\Big)^{\mathrm{T}}$

Proof: 1. By definition, $xy \in R^{\mathrm{T}\mathrm{T}} \longleftrightarrow yx \in R^{\mathrm{T}} \longleftrightarrow xy \in R$; therefore, by the axiom of extensionality, $R^{\mathrm{T}\mathrm{T}} = R$.

2. For arbitrary x, y we take the following crisp cases:[3] if $x = y$, then $Rxy \leftrightarrow Ryx$ by the axiom of identity (=2); if $x \neq y$, then $\triangle\neg Rxy \,\&\, \triangle\neg Ryx$ by the assumption $R \subseteq^{\triangle} \mathrm{Id}$, hence $Rxy \leftrightarrow Ryx$ by Lemma 2.3(2). In both cases we have $Rxy \leftrightarrow R^{\mathrm{T}}xy$, so by $\triangle$-necessitation, generalization, and the axiom of extensionality we get $R = R^{\mathrm{T}}$.

3. By renaming bound variables we get $(\forall xy)\varphi(R_1yx, \ldots, R_nyx) \leftrightarrow (\forall yx)\varphi(R_1xy, \ldots, R_nxy)$, and similarly for $\mathrm{Rel}^{\exists}_{\varphi}$.

4. By expanding the definitions we get $xy \in (\mathrm{Op}_{\varphi}(R_1, \ldots, R_n))^{\mathrm{T}} \longleftrightarrow \varphi(R_1yx, \ldots, R_nyx) \longleftrightarrow \varphi(R_1^{\mathrm{T}}xy, \ldots, R_n^{\mathrm{T}}xy) \longleftrightarrow xy \in \mathrm{Op}_{\varphi}(R_1^{\mathrm{T}}, \ldots, R_n^{\mathrm{T}})$.

5. $xy \in \bigcup_{R\in\mathcal{A}} R^{\mathrm{T}} \longleftrightarrow (\exists R \in \mathcal{A})(yx \in R) \longleftrightarrow yx \in \bigcup_{R\in\mathcal{A}} R \longleftrightarrow xy \in \big(\bigcup_{R\in\mathcal{A}} R\big)^{\mathrm{T}}$, and analogously for $\bigcap$. QED

[2] Formally, we should explicitly mark the arities of variables in all formulae. We omit the arity marks for better readability, since usually the arities are either immaterial or determined by the context. If needed, the arity of a variable can be expressed by the formula $x \in \mathrm{V}^n$ if x is a variable just for n-tuples of objects, or $x \in \mathrm{V}$ if x is a variable for objects of any arity. The lowercase variables in Definitions 2.8–2.13 are universal (i.e., represent any tuples of objects), the defined notions can therefore be applied to fuzzy relations as well as classes.

[3] Recall that the soundness of proofs by cases follows from the provability of $(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$ in MTL.

3 Representation of fuzzy classes and truth values by fuzzy relations

Fuzzy classes and truth values can be represented as fuzzy relations of a certain form, as described below. This representation will allow us straightforwardly to apply the properties of fuzzy relational compositions to many derived concepts which involve fuzzy classes or truth values.

The identification of fuzzy classes and truth values with certain fuzzy relations will in this paper be described only informally. It can, nevertheless, be carried out in a rigorous formal way by means of syntactic interpretations of formal theories in FCT. We do not elaborate the apparatus of interpretations here as it would make the paper too much loaded with formalism, and simpler methods are sufficient for theorems stated in this paper. Technical details on syntactic interpretations in FCT, including the interpretations used for the identifications made in this paper, can be found in [9].

Convention 3.1 Let $\underline{0}$ be an arbitrarily chosen, but fixed, atomic object (i.e., an element of V^1). The fuzzy class $\{\underline{0}\}$ (i.e., the crisp singleton of the urelement $\underline{0}$) will be denoted by $\underline{1}$.

Convention 3.2 A fuzzy class $A \subseteq^{\triangle} \mathrm{V}^1$ will be identified with the fuzzy relation $A \times \underline{1} = \{\langle x, \underline{0}\rangle \mid x \in A\}$. When representing the fuzzy class A, the fuzzy relation $A \times \underline{1}$ will be written as $\boldsymbol{A}$ (the same letter in boldface).

Obviously, the relation $A \times \underline{1}$ is isomorphic in a very natural sense to the original fuzzy class A: each of the original elements x got replaced by a pair $x\underline{0}$, but its membership degree has not changed ($\boldsymbol{A}x\underline{0} \equiv Ax$); thus the *structure* of the fuzzy class has been preserved. Consequently, all of its properties that do not refer to the actual names of its elements have been preserved as well under this identification. Furthermore, the original class A can uniquely be reconstructed from the relation $A \times \underline{1}$ as $A = \{x \mid \langle x, \underline{0}\rangle \in A \times \underline{1}\}$. Also the identity of classes is preserved under the translation, since $A = B$ iff $A \times \underline{1} = B \times \underline{1}$ (which follows easily from $\langle x, \underline{0}\rangle = \langle y, \underline{0}\rangle \leftrightarrow x = y$, one of the axioms for tuples). The relations of the form $A \times \underline{1}$ thus faithfully represent fuzzy classes among fuzzy relations.[4]

This identification is quite natural and well-known. If the universe of discourse is finite, consisting of elements $x_1, \ldots, x_n$, fuzzy relations can be represented by $(n \times n)$-matrices of truth values, $R = (Rx_ix_j)_{ij}$:

$$R = \begin{pmatrix} Rx_1x_1 & Rx_1x_2 & \cdots & Rx_1x_n \\ Rx_2x_1 & Rx_2x_2 & \cdots & Rx_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ Rx_nx_1 & Rx_nx_2 & \cdots & Rx_nx_n \end{pmatrix}$$

Assume that $\underline{0}$ denotes the element x_1. The fuzzy class A is then identified with the relation

$$\boldsymbol{A} = \begin{pmatrix} A\underline{0} & 0 & \cdots & 0 \\ Ax_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Ax_n & 0 & \cdots & 0 \end{pmatrix}$$

[4] In the language of formal interpretations we can describe this fact rigorously by observing that $A \mapsto A \times \underline{1}$ is a faithful interpretation of the theory of fuzzy classes $\mathrm{FCT}_{2,2}$ (i.e., a fragment of FCT containing only variables for atomic individuals and fuzzy classes) in the theory of binary fuzzy relations $\mathrm{FCT}_{2,3}$ (i.e., a fragment of FCT containing only variables for atomic individuals, pairs of atomic individuals, and fuzzy classes). The interpretation provides a faithful translation between the properties of fuzzy classes and the corresponding fuzzy relations. For details see [9].

which by the usual convention of linear algebra can be written as the (file) vector $n \times 1$,

$$\boldsymbol{A} = \begin{pmatrix} A\underline{0} \\ Ax_2 \\ \vdots \\ Ax_n \end{pmatrix}$$

Notice that Convention 3.2 just extends this representation in a formal way to arbitrary (not only finite) fuzzy classes.

A similar trick will allow us to represent truth values as certain relations. First observe that truth values can be internalized in FCT as subclasses of an arbitrary crisp singleton, e.g., of $\underline{1}$, in the following way:

- The truth value of a formula φ is represented by the class $\overline{\varphi} =_{\mathrm{df}} \{\underline{0} \mid \varphi\}$. Then by definition, $\overline{\varphi} \subseteq^{\triangle} \underline{1}$ and $\varphi \leftrightarrow (\underline{0} \in \overline{\varphi})$.
- Vice versa, every $\alpha \subseteq^{\triangle} \underline{1}$ represents the truth value of a formula—e.g., of $\underline{0} \in \alpha$, since $(\forall \alpha \subseteq^{\triangle} \underline{1})(\overline{\underline{0} \in \alpha} = \alpha)$ by Proposition 3.4(1) below.

The truth values are thus represented by subclasses of $\underline{1}$, where the truth value represented is the degree of membership of $\underline{0}$ in the subclass. We shall therefore call the elements of $\operatorname{Ker}\operatorname{Pow}\underline{1}$ the *inner* (or *formal*) *truth values* and denote them by lowercase Greek letters $\alpha, \beta, \ldots$ The system of formal truth values will for brevity's sake be denoted by L:

$$\mathrm{L} =_{\mathrm{df}} \operatorname{Ker}\operatorname{Pow}\underline{1}$$

The ordering of truth values is represented by the relation $\subseteq^{\triangle}$ between their formal counterparts: by Proposition 3.4(2) below, $(\varphi \to \psi) \leftrightarrow (\overline{\varphi} \subseteq \overline{\psi})$ and $(\varphi \leftrightarrow \psi) \leftrightarrow (\overline{\varphi} \approx \overline{\psi})$ for any formulae φ and ψ. Furthermore, there is the following correspondence between the propositional connectives and class operations on L:

$$\begin{aligned} \overline{\varphi \,\&\, \psi} &= \overline{\varphi} \cap \overline{\psi} \\ \overline{\varphi \wedge \psi} &= \overline{\varphi} \cap_{\wedge} \overline{\psi} \\ \overline{\varphi \vee \psi} &= \overline{\varphi} \cup_{\vee} \overline{\psi} \\ \overline{\neg\varphi} &= \underline{1} \setminus \overline{\varphi} \\ \overline{0} &= \emptyset, \quad \text{etc., in general:} \\ \overline{c(\psi_1, \ldots, \psi_n)} &= \underline{1} \cap \mathrm{Op}_{c(p_1,\ldots,p_n)}(\overline{\psi_1}, \ldots, \overline{\psi_n}) \end{aligned}$$

for any definable n-ary propositional connective c, by Proposition 3.4(3) below. Using this correspondence, we can also denote the operations $\cap, \cap_{\wedge}, \cup_{\vee}, \ldots$ on L by $\overline{\&}, \overline{\wedge}, \overline{\vee}, \ldots$ and call them *formal connectives* on inner truth values.

Since inner truth values represent the semantical concept of truth value within the theory, we shall occasionally use the lattice-theoretical notation $\bigvee_{\alpha \in \mathcal{A}} \alpha$ and $\bigwedge_{\alpha \in \mathcal{A}} \alpha$ instead of $(\exists \alpha \in \mathcal{A})(\underline{0} \in \alpha)$ and $(\forall \alpha \in \mathcal{A})(\underline{0} \in \alpha)$, respectively, for $\mathcal{A} \subseteq^{\triangle} \mathrm{L}$. Proposition 3.4(4) below shows that $\bigvee_{\alpha \in \mathcal{A}} \alpha$ and $\bigwedge_{\alpha \in \mathcal{A}} \alpha$ respectively correspond to the union and intersection of the class $\mathcal{A} \subseteq^{\triangle} \mathrm{L}$.

Remark 3.3 It should be noticed that in an $\mathbf{L}$-valued model $\mathcal{M}$ of FCT (for an $\mathrm{MTL}_{\triangle}$-chain $\mathbf{L}$), the lattice L of inner truth values need not coincide with the lattice $\mathbf{L}$ of semantic truth values, but can be a proper sublattice of $\mathbf{L}$: in general, only those elements of $\mathbf{L}$ are represented in L which are the truth values of FCT-formulae in $\mathcal{M}$. Thus, for instance, in any *standard* model of FCT the system $\mathbf{L}$ of semantic truth values is the real unit interval $[0, 1]$; however, *crisp* standard models of FCT (cf. footnote 1 on page 8) have only two inner truth values, $\emptyset$ and $\underline{1}$.

It can also be observed that by the axioms of comprehension, $\bigvee_{\alpha \in \mathcal{A}} \alpha$ and $\bigwedge_{\alpha \in \mathcal{A}} \alpha$ exist for any class $\mathcal{A} \subseteq^{\triangle} \mathrm{L}$; thus FCT proves that L is a complete lattice, even though the system $\mathbf{L}$

of semantic truth values need not in general be complete: recall that only the *safeness* of the structure is required in the semantics of first-order fuzzy logic, i.e., the existence of all suprema and infima that are the truth values of formulae (see [30] for details). The difference is again due to the fact that the existence of suprema and infima is only ensured for such subsets $\mathcal{A}$ of L which are represented in the model, rather than *all* subsets of L.

Nevertheless, in the intended full models of FCT, i.e., those formed by *all* fuzzy subsets, inner truth values correspond exactly to the semantical ones.

Now we give proofs of the statements mentioned above:

Proposition 3.4 FCT *proves:*

1. $(\forall\alpha \subseteq^{\triangle} \underline{1})(\alpha = \{\underline{0} \mid \underline{0} \in \alpha\})$
2. $(\varphi \to \psi) \leftrightarrow (\overline{\varphi} \subseteq \overline{\psi})$ *for any formulae* φ *and* ψ
3. $\overline{\varphi(\psi_1, \dots, \psi_n)} = \underline{1} \cap \mathrm{Op}_\varphi(\overline{\psi_1}, \dots, \overline{\psi_n})$, *for any propositional formula* $\varphi(p_1, \dots, p_n)$
4. $\overline{\bigvee_{\alpha\in\mathcal{A}} \alpha} = \bigcup_{\alpha\in\mathcal{A}} \alpha$ *and* $\overline{\bigwedge_{\alpha\in\mathcal{A}} \alpha} = \bigcap_{\alpha\in\mathcal{A}} \alpha$ *for any* $\mathcal{A} \subseteq^{\triangle} \mathrm{L}$

Proof: 1. It is sufficient to prove $x \in \alpha \leftrightarrow (x = \underline{0} \mathbin{\&} \underline{0} \in \alpha)$ from the assumption $\alpha \subseteq \{\underline{0}\}$; the result then follows by $\triangle$-necessitation and generalization. Now $(x = \underline{0} \mathbin{\&} \underline{0} \in \alpha) \to x \in \alpha$ follows directly from the identity axioms, and $x \in \alpha \to (x = \underline{0} \mathbin{\&} \underline{0} \in \alpha)$ follows (by taking crisp cases $x = \underline{0}$ and $x \neq \underline{0}$) from the assumption $(\forall x \in \alpha)(x = \underline{0})$.

2. By definitions, $\overline{\varphi} \subseteq \overline{\psi} \longleftrightarrow \{\underline{0} \mid \varphi\} \subseteq \{\underline{0} \mid \psi\} \longleftrightarrow \{x \mid x = \underline{0} \mathbin{\&} \varphi\} \subseteq \{x \mid x = \underline{0} \mathbin{\&} \psi\} \longleftrightarrow (\forall x)((x = \underline{0} \mathbin{\&} \varphi) \to (x = \underline{0} \mathbin{\&} \psi))$; thus it is sufficient to prove

$$(\varphi \to \psi) \;\leftrightarrow\; (\forall x)((x = \underline{0} \mathbin{\&} \varphi) \to (x = \underline{0} \mathbin{\&} \psi)) \tag{9}$$

Now $(\varphi \to \psi) \to ((x = \underline{0} \mathbin{\&} \varphi) \to (x = \underline{0} \mathbin{\&} \psi))$, from which the left-to-right direction of (9) follows by generalization; vice versa, by specifying $\underline{0}$ for x in (9) we get: (9) $\longrightarrow ((\underline{0} = \underline{0} \mathbin{\&} \varphi) \to (\underline{0} = \underline{0} \mathbin{\&} \psi)) \longleftrightarrow (\varphi \to \psi)$.

3. By definitions,

$$\underline{1} \cap \mathrm{Op}_\varphi(\overline{\psi_1}, \dots, \overline{\psi_n}) = \{x \mid (x = \underline{0}) \mathbin{\&} \varphi((x = \underline{0} \mathbin{\&} \psi_1), \dots, (x = \underline{0} \mathbin{\&} \psi_n))\}$$

Denote the latter class by A and take crisp cases on x: if $x \neq \underline{0}$, then $Ax \leftrightarrow 0$ since $(x = \underline{0}) \leftrightarrow 0$; if $x = \underline{0}$, then $Ax \leftrightarrow (x = \underline{0}) \mathbin{\&} \varphi(\psi_1, \dots, \psi_n)$ since $(x = \underline{0} \mathbin{\&} \psi_i) \leftrightarrow \psi_i$ for all i. Thus in both cases $Ax \leftrightarrow (x = \underline{0}) \mathbin{\&} \varphi(\psi_1, \dots, \psi_n)$, i.e., $A = \{\underline{0} \mid \varphi(\psi_1, \dots, \psi_n)\} = \overline{\varphi(\psi_1, \dots, \psi_n)}$.

4. If $x = \underline{0}$, then $(\exists\alpha \in \mathcal{A})(x \in \alpha) \leftrightarrow x = \underline{0} \mathbin{\&} (\exists\alpha \in \mathcal{A})(x \in \alpha)$; if $x \neq \underline{0}$, then $(\exists\alpha \in \mathcal{A})(x \in \alpha) \leftrightarrow 0$, since $\alpha \in \mathcal{A} \mathbin{\&} x \in \alpha \longrightarrow \alpha \in \mathrm{L} \mathbin{\&} x \in \alpha \longrightarrow x = \underline{0}$ by $\mathcal{A} \subseteq^{\triangle} \mathrm{L}$ and $(\forall\alpha \in \mathrm{L})(\alpha \subseteq^{\triangle} \{\underline{0}\})$. In both cases we have $(\exists\alpha \in \mathcal{A})(x \in \alpha) \leftrightarrow x = \underline{0} \mathbin{\&} (\exists\alpha \in \mathcal{A})(x \in \alpha)$, thus

$$\textstyle\bigcup_{\alpha\in\mathcal{A}} \alpha = \{x \mid (\exists\alpha \in \mathcal{A})(x \in \alpha)\} = \{x \mid x = \underline{0} \mathbin{\&} (\exists\alpha \in \mathcal{A})(x \in \alpha)\} = \overline{\bigvee_{\alpha\in\mathcal{A}} \alpha}$$

The proof for $\bigwedge$ is analogous. QED

Remark 3.5 Inner truth values are an important construction in FCT (and generally in any formal theory of fuzzy sets), neither limited to nor motivated by the purposes of the present paper. The construction presented here is rather standard (cf., e.g., [41]) and shows, i.a., that FCT is strong enough to internalize its own semantics. By means of inner truth values, usual semantical notions like membership functions can be defined and investigated within the formal theory. However, since this is not the aim of the present paper, we leave this topic aside and turn back to the representation of truth values by fuzzy relations.

Now as the truth values are represented by special *fuzzy classes* (viz, the subclasses of $\underline{1}$), they can be identified with certain fuzzy relations by Convention 3.2. Namely, an inner truth value $\alpha \subseteq^{\triangle} \underline{1}$ is identified with the fuzzy relation $\alpha \times \underline{1} = \{\langle \underline{0}, \underline{0} \rangle \mid \underline{0} \in \alpha\}$. By the same convention, when representing the truth value α, the fuzzy relation $\alpha \times \underline{1}$ can be denoted by boldface $\boldsymbol{\alpha}$.

Again, if the universe of discourse is finite and consists of elements $\underline{0}, x_2, \ldots, x_n$, an inner truth value α is identified with the relation

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha\underline{0} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} \alpha\underline{0} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

which by usual conventions of linear algebra can be identified with the (1×1)-matrix (or *scalar*) $\big(\alpha\underline{0}\big)$. (Recall that $\alpha\underline{0}$, i.e., $\underline{0} \in \alpha$, has the truth value that is represented by α. In the informal matrix expressions, we shall write just α instead of $\alpha\underline{0}$ further on.)

It can again be noticed that the apparatus of Fuzzy Class Theory employed here just extends the usual correspondence between fuzzy relations, sets, and truth values on the one hand and matrices, vectors, and scalars of truth values on the other hand, to arbitrary (not only finite) fuzzy relations and classes, and provides a uniform way of formal handling thereof. In particular, the reduction of fuzzy classes and truth values to fuzzy relations will allow us to extend the apparatus of sup-T and inf-R compositions of fuzzy relations to fuzzy classes and truth values, apply the results on compositions to a rich variety of derived notions, and get the proofs of their properties for free.

We conclude this section with some conventions and observations that will be useful later.

Convention 3.6 Unless explicitly said otherwise, we shall always assume that R, S, or T (possibly subscripted) denote fuzzy relations $\subseteq^{\triangle} \mathrm{V}^2$; A, B, or C (possibly subscripted) denote unary classes $\subseteq^{\triangle} \mathrm{V}^1$; and α, β, γ (possibly subscripted) denote inner truth values $\subseteq^{\triangle} \underline{1}$.

Proposition 3.7 FCT *proves that* $(\forall \alpha \subseteq^{\triangle} \underline{1})(\boldsymbol{\alpha} \subseteq^{\triangle} \mathrm{Id})$*; therefore,* $(\forall \alpha \subseteq^{\triangle} \underline{1})(\boldsymbol{\alpha}^{\mathrm{T}} = \boldsymbol{\alpha})$ *by Proposition 2.14(2).*

Proof: From $(x \in \alpha \to x = \underline{0}) \to (x \in \alpha \,\&\, y = \underline{0} \to x = y)$, which follows from the axioms of identity, we get by generalization and distribution of quantifiers $(\forall x)(x \in \alpha \to x = \underline{0}) \to (\forall xy)(x \in \alpha \,\&\, y = \underline{0} \to x = y)$, i.e., $\alpha \subseteq \underline{1} \to \{x\underline{0} \mid x \in \alpha\} \subseteq \{xy \mid x = y\}$. Then $\triangle$-necessitation finishes the proof. QED

4 Sup-T-composition and derived notions

The usual definition of composition of fuzzy relations R and S is as follows:

Definition 4.1 $R \circ S =_{\mathrm{df}} \{xy \mid (\exists z)(Rxz \,\&\, Szy)\}$

Since & is interpreted by a (left-continuous) t-norm and $\exists$ by the supremum, $\circ$ is also called the *sup-T-composition* of R and S. It generalizes Zadeh's original definition [42] of max-min-composition to infinite domains and arbitrary left-continuous t-norms. Notice that the defining formula is the same as the defining formula of the relational composition in classical mathematics, the fuzziness being introduced only by the semantics of the logical symbols $\exists$ and &. This makes it the "default" definition of fuzzy relational composition according to the methodology of [15].

The following properties of sup-T-compositions are well-known (see, e.g., [21], [18], etc.). We repeat them here for reference and give their proofs in FCT.

Theorem 4.2 FCT *proves the following properties of sup-T-compositions:*

1. *Transposition:* $(R \circ S)^{\mathrm{T}} = S^{\mathrm{T}} \circ R^{\mathrm{T}}$

2. *Monotony:* $R_1 \subseteq R_2 \rightarrow R_1 \circ S \subseteq R_2 \circ S$

3. *Union:* $\Big(\bigcup_{R\in\mathcal{A}} R\Big) \circ S = \bigcup_{R\in\mathcal{A}} (R \circ S)$

4. *Intersection:* $\Big(\bigcap_{R\in\mathcal{A}} R\Big) \circ S \subseteq \bigcap_{R\in\mathcal{A}} (R \circ S)$

   (The converse inclusion has well-known crisp counter-examples.)

5. *Associativity:* $(R \circ S) \circ T = R \circ (S \circ T)$

Proof: 1. $(R \circ S)^{\mathrm{T}} = \{xy \mid (\exists z)(Ryz \mathbin{\&} Szx)\} = \{xy \mid (\exists z)(S^{\mathrm{T}}xz \mathbin{\&} R^{\mathrm{T}}zy)\} = S^{\mathrm{T}} \circ R^{\mathrm{T}}$.

2. $(R_1xz \rightarrow R_2xz) \longleftrightarrow (R_1xz \rightarrow R_2xz) \mathbin{\&} (Szy \rightarrow Szy) \longrightarrow ((R_1xz \mathbin{\&} Szy) \rightarrow (R_2xz \mathbin{\&} Szy))$, followed by generalization and distribution of quantifiers.

3. $(\exists z)[(\exists R \in \mathcal{A})(Rxz) \mathbin{\&} Szy] \longleftrightarrow (\exists z)(\exists R \in \mathcal{A})(Rxz \mathbin{\&} Szy) \longleftrightarrow (\exists R \in \mathcal{A})(\exists z)(Rxz \mathbin{\&} Szy)$.

4. The claim is proved by the following chain of implications (see Lemma 2.4 for the shifts of relativized quantifiers needed here):

$$(\exists z)[(\forall R \in \mathcal{A})(Rxz) \mathbin{\&} Szy] \longrightarrow (\exists z)(\forall R \in \mathcal{A})(Rxz \mathbin{\&} Szy) \longrightarrow (\forall R \in \mathcal{A})(\exists z)(Rxz \mathbin{\&} Szy) \quad (10)$$

The existence of crisp counter-examples to the converse inclusion follows from the fact that even though the first implication in (10) can be converted in classical logic, the second one cannot (the quantifiers do not commute).

5. $\{xy \mid (\exists w)((\exists z)(Rxz \mathbin{\&} Szw) \mathbin{\&} Twy)\} = \{xy \mid (\exists z)(Rxz \mathbin{\&} (\exists w)(Szw \mathbin{\&} Twy))\}$ QED

Corollary 4.3 *By Theorem 4.2(1) and Proposition 2.14(1, 3, 5),* FCT *proves the mirror variants of Theorem 4.2(2,3,4), too:*

1. $S_1 \subseteq S_2 \rightarrow R \circ S_1 \subseteq R \circ S_2$

2. $R \circ \bigcup_{S\in\mathcal{A}} S = \bigcup_{S\in\mathcal{A}} (R \circ S)$

3. $R \circ \bigcap_{S\in\mathcal{A}} S \subseteq \bigcap_{S\in\mathcal{A}} (R \circ S)$*, with crisp counter-examples to the converse inclusion.*

By means of the identification of fuzzy classes with fuzzy relations by Convention 3.2, the statements of Theorem 4.2 and Corollary 4.3 can be transferred to further relational notions besides sup-T-composition, by the following method.

Comparing, e.g., the (equivalent variant of the) definition of the preimage of a fuzzy class A under a fuzzy relation R with the definition of relational composition,

$$\begin{aligned} R^{\leftarrow} A &=_{\mathrm{df}} \{x \mid (\exists z)(Rxz \mathbin{\&} Az)\} \\ R \circ S &=_{\mathrm{df}} \{xy \mid (\exists z)(Rxz \mathbin{\&} Szy)\} \end{aligned}$$

one can recognize the same pattern of the defining expression: the only difference is that in the definition of the preimage, the second argument as well as the result are unary rather than binary (the variable y is missing). However, our identification of the fuzzy classes A and $R^{\leftarrow} A$ with the fuzzy relations $\boldsymbol{A} = A \times \underline{1}$ and $(R^{\leftarrow} A) \times \underline{1}$, respectively, reduces the definition of preimage exactly to that of composition, by supplying the dummy argument $\underline{0}$ for the missing variable y:

$$(R^{\leftarrow} A) \times \underline{1} = \{x\underline{0} \mid (\exists z)(Rxz \mathbin{\&} Az)\} = \{x\underline{0} \mid (\exists z)(Rxz \mathbin{\&} (A \times \underline{1})z\underline{0})\} = R \circ (A \times \underline{1})$$

Thus $B = R^{\leftarrow} A$ iff $\boldsymbol{B} = R \circ \boldsymbol{A}$.[5]

[5] Having adopted Convention 3.6, we could abandon the distinction between A and $\boldsymbol{A}$ altogether and simply equate $R^{\leftarrow} A = R \circ A$, since the convention ensures that A is a unary class even if $R \circ A$ is written out of any context. We keep the distinction here only for the sake of clarity.

Consequently, the properties of compositions stated in Theorem 4.2(2–4) and Corollary 4.3 automatically translate to properties of preimages:

$$\begin{array}{rcl} R_1 \subseteq R_2 & \rightarrow & R_1{}^{\leftarrow}A \subseteq R_2{}^{\leftarrow}A \\ A_1 \subseteq A_2 & \rightarrow & R^{\leftarrow}A_1 \subseteq R^{\leftarrow}A_2 \\ (\bigcup_{R\in\mathcal{A}} R)^{\leftarrow}A & = & \bigcup_{R\in\mathcal{A}}(R^{\leftarrow}A) \\ R^{\leftarrow}\bigcup_{A\in\mathcal{A}} A & = & \bigcup_{A\in\mathcal{A}}(R^{\leftarrow}A) \\ (\bigcap_{R\in\mathcal{A}} R)^{\leftarrow}A & \subseteq & \bigcap_{R\in\mathcal{A}}(R^{\leftarrow}A) \\ R^{\leftarrow}\bigcap_{A\in\mathcal{A}} A & \subseteq & \bigcap_{R\in\mathcal{A}}(R^{\leftarrow}A) \end{array}$$

Again, the converse inclusions for intersection are not generally valid even for crisp relations and classes, since there are crisp counter-examples even with relations of the form $A \times \underline{1}$.

For a proof of the properties, one only needs to realize that the predicates involved ($\subseteq, =$) are invariant under the transformation $\cdot \times \underline{1}$ as well as under its inverse, the operations involved ($\bigcup, \bigcap$) commute with both of these transformations, and that $(R^{\leftarrow}A) \times \underline{1}$ is $R \circ \boldsymbol{A}$, to which Theorem 4.2 applies. Another proof consists in the observation that the proof of Theorem 4.2 remains sound when deleting all occurrences of the variable y. A general method for proving the invariance of theorems of certain forms under translations like our identification of A with $A \times \underline{1}$ is available, in virtue of theorems on formal interpretations of theories over fuzzy logic (cf. footnote 4 and see [9]). Here we shall take these results for granted, since the method of inspecting the proofs and verifying their invariance under the substitution of $\underline{0}$ for some variables is always available and sufficiently simple for all theorems listed in this paper.

In the same manner, the notion of image of a fuzzy class under a fuzzy relation, $R^{\rightarrow}A =_{\mathrm{df}} \{y \mid (\exists z)(Az \mathbin{\&} Rzy)\}$, is obtained by substituting $\underline{0}$, only this time for x rather than y, in the definition of fuzzy relational composition, as

$$\begin{array}{rcl} (R^{\rightarrow}A) \times \underline{1} & = & \{y\underline{0} \mid (\exists z)(Az \mathbin{\&} Rzy)\} \\ & = & \{y\underline{0} \mid (\exists z)(\boldsymbol{A}z\underline{0} \mathbin{\&} Rzy)\} \\ & = & \{y\underline{0} \mid (\exists z)(R^{\mathrm{T}}yz \mathbin{\&} \boldsymbol{A}z\underline{0})\} \\ & = & R^{\mathrm{T}} \circ \boldsymbol{A} \end{array}$$

Thus $B = R^{\rightarrow}A$ iff $\boldsymbol{B} = R^{\mathrm{T}} \circ \boldsymbol{A}$, so the image of A under R can simply be equated with $R^{\mathrm{T}} \circ \boldsymbol{A}$. Again the above properties of compositions translate into those of images. (Notice that this time, we also need to employ Proposition 2.14(5) to get the preservation of unions and intersections under images, since R is transposed in $R^{\mathrm{T}} \circ \boldsymbol{A}$.)

As mentioned in the Introduction, the method of transferring the results on relational compositions to related notions like images or preimages has already been suggested in [18, Remark 6.16]. In our formal setting we can exploit the method systematically:

There are three variables in the definition of sup-T-composition and each of them can be replaced by the dummy value $\underline{0}$. This yields seven relational operations derived from sup-T-composition of fuzzy relations: they are summarized in Table 1.

We shall comment on the notions in the table. The first three lines have been described in detail above. The image and preimage have also been called the *inclusive afterset* and *inclusive foreset,* respectively, by Bandler and Kohout [5].

The fourth notion, arising from setting z to $\underline{0}$, is the usual *Cartesian product* of the classes A and B. Notice that fixing $z = \underline{0}$ makes the quantification over z void, so the comprehension term indeed equals $\{xy \mid Ax \mathbin{\&} By\}$. The resulting term $\boldsymbol{A} \circ \boldsymbol{B}^{\mathrm{T}}$ just reflects the valid equation $A \times B = (A \times \underline{1}) \circ (\underline{1} \times B)$.

Setting both x and y to $\underline{0}$ in the fifth line of Table 1 makes the result a fuzzy singleton—a class to which only the pair $\langle \underline{0}, \underline{0} \rangle$ belongs to the degree $(\exists z)(Az \mathbin{\&} Bz)$. The latter formula expresses the *compatibility* $A \parallel B$ of the fuzzy properties (or classes) A and B, i.e., the height of their

	$\{xy \mid (\exists z)(Rxz \mathbin{\&} Szy)\}$	$=$	$R \circ S$	$\ldots$	*composition*	$R \circ S$
$x = \underline{0}$	$\{\underline{0}y \mid (\exists z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \mathbin{\&} Rzy)\}^{\mathrm{T}} = (\boldsymbol{A}^{\mathrm{T}} \circ R)^{\mathrm{T}}$	$=$	$R^{\mathrm{T}} \circ \boldsymbol{A}$	$\ldots$	*image*	$R^{\rightarrow} A$
$y = \underline{0}$	$\{x\underline{0} \mid (\exists z)(Rxz \mathbin{\&} \boldsymbol{A}z\underline{0})\}$	$=$	$R \circ \boldsymbol{A}$	$\ldots$	*pre-image*	$R^{\leftarrow} A$
$z = \underline{0}$	$\{xy \mid (\exists \underline{0})(\boldsymbol{A}x\underline{0} \mathbin{\&} \boldsymbol{B}^{\mathrm{T}}\underline{0}y)\}$	$=$	$\boldsymbol{A} \circ \boldsymbol{B}^{\mathrm{T}}$	$\ldots$	*Cartesian product*	$A \times B$
$x, y = \underline{0}$	$\{\underline{00} \mid (\exists z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \mathbin{\&} \boldsymbol{B}z\underline{0})\}$	$=$	$\boldsymbol{A}^{\mathrm{T}} \circ \boldsymbol{B}$	$\ldots$	*compatibility*	$A \mathbin{\overline{\|}} B$
$x, z = \underline{0}$	$\{\underline{0}y \mid (\exists \underline{0})(\boldsymbol{\alpha}^{\mathrm{T}}\underline{00} \mathbin{\&} \boldsymbol{A}^{\mathrm{T}}\underline{0}y)\}^{\mathrm{T}} = (\boldsymbol{\alpha}^{\mathrm{T}} \circ \boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}}$	$=$	$\boldsymbol{A} \circ \boldsymbol{\alpha}$	$\ldots$	*α-resize*	αA
$y, z = \underline{0}$	$\{x\underline{0} \mid (\exists \underline{0})(\boldsymbol{A}x\underline{0} \mathbin{\&} \boldsymbol{\alpha}\underline{00})\}$	$=$	$\boldsymbol{A} \circ \boldsymbol{\alpha}$	$\ldots$	*α-resize*	αA
$x, y, z = \underline{0}$	$\{\underline{00} \mid (\exists \underline{0})(\boldsymbol{\alpha}\underline{00} \mathbin{\&} \boldsymbol{\beta}\underline{00})\}$	$=$	$\boldsymbol{\alpha} \circ \boldsymbol{\beta}$	$\ldots$	*conjunction*	$\alpha \mathbin{\overline{\&}} \beta$

Table 1: Operations derived from the sup-T-composition

intersection. Since fuzzy singletons internalize truth values,[6] the resulting expression represents the truth value of $A \parallel B$; thus $\boldsymbol{A}^{\mathrm{T}} \circ \boldsymbol{B} = \overline{A \parallel B} = \overline{\mathrm{Hgt}(A \cap B)}$. We shall denote the operation $\overline{\parallel}$, since the result is a *formal* truth value—the fuzzy singleton—rather than the semantical truth value of $A \parallel B$.

The sixth notion in Table 1, which for the lack of an established name we call the *α-resize* of A and denote by αA, is derived from composition by fixing $x, z = \underline{0}$ (notice that the same notion is obtained also by fixing $y, z = \underline{0}$). The operation is widely applicable in fuzzy set theory and often is used implicitly or without notice (see Examples 5.13 and 5.14 below).

Finally, fixing all x, y, z to $\underline{0}$ yields the operation of *formal conjunction* of two formal truth values (i.e., the intersection of the two fuzzy singletons that represent them).

Remark 4.4 It has already been observed by Zadeh in [42] that in the finite case, the sup-T-composition of fuzzy relations is computed in the same manner as the product of the corresponding matrices, only performing & instead of multiplication and taking the supremum ($\exists$) instead of the sum: $(\,\|(R \circ S)x_i x_j\|\,)_{ij} = (\,\|(\exists x_k)(Rx_i x_k \mathbin{\&} Sx_k x_j)\|\,)_{ij} = (\,\sup_k(\|Rx_i x_k\| * \|Sx_k x_j\|)\,)_{ij}$. The calculation is represented by the following diagram:[7]

$$
\begin{array}{c|c}
\circ & \begin{pmatrix} Sx_1x_1 & \cdots & Sx_1x_n \\ \vdots & \ddots & \vdots \\ Sx_nx_1 & \cdots & Sx_nx_n \end{pmatrix} \\
\hline
\begin{pmatrix} Rx_1x_1 & \cdots & Rx_1x_n \\ \vdots & \ddots & \vdots \\ Rx_nx_1 & \cdots & Rx_nx_n \end{pmatrix} & \begin{pmatrix} (R \circ S)x_1x_1 & \cdots & (R \circ S)x_1x_n \\ \vdots & \ddots & \vdots \\ (R \circ S)x_nx_1 & \cdots & (R \circ S)x_nx_n \end{pmatrix}
\end{array}
$$

Because of this correspondence, the sup-T-composition is by some authors also called the *sup-T-product* of fuzzy relations. The correspondence extends to the derived notions (since after all, file and row vectors as well as scalars are just special cases of matrices). Thus, e.g., taking the pre-image of a fuzzy class A in a fuzzy relation R can in the finite case be calculated as the

[6] According to the conventions of Section 3, fuzzy truth values are represented by fuzzy singletons $\alpha \subseteq^{\triangle} \{\underline{0}\}$, which classes we have identified with fuzzy relations $\boldsymbol{\alpha} = \alpha \times \underline{1} \subseteq^{\triangle} \{\langle \underline{0}, \underline{0} \rangle\}$. Thus among fuzzy relations, formal truth values are indeed represented by fuzzy singletons of $\underline{00}$.

[7] The element in the i-th row and j-th file in the resulting matrix is obtained as the supremum over the values (for all k) of the conjunction of the k-th element in the row and the k-th element in the file, respectively. The diagram just shows the usual way of calculating the matrix product, in which we now take suprema and conjunctions instead sums and products.

sup-T-product of the matrix $(Rx_i x_j)_{ij}$ and the vector $(Ax_j)_j$:

$$\begin{array}{c|c} \circ & \begin{pmatrix} Ax_1 \\ \vdots \\ Ax_n \end{pmatrix} \\ \hline \begin{pmatrix} Rx_1x_1 & \cdots & Rx_1x_n \\ \vdots & \ddots & \vdots \\ Rx_nx_1 & \cdots & Rx_nx_n \end{pmatrix} & \begin{pmatrix} (R^{\leftarrow} A)x_1 \\ \vdots \\ (R^{\leftarrow} A)x_n \end{pmatrix} \end{array}$$

Similarly, the α-resize of a class A is the product of the $(n \times 1)$-vector $\boldsymbol{A}$ and the scalar $\boldsymbol{\alpha}$:

$$\begin{array}{c|c} \circ & \begin{pmatrix} \alpha \end{pmatrix} \\ \hline \begin{pmatrix} Ax_1 \\ \vdots \\ Ax_n \end{pmatrix} & \begin{pmatrix} (\alpha A)x_1 \\ \vdots \\ (\alpha A)x_n \end{pmatrix} \end{array}$$

By the usual convention, we write transposed file vectors as row vectors; thus, e.g., for a fuzzy class A over a finite domain we can write $\boldsymbol{A}^{\mathrm{T}} = (Ax_1, \ldots, Ax_n)$. The difference between the Cartesian product $\boldsymbol{A} \circ \boldsymbol{B}^{\mathrm{T}}$ and the compatibility $\boldsymbol{A}^{\mathrm{T}} \circ \boldsymbol{B}$ illustrates the importance of distinguishing transposed classes from non-transposed ones:

$$\begin{array}{c|c} \circ & \begin{pmatrix} Bx_1 & \cdots & Bx_n \end{pmatrix} \\ \hline \begin{pmatrix} Ax_1 \\ \vdots \\ Ax_n \end{pmatrix} & \begin{pmatrix} (A \times B)x_1x_1 & \cdots & (A \times B)x_1x_n \\ \vdots & \ddots & \vdots \\ (A \times B)x_nx_1 & \cdots & (A \times B)x_nx_n \end{pmatrix} \end{array} \qquad \begin{array}{c|c} \circ & \begin{pmatrix} Bx_1 \\ \vdots \\ Bx_n \end{pmatrix} \\ \hline \begin{pmatrix} Ax_1 & \cdots & Ax_n \end{pmatrix} & \begin{pmatrix} A \parallel B \end{pmatrix} \end{array}$$

Notice that compatibility corresponds to the *scalar (sup-T-)product* of the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. Finally, conjunction is the product of two scalars,

$$\begin{array}{c|c} \circ & \begin{pmatrix} \alpha \end{pmatrix} \\ \hline \begin{pmatrix} \beta \end{pmatrix} & \begin{pmatrix} \alpha \mathbin{\&} \beta \end{pmatrix} \end{array}$$

Obviously, infinite matrices can be considered as well as finite ones (matrices of arbitrary cardinalities have been used, e.g., in [4]). Thus it can be seen that the apparatus of FCT just formalizes the natural correspondence between fuzzy relations, classes, and truth values on the one hand and (finite or infinite) matrices, vectors, and scalars on the other hand. This will be reflected by the following convention:

Convention 4.5 For the sake of convenience, we shall sometimes employ the matrix terminology and even in the formal theory of FCT call the relations of the form $A \times \underline{1}$ *(file) vectors,* $\underline{1} \times A$ *row vectors,* and fuzzy singletons $\boldsymbol{\alpha} \subseteq^{\triangle} \{\underline{00}\}$ *scalars,* for arbitrary (not only finite) classes A and α. We shall sometimes speak of the *type* of a fuzzy relation, meaning one of these four categories which the relation belongs to.

Remark 4.6 In the graph-theoretical representation of fuzzy relations, a binary fuzzy relation R is identified with a (possibly infinite) weighted node graph, where nodes represent the elements of the domain V^1 of R, and weighted arrows between the nodes indicate the truth values of the relation R between pairs of the elements. Our representation $\boldsymbol{A}$ of a fuzzy class A among fuzzy relations can thus be visualized as a (possibly infinite) graph with arrows from elements x of V^1 to $\underline{0}$ weighted by the values of Ax, and all other arrows weighted by 0 (see Figure 1). Similarly, the transposed class $\boldsymbol{A}^{\mathrm{T}}$ is represented by a graph with arrows from $\underline{0}$ to the elements of V^1 weighted by Ax. Inner truth values are represented by graphs with the only non-zero arrow between $\underline{0}$ and itself, weighted with the truth value it represents.

Sup-T-compositions of the derived notions then work as expected in such node graphs. For instance it can be seen in Figure 1 that the composition of $\boldsymbol{A}^{\mathrm{T}}$ and $\boldsymbol{B}$ is an arrow from $\underline{0}$ to $\underline{0}$ aggregating all values $Ax \mathbin{\&} Bx$, which indeed represents the compatibility of A and B, while the composition of $\boldsymbol{A}$ and $\boldsymbol{B}^{\mathrm{T}}$ is a relation between all pairs xy weighted by $Ax \mathbin{\&} By$ (as the only non-zero path from x to y goes through $\underline{0}$), which represents the Cartesian product $A \times B$.

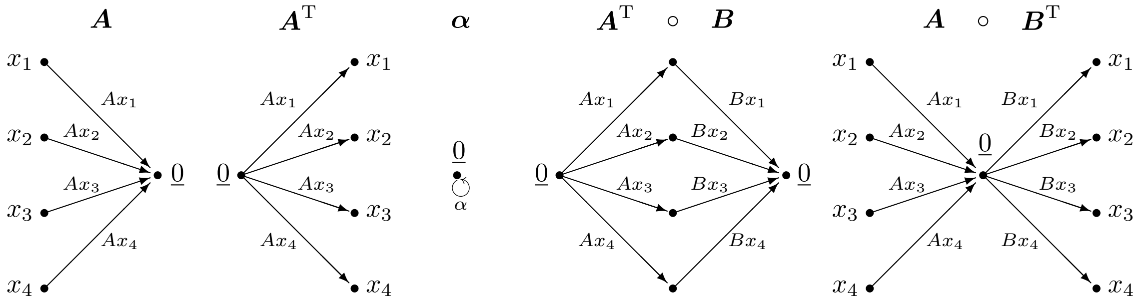

Figure 1: Graph representations of $\boldsymbol{A}$, $\boldsymbol{A}^{\mathrm{T}}$, $\boldsymbol{\alpha}$, $\boldsymbol{A} \circ \boldsymbol{B}^{\mathrm{T}}$, and $\boldsymbol{A}^{\mathrm{T}} \circ \boldsymbol{B}$ (zero-weighted arrows not indicated)

Besides the operations listed in Table 1, further important relational operations are definable from compositions—e.g., by taking the universal class V for an argument in some of the derived notions. Some of such derived notions are listed in Table 2.

domain	$\mathrm{Dom}\, R$	$=$	$R \,{}^{\leftarrow}\, \mathrm{V}$	...	$R \circ \mathbf{V}$
range	$\mathrm{Rng}\, R$	$=$	$R \,{}^{\rightarrow}\, \mathrm{V}$	...	$R^{\mathrm{T}} \circ \mathbf{V}$
height	$\overline{\mathrm{Hgt}}\, A$	$=$	$A \mathbin{\overline{\|}} \mathrm{V} = \mathrm{V} \mathbin{\overline{\|}} A$	...	$\boldsymbol{A}^{\mathrm{T}} \circ \mathbf{V} = \mathbf{V}^{\mathrm{T}} \circ \boldsymbol{A}$

Table 2: Further operations derived from sup-T-compositions

Indeed, our conventions identify the domain $\mathrm{Dom}\, R = \{x \mid (\exists z) Rxz\}$ of a fuzzy relation R with the vector $\{x\underline{0} \mid (\exists z)(Rxz \mathbin{\&} 1)\} = \{x\underline{0} \mid (\exists z)(Rxz \mathbin{\&} \mathbf{V}z\underline{0})\} = R \circ \mathbf{V}$, and similarly for Rng.

The third operation in Table 2 yields the formal truth value of the *height* $\mathrm{Hgt}\, A \equiv_{\mathrm{df}} (\exists z) Az$ of a fuzzy class A, which our conventions indeed identify with the scalar $\{\underline{00} \mid (\exists z)(\mathbf{V}z\underline{0} \mathbin{\&} \boldsymbol{A}z\underline{0})\} = \mathbf{V}^{\mathrm{T}} \circ \boldsymbol{A}$. In other words, $(\underline{00} \in \mathbf{V}^{\mathrm{T}} \circ \boldsymbol{A}) \leftrightarrow \mathrm{Hgt}\, A$, and therefore we can equate the height of A with the scalar $\mathbf{V}^{\mathrm{T}} \circ \boldsymbol{A}$. Like with $\overline{\|}$ or $\overline{\&}$, we denote the operation by $\overline{\mathrm{Hgt}}$ (overlined) as it yields an *inner* truth value (i.e., a fuzzy singleton) and needs to be formally distinguished from Hgt (which is a defined *predicate* and evaluates to *semantic* truth values in a model).

The point of the reduction of the above notions to compositions is of course that the properties of sup-T-compositions automatically transfer to all of them. Thus we now get dozens of theorems on fuzzy relational operations entirely for free.

First we apply Theorem 4.2(2) and Corollary 4.3(1) to the derived notions:

Corollary 4.7 FCT *proves the monotony of all notions listed in Tables 1 and 2 w.r.t. inclusion.*

In particular,

$$\begin{array}{rcl@{\qquad}rcl}
R_1 \subseteq R_2 & \to & R_1 \circ S \subseteq R_2 \circ S & S_1 \subseteq S_2 & \to & R \circ S_1 \subseteq R \circ S_2 \\
R_1 \subseteq R_2 & \to & R_1{}^{\rightharpoonup} A \subseteq R_2{}^{\rightharpoonup} A & A_1 \subseteq A_2 & \to & R^{\rightharpoonup} A_1 \subseteq R^{\rightharpoonup} A_2 \\
R_1 \subseteq R_2 & \to & R_1{}^{\leftharpoonup} A \subseteq R_2{}^{\leftharpoonup} A & A_1 \subseteq A_2 & \to & R^{\leftharpoonup} A_1 \subseteq R^{\leftharpoonup} A_2 \\
A_1 \subseteq A_2 & \to & A_1 \times B \subseteq A_2 \times B & B_1 \subseteq B_2 & \to & A \times B_1 \subseteq A \times B_2 \\
A_1 \subseteq A_2 & \to & (A_1 \parallel B \to A_2 \parallel B) & B_1 \subseteq B_2 & \to & (A \parallel B_1 \to A \parallel B_2) \\
A_1 \subseteq A_2 & \to & \alpha A_1 \subseteq \alpha A_2 & (\alpha_1 \to \alpha_2) & \to & \alpha_1 A \subseteq \alpha_2 A \\
(\alpha_1 \to \alpha_2) & \to & (\alpha_1 \mathbin{\&} \beta \to \alpha_2 \mathbin{\&} \beta) & (\beta_1 \to \beta_2) & \to & (\alpha \mathbin{\&} \beta_1 \to \alpha \mathbin{\&} \beta_2)
\end{array}$$

$$\begin{array}{rcl}
R_1 \subseteq R_2 & \to & \operatorname{Rng} R_1 \subseteq \operatorname{Rng} R_2 \\
R_1 \subseteq R_2 & \to & \operatorname{Dom} R_1 \subseteq \operatorname{Dom} R_2 \\
A_1 \subseteq A_2 & \to & (\operatorname{Hgt} A_1 \to \operatorname{Hgt} A_2)
\end{array}$$

Some comments (which apply to subsequent corollaries as well) are in order here:

Remark 4.8 Notice that, as usual in FCT, the theorems have the form of *provable implications.* Thus they are effective even if the antecedent is only partially valid: due to the semantics of implication, they express the fact that the consequent is *at least as true* as the antecedent. Therefore the theorems are stronger than the assertions of the form "if the antecedent is fully true (to degree 1), then so is the consequent", which are more usual in traditional fuzzy mathematics. The traditional theorems, which in formal fuzzy logic would have the form $\triangle\varphi \to \triangle\psi$ rather than $\varphi \to \psi$, follow from those proved in FCT as their special cases with the antecedents true to degree 1. Recall further that in FCT, not only the membership predicate $\in$, but all defined predicates are in general fuzzy (unless they are defined as provably crisp). Thus, e.g., $A \subseteq B$ does not express the fact that the membership function of B majorizes that of A (although this is the meaning of its being true to degree 1): according to its definition, $A \subseteq B$ yields the truth value of the formula $(\forall x)(Ax \to Bx)$, i.e., the infimum of all values $Ax \to Bx$. This kind of *gradual* inclusion has already been considered by Klaua in the 1960's (as reported in [29]) with Łukasiewicz implication; by Bandler and Kohout [3] with a broader class of implicational operators; and by many authors afterwards.

Remark 4.9 Many of the particular theorems listed here are known, even in their gradual forms (see esp. [27, 18]), and all of them have rather simple proofs in FCT. Therefore the main contribution of the present approach is rather the systematic method by which these propositions can be proved *all at once,* as corollaries of the simple statements of Theorem 4.2.

Remark 4.10 Although we present our methods for homogeneous relations only, they can be extended to heterogeneous relations in the following way. Heterogeneous fuzzy relations $R \subseteq^{\triangle} X \times Y$ (for crisp $X, Y \subseteq^{\triangle} \mathrm{V}$) can always be understood as homogeneous fuzzy relations $R \subseteq^{\triangle} \mathrm{V} \times \mathrm{V}$ by taking for V the disjoint union of the two domains X, Y and defining Rxy as 0 outside the domain $X \times Y$ of R. Since 0 is neutral w.r.t. $\exists$, the values of sup-T-compositions are not changed by this extension to V^2. The result of composition $R \circ S \subseteq^{\triangle} \mathrm{V}^2$ of heterogeneous fuzzy relations $R \subseteq^{\triangle} X \times Y$ and $S \subseteq^{\triangle} Y \times Z$ can then again be interpreted as the heterogeneous fuzzy relation $R \circ S \subseteq^{\triangle} X \times Z$, since it is easily proved in FCT that $R \subseteq^{\triangle} X \times Y \mathbin{\&} S \subseteq^{\triangle} Y \times Z \to R \circ S \subseteq^{\triangle} X \times Z$. Although the theory of heterogeneous relations is not exhausted by this reduction to homogeneous relations (as, i.a., the domains of relations are lost by the reduction), at least it enables to apply the results of the present paper to heterogeneous fuzzy relations.

The following two remarks regard formal and notational aspects of the presented results. Readers that are not interested in formalistic details can safely skip them.

Remark 4.11 We translate the theorems directly into their variants with fuzzy classes A and inner truth values α rather than their relational counterparts $\boldsymbol{A}, \boldsymbol{\alpha}$, although the latter are more direct corollaries of Theorem 4.2. The translation is made possible by the "isomorphism" of A

and $A \times \underline{1}$ mentioned in Section 3 and can be made precise by the methods of faithful formal interpretations described in [9]. We do not elaborate on these details here since for the theorems listed in the present paper, their preservation under the translation is perspicuous enough in each particular case.

Remark 4.12 We use the operations Hgt, $\|$ in Corollary 4.7, although more direct corollaries of Theorem 4.2 would contain their counterparts operating on inner truth values ($\overline{\mathrm{Hgt}}, \overline{\|}$). This is allowed by the fact that they directly correspond to each other, as $(\underline{00} \in \overline{\mathrm{Hgt}}\, A) \leftrightarrow \mathrm{Hgt}\, A$, and similarly for other scalar notions. Consequently, by Proposition 3.4(2), the inclusion $\overline{\mathrm{Hgt}}\, A_1 \subseteq \overline{\mathrm{Hgt}}\, A_2$ translates to implication $\mathrm{Hgt}\, A_1 \to \mathrm{Hgt}\, A_2$ (and similarly for $\overline{\|}$ and other scalar operations).

In particular, formal conjunction (i.e., the intersection of fuzzy singletons, $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$) translates into usual conjunction (as $\underline{00} \in \boldsymbol{\alpha} \cap \boldsymbol{\beta} \leftrightarrow \varphi \mathbin{\&} \psi$ for $\alpha = \overline{\varphi}$ and $\beta = \overline{\psi}$), and similarly inclusion of formal truth values translates into implication (both by Proposition 3.4). The monotony of $\boldsymbol{\alpha} \circ \boldsymbol{\beta}$ w.r.t. inclusion thus expresses the monotony of conjunction w.r.t. implication—a theorem that can of course be proved in a much simpler way (even propositionally). We include it here for the sake of completeness, and to show that FCT "knows" the formal counterpart of this propositional law (i.e., that its internalization operating on inner truth values is provable in FCT).

Now we shall continue listing (some of) the corollaries of Theorem 4.2 and Corollary 4.3 for the derived notions.

Corollary 4.13 FCT *proves the following relational properties w.r.t. unions and intersections:*

$$
\begin{array}{rcl@{\qquad}rcl}
(\bigcup_{R\in\mathcal{A}} R) \circ S & = & \bigcup_{R\in\mathcal{A}} (R \circ S) & R \circ \bigcup_{S\in\mathcal{A}} S & = & \bigcup_{S\in\mathcal{A}} (R \circ S) \\
(\bigcup_{R\in\mathcal{A}} R)^{\to} A & = & \bigcup_{R\in\mathcal{A}} (R^{\to} A) & R^{\to} \bigcup_{A\in\mathcal{A}} A & = & \bigcup_{A\in\mathcal{A}} (R^{\to} A) \\
(\bigcup_{R\in\mathcal{A}} R)^{\leftarrow} A & = & \bigcup_{R\in\mathcal{A}} (R^{\leftarrow} A) & R^{\leftarrow} \bigcup_{A\in\mathcal{A}} A & = & \bigcup_{A\in\mathcal{A}} (R^{\leftarrow} A) \\
(\bigcup_{A\in\mathcal{A}} A) \times B & = & \bigcup_{A\in\mathcal{A}} (A \times B) & A \times \bigcup_{B\in\mathcal{A}} B & = & \bigcup_{B\in\mathcal{A}} (A \times B) \\
(\bigcup_{A\in\mathcal{A}} A) \parallel B & \leftrightarrow & (\exists A \in \mathcal{A})(A \parallel B) & A \parallel \bigcup_{B\in\mathcal{A}} B & \leftrightarrow & (\exists B \in \mathcal{A})(A \parallel B) \\
\alpha \bigcup_{A\in\mathcal{A}} A & = & \bigcup_{A\in\mathcal{A}} (\alpha A) & (\bigvee_{\alpha\in\mathcal{A}} \alpha) A & = & \bigcup_{\alpha\in\mathcal{A}} (\alpha A) \\
(\bigvee_{\alpha\in\mathcal{A}} \alpha) \mathbin{\&} \beta & \leftrightarrow & \bigvee_{\alpha\in\mathcal{A}} (\alpha \mathbin{\&} \beta) & \alpha \mathbin{\&} \bigvee_{\beta\in\mathcal{A}} \beta & \leftrightarrow & \bigvee_{\beta\in\mathcal{A}} (\alpha \mathbin{\&} \beta)
\end{array}
$$

$$
\begin{array}{rcl}
\mathrm{Dom}(\bigcup_{R\in\mathcal{A}} R) & = & \bigcup_{R\in\mathcal{A}} \mathrm{Dom}\, R \\
\mathrm{Rng}(\bigcup_{R\in\mathcal{A}} R) & = & \bigcup_{R\in\mathcal{A}} \mathrm{Rng}\, R \\
\mathrm{Hgt}(\bigcup_{A\in\mathcal{A}} A) & \leftrightarrow & (\exists A \in \mathcal{A})(\mathrm{Hgt}\, A)
\end{array}
$$

$$
\begin{array}{rcl@{\qquad}rcl}
(\bigcap_{R\in\mathcal{A}} R) \circ S & \subseteq & \bigcap_{R\in\mathcal{A}} (R \circ S) & R \circ \bigcap_{S\in\mathcal{A}} S & \subseteq & \bigcap_{S\in\mathcal{A}} (R \circ S) \\
(\bigcap_{R\in\mathcal{A}} R)^{\to} A & \subseteq & \bigcap_{R\in\mathcal{A}} (R^{\to} A) & R^{\to} \bigcap_{A\in\mathcal{A}} A & \subseteq & \bigcap_{A\in\mathcal{A}} (R^{\to} A) \\
(\bigcap_{R\in\mathcal{A}} R)^{\leftarrow} A & \subseteq & \bigcap_{R\in\mathcal{A}} (R^{\leftarrow} A) & R^{\leftarrow} \bigcap_{A\in\mathcal{A}} A & \subseteq & \bigcap_{A\in\mathcal{A}} (R^{\leftarrow} A) \\
(\bigcap_{A\in\mathcal{A}} A) \times B & \subseteq & \bigcap_{A\in\mathcal{A}} (A \times B) & A \times \bigcap_{B\in\mathcal{A}} B & \subseteq & \bigcap_{B\in\mathcal{A}} (A \times B) \\
(\bigcap_{A\in\mathcal{A}} A) \parallel B & \to & (\forall A \in \mathcal{A})(A \parallel B) & A \parallel \bigcap_{B\in\mathcal{A}} B & \to & (\forall B \in \mathcal{A})(A \parallel B) \\
\alpha \bigcap_{A\in\mathcal{A}} A & \subseteq & \bigcap_{A\in\mathcal{A}} (\alpha A) & (\bigwedge_{\alpha\in\mathcal{A}} \alpha) A & \subseteq & \bigcap_{\alpha\in\mathcal{A}} (\alpha A) \\
(\bigwedge_{\alpha\in\mathcal{A}} \alpha) \mathbin{\&} \beta & \to & \bigwedge_{\alpha\in\mathcal{A}} (\alpha \mathbin{\&} \beta) & \alpha \mathbin{\&} \bigwedge_{\beta\in\mathcal{A}} \beta & \to & \bigwedge_{\beta\in\mathcal{A}} (\alpha \mathbin{\&} \beta)
\end{array}
$$

$$
\begin{array}{rcl}
\mathrm{Dom}(\bigcap_{R\in\mathcal{A}} R) & \subseteq & \bigcap_{R\in\mathcal{A}} \mathrm{Dom}\, R \\
\mathrm{Rng}(\bigcap_{R\in\mathcal{A}} R) & \subseteq & \bigcap_{R\in\mathcal{A}} \mathrm{Rng}\, R \\
\mathrm{Hgt}(\bigcap_{A\in\mathcal{A}} A) & \to & (\forall A \in \mathcal{A})(\mathrm{Hgt}\, A)
\end{array}
$$

The converse inclusions and implications have (well-known) crisp counter-examples, except those with the Cartesian product, resize, and conjunction, which only have fuzzy counter-examples in MTL *and do hold in stronger fuzzy logics like Gödel or Łukasiewicz.*

Proof: Since all of the inclusions and implications are direct corollaries of Theorem 4.2(3,4) and Corollary 4.3(2,3), we only need to prove the claim about counter-examples to converse inclusions and implications:

As can be seen from the proof of Theorem 4.2(4), the crisp counter-examples can be found whenever the quantification over z in formula (10) in the proof is not void, which (by definitions in Tables 1–2) is the case for all operations in Tables 1–2 except the resize, $\times$, and &. For the latter three operations, the second implication in (10) can be converted (thus they do not have crisp counter-examples), but still the converse to the first implication of (10) is not generally valid in MTL.[8] The first implication of (10) is nevertheless convertible (and so the converse inclusions and implications do hold for the resize, $\times$, and &) in stronger logics like Łukasiewicz or Gödel. QED

Relational operations can also be nested, whenever the types of their results permit. The associativity and transposition properties of sup-T-compositions proved in Theorem 4.2(1,5), Proposition 2.14(1), Proposition 3.7, and Lemma 4.15 (below) then yield an infinite number of identities between expressions composed of the operations from Tables 1 and 2: some of these are listed in the following corollary. We abandon the distinction between A and $\boldsymbol{A}$ here in order to make the chains of identities more compact (cf. footnote 5); similarly we do not distinguish scalar operations from the defined predicates they represent, e.g., $\overline{\mathrm{Hgt}}$ from Hgt (cf. Remark 4.12).

Corollary 4.14 FCT *proves the following identities:*

$$
\begin{array}{rcl}
(A \times B)^{\mathrm{T}} & = & (A \circ B^{\mathrm{T}})^{\mathrm{T}} = B \circ A^{\mathrm{T}} \quad = \quad B \times A \\
R \circ (A \times B) & = & R \circ (A \circ B^{\mathrm{T}}) = (R \circ A) \circ B^{\mathrm{T}} \quad = \quad (R^{\leftarrow} A) \times B \\
(A \times B) \circ R & = & A \circ B^{\mathrm{T}} \circ R = A \circ (R^{\mathrm{T}} \circ B)^{\mathrm{T}} \quad = \quad A \times (R^{\rightarrow} B) \\
A \times \alpha B & = & A \circ (B \circ \alpha)^{\mathrm{T}} = A \circ \alpha \circ B^{\mathrm{T}} \quad = \quad \alpha A \times B \\
A \times \operatorname{Rng} R & = & A \circ (R^{\mathrm{T}} \circ \mathrm{V})^{\mathrm{T}} = A \circ \mathrm{V}^{\mathrm{T}} \circ R \quad = \quad (A \times \mathrm{V}) \circ R
\end{array}
$$

$$
\begin{array}{rcl}
R^{\rightarrow}(S^{\rightarrow} A) & = & R^{\mathrm{T}} \circ (S^{\mathrm{T}} \circ A) = (S \circ R)^{\mathrm{T}} \circ A \quad = \quad (S \circ R)^{\rightarrow} A \\
R^{\rightarrow} \alpha A & = & R^{\mathrm{T}} \circ A \circ \alpha \quad = \quad \alpha(R^{\rightarrow} A) \\
R^{\rightarrow} \operatorname{Rng} S & = & R^{\mathrm{T}} \circ S^{\mathrm{T}} \circ \mathrm{V} = (S \circ R)^{\mathrm{T}} \circ \mathrm{V} \quad = \quad \operatorname{Rng}(S \circ R) \\
(A \times B)^{\rightarrow} C & = & (A \circ B^{\mathrm{T}})^{\mathrm{T}} \circ C = B \circ A^{\mathrm{T}} \circ C \quad = \quad (A \parallel C) B \\
R^{\leftarrow}(S^{\leftarrow} A) & = & R \circ S \circ A \quad = \quad (S \circ R)^{\leftarrow} A \\
R^{\leftarrow} \alpha A & = & R \circ A \circ \alpha \quad = \quad \alpha(R^{\leftarrow} A) \\
R^{\leftarrow} \operatorname{Dom} S & = & R \circ S \circ \mathrm{V} \quad = \quad \operatorname{Dom}(R \circ S) \\
\alpha(\beta A) & = & (A \circ \beta) \circ \alpha = A \circ (\alpha \circ \beta) \quad = \quad (\alpha \mathbin{\&} \beta) A \\
\alpha(\operatorname{Dom} R) & = & R \circ \mathrm{V} \circ \alpha \quad = \quad R^{\rightarrow} \alpha \mathrm{V} \\
\alpha(\operatorname{Rng} R) & = & R^{\mathrm{T}} \circ \mathrm{V} \circ \alpha \quad = \quad R^{\leftarrow} \alpha \mathrm{V} \\
\operatorname{Dom}(A \times B) & = & A \circ B^{\mathrm{T}} \circ \mathrm{V} \quad = \quad (\operatorname{Hgt} B) A \\
\operatorname{Rng}(A \times B) & = & (A \circ B^{\mathrm{T}})^{\mathrm{T}} \circ \mathrm{V} = B \circ A^{\mathrm{T}} \circ \mathrm{V} \quad = \quad (\operatorname{Hgt} A) B
\end{array}
$$

$$
\begin{array}{rcl}
A \parallel B & = & A^{\mathrm{T}} \circ B = (A^{\mathrm{T}} \circ B)^{\mathrm{T}} = B^{\mathrm{T}} \circ A \quad = \quad B \parallel A \\
\alpha \mathbin{\&} \beta & = & (\alpha \circ \beta)^{\mathrm{T}} = \beta^{\mathrm{T}} \circ \alpha^{\mathrm{T}} \quad = \quad \beta \mathbin{\&} \alpha \\
A \parallel (R^{\rightarrow} B) & = & A^{\mathrm{T}} \circ R^{\mathrm{T}} \circ B = (R \circ A)^{\mathrm{T}} \circ B \quad = \quad (R^{\leftarrow} A) \parallel B \\
A \parallel \alpha B & = & A^{\mathrm{T}} \circ B \circ \alpha \quad = \quad \alpha \mathbin{\&} (A \parallel B) \\
A \parallel \operatorname{Dom} R & = & A^{\mathrm{T}} \circ R \circ \mathrm{V} = (R^{\mathrm{T}} \circ A)^{\mathrm{T}} \circ \mathrm{V} \quad = \quad \operatorname{Hgt}(R^{\rightarrow} A) \\
A \parallel \operatorname{Rng} R & = & A^{\mathrm{T}} \circ R^{\mathrm{T}} \circ \mathrm{V} = (R \circ A)^{\mathrm{T}} \circ \mathrm{V} \quad = \quad \operatorname{Hgt}(R^{\leftarrow} A) \\
\alpha \mathbin{\&} (\beta \mathbin{\&} \gamma) & = & \alpha \circ \beta \circ \gamma \quad = \quad (\alpha \mathbin{\&} \beta) \mathbin{\&} \gamma \\
\operatorname{Hgt} \alpha A & = & \mathrm{V}^{\mathrm{T}} \circ A \circ \alpha \quad = \quad \alpha \mathbin{\&} \operatorname{Hgt} A \\
\operatorname{Hgt} \operatorname{Dom} R & = & \mathrm{V}^{\mathrm{T}} \circ R \circ \mathrm{V} = (R^{\mathrm{T}} \circ \mathrm{V})^{\mathrm{T}} \circ \mathrm{V} \quad = \quad \operatorname{Hgt} \operatorname{Rng} R
\end{array}
$$

[8] A (well-known) counter-example in MTL is, e.g., a $[0,1]$-model with $\alpha = 0.5$, $\beta_n = 0.5 + \frac{1}{n}$ for all natural n, and the nilpotent minimum [25] for &; then $\alpha \mathbin{\&} \bigwedge \beta_n$ is 0, while $\bigwedge(\alpha \mathbin{\&} \beta_n)$ is 0.5. (The counter-examples for the resize and Cartesian product are similar.)

Corollary 4.14 actually lists provable identities between almost all terms with two nested sup-T-operations: it only omits some uninteresting cases like $(A \parallel B)^{\mathrm{T}} = A \parallel B$, formal artifacts like $\operatorname{Hgt}(\operatorname{Hgt} A) = \operatorname{Hgt} A$, and identities easily reducible to those above by the commutativity of $\parallel$ and & or the interdefinability $R^{\rightarrow} A = (R^{\mathrm{T}})^{\leftarrow} A$ and $\operatorname{Rng} R = \operatorname{Dom} R^{\mathrm{T}}$. Identities between more complex terms composed of sup-T-operations can be derived by similar simple calculations like those above. For proving some of them, also the following lemma is needed:

Lemma 4.15 FCT *proves the following identities:*

1. $\mathbf{V}^{\mathrm{T}} \circ \mathbf{V} = \underline{\mathbf{1}}$
2. $\boldsymbol{A} \circ \underline{\mathbf{1}} = \boldsymbol{A}, \quad \boldsymbol{\alpha} \circ \underline{\mathbf{1}} = \boldsymbol{\alpha}$

Proof: 1. $\mathbf{V}^{\mathrm{T}} \circ \mathbf{V} = \{\underline{00} \mid (\exists z)(\mathbf{V}^{\mathrm{T}}\underline{0}z \,\&\, \mathbf{V}z\underline{0})\} = \{\underline{00} \mid (\exists z)(\mathrm{V}z \,\&\, \mathrm{V}z)\} = \{\underline{00} \mid 1\} = \underline{\mathbf{1}}$.
2. follows similarly from the provability in MTL of $\alpha \,\&\, 1 \leftrightarrow 1$ and $(\exists z)1 \leftrightarrow 1$. QED

Example 4.16 By Lemma 4.15 we get $(A \times \mathrm{V})^{\leftarrow}\alpha\mathrm{V} = A \circ \mathrm{V}^{\mathrm{T}} \circ \mathrm{V} \circ \alpha = A \circ \underline{1} \circ \alpha = A \circ \alpha = \alpha A$.

5 BK-products and derived notions

Besides sup-T-composition, many other products of fuzzy relations have been defined in the literature. Perhaps the most notable among these is the relational product which can be called *inf-R-composition,* as it replaces the supremum in the definition of composition by infimum and the t-norm by its residuum.[9] It has been introduced by Bandler and Kohout in [1] for crisp relations and generalized to fuzzy relations in [2]; referring to the initials of the authors, inf-R-composition is also known as the *BK-product* of fuzzy or crisp relations. Depending on the direction of the residual implication (left-to-right, right-to-left, or both) we get three variants of BK-products:

Definition 5.1 We define the following three products of fuzzy relations R, S:

$$\begin{array}{rcll} R \lhd S & =_{\mathrm{df}} & \{xy \mid (\forall z)(Rxz \rightarrow Szy)\} & \ldots \textit{BK-subproduct} \\ R \rhd S & =_{\mathrm{df}} & \{xy \mid (\forall z)(Rxz \leftarrow Szy)\} & \ldots \textit{BK-superproduct} \\ R \,\square\, S & =_{\mathrm{df}} & \{xy \mid (\forall z)(Rxz \leftrightarrow Szy)\} & \ldots \textit{BK-squareproduct} \end{array}$$

The prefix BK may be omitted if no confusion can arise. By the *BK-product* (simpliciter) we shall mean the BK-subproduct.

For the motivation and utility of BK-products see [35, 36]. In this paper we give further illustrations of their importance and ubiquity in the theory of fuzzy relations.

Remark 5.2 BK-products have some properties that are felt undesirable in certain kinds of applications of fuzzy relations. As an especially problematic property is by many authors seen the fact that $(R \lhd S)xy$ is 1 whenever $(\exists z)(Rxz)$ is 0. To avoid this particular feature of BK-products, De Baets and Kerre proposed a redefinition of the same notion in [21]: in our notation, De Baets and Kerre's modified definition of $R \lhd S$ reads $\{xy \mid (\exists z)(Rxz) \,\&\, (\forall z)(Rxz \rightarrow Szy)\}$, and similarly for $\rhd$ and $\square$. Following De Baets and Kerre's paper, some authors when speaking about BK-products refer to the modified definition rather than Bandler and Kohout's original definition. As this may lead to confusion, we need to stress that in the present paper, we always refer to the *original definitions* by Bandler and Kohout (i.e., those of Definition 5.1), and never to the modification by De Baets and Kerre.

Our sticking to Bandler and Kohout's original definition is justified not only by the suitability for our needs, but also by the fact that De Baets and Kerre's elimination of the "useless pairs" from the product is only suitable in certain applications of fuzzy relational products. In other areas of fuzzy mathematics (e.g., the theory of fuzzy orderings, as shown below), the original

[9] The relationship between sup-T and inf-R composition is an instance of Morsi's duality [40].

notion of BK-product is well-motivated, and the "useless pairs" play important roles in various manifestations of BK-products throughout the theory. This suggests that the emended definition by De Baets and Kerre should not *replace* the original definition by Bandler and Kohout, but only complement it; from this point of view it seems unfortunate that the authors of [21] chose to overload the definition and notation of BK-products rather than to use a modified name and symbols.

In what follows, we shall need the following (well-known) properties of BK-products.

Theorem 5.3 FCT *proves the following properties of BK-products:*

1. *Transposition:* $(R \lhd S)^{\mathrm{T}} = S^{\mathrm{T}} \rhd R^{\mathrm{T}}$
2. *Monotony:* $R_1 \subseteq R_2 \to R_2 \lhd S \subseteq R_1 \lhd S, \quad S_1 \subseteq S_2 \to R \lhd S_1 \subseteq R \lhd S_2$
3. *Intersection:* $\displaystyle\bigcap_{R \in \mathcal{A}} (R \lhd S) = \Big(\bigcup_{R \in \mathcal{A}} R\Big) \lhd S, \quad \bigcap_{S \in \mathcal{A}} (R \lhd S) = R \lhd \bigcap_{S \in \mathcal{A}} S$
4. *Union:* $\displaystyle\bigcup_{R \in \mathcal{A}} (R \lhd S) \subseteq \Big(\bigcap_{R \in \mathcal{A}} R\Big) \lhd S, \quad \bigcup_{S \in \mathcal{A}} (R \lhd S) \subseteq R \lhd \bigcup_{S \in \mathcal{A}} S$
   (Converse inclusions have crisp counter-examples.)
5. *Residuation:* $R \lhd (S \lhd T) = (R \circ S) \lhd T$
6. *Exchange:* $R \lhd (S \rhd T) = (R \lhd S) \rhd T$
7. *Interdefinability:* $R \mathbin{\square} S = (R \lhd S) \cap_\wedge (R \rhd S)$

Proof: Claims 1–3 are proved similarly as the corresponding statements of Theorem 4.2 (for the shifts of relativized quantifiers needed here, see [16] and Lemma 2.4). The two inclusions of claim 4 are respectively proved by the following chains of implications:

$$(\exists R \in \mathcal{A})(\forall z)(Rxz \to Szy) \longrightarrow (\forall z)(\exists R \in \mathcal{A})(Rxz \to Szy) \longrightarrow (\forall z)[(\forall R \in \mathcal{A})Rxz \to Szy] \quad (11)$$
$$(\exists S \in \mathcal{A})(\forall z)(Rxz \to Szy) \longrightarrow (\forall z)(\exists S \in \mathcal{A})(Rxz \to Szy) \longrightarrow (\forall z)[Rxz \to (\exists S \in \mathcal{A})Szy] \quad (12)$$

The existence of crisp counter-examples to the converse inclusions follows from the fact that the first implications in (11)–(12) cannot be converted in classical logic (as the quantifiers do not commute), while the second implications can.

5. $xy \in R \lhd (S \lhd T) \longleftrightarrow (\forall z)(Rxz \to (\forall t)(Szt \to Tty)) \longleftrightarrow (\forall zt)(Rxz \to (Szt \to Tty)) \longleftrightarrow (\forall zt)(Rxz \mathbin{\&} Szt \to Tty) \longleftrightarrow (\forall t)((\exists z)(Rxz \mathbin{\&} Szt) \to Tty) \longleftrightarrow xy \in (R \circ S) \lhd T$, and similarly for 6.

7. $xy \in R \mathbin{\square} S \longleftrightarrow (\forall z)(Rxz \leftrightarrow Szy) \longleftrightarrow (\forall z)[(Rxz \to Szy) \wedge (Rxz \leftarrow Szy)] \longleftrightarrow (\forall z)(Rxz \to Szy) \wedge (\forall z)(Rxz \leftarrow Szy) \longleftrightarrow xy \in (R \lhd S) \cap_\wedge (R \rhd S)$. QED

By transposition of the statements of Theorem 5.3 we get the following properties of BK-products:

Corollary 5.4 FCT *proves:*

1. *Transposition:* $(R \rhd S)^{\mathrm{T}} = S^{\mathrm{T}} \lhd R^{\mathrm{T}}, \quad (R \mathbin{\square} S)^{\mathrm{T}} = S^{\mathrm{T}} \mathbin{\square} R^{\mathrm{T}}$
2. *Monotony:* $R_1 \subseteq R_2 \to R_1 \rhd S \subseteq R_2 \rhd S, \quad S_1 \subseteq S_2 \to R \rhd S_2 \subseteq R \rhd S_1$
3. *Intersection:* $\displaystyle\bigcap_{R \in \mathcal{A}} (R \rhd S) = \Big(\bigcap_{R \in \mathcal{A}} R\Big) \rhd S, \quad \bigcap_{S \in \mathcal{A}} (R \rhd S) = R \rhd \bigcup_{S \in \mathcal{A}} S$
4. *Union:* $\displaystyle\bigcup_{R \in \mathcal{A}} (R \rhd S) \subseteq \Big(\bigcup_{R \in \mathcal{A}} R\Big) \rhd S, \quad \bigcup_{S \in \mathcal{A}} (R \rhd S) \subseteq R \rhd \bigcap_{S \in \mathcal{A}} S$
   (Converse inclusions have crisp counter-examples.)

5. *Residuation:* $(R \rhd S) \rhd T = R \rhd (S \circ T)$

Applying the identifications of the previous section to BK-products in the same way as we did to sup-T-products, we get the derived notions listed in Tables 3–5. We write just $\subseteq, \rightarrow, \leftrightarrow, \mathrm{Plt}$, instead of the more correct $\overline{\subseteq}, \overline{\rightarrow}, \overline{\leftrightarrow}, \overline{\mathrm{Plt}}$ (cf. Remark 4.12).

	$\{xy \mid (\forall z)(Rxz \rightarrow Szy)\}$	$=$	$R \lhd S$	$\ldots$	$\lhd$-*product*	$R \lhd S$
$x = \underline{0}$	$\{\underline{0}y \mid (\forall z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \rightarrow Rzy)\}^{\mathrm{T}} = (\boldsymbol{A}^{\mathrm{T}} \lhd R)^{\mathrm{T}}$	$=$	$R^{\mathrm{T}} \rhd \boldsymbol{A}$	$\ldots$	$\lhd$-*image*	$R^{\lhd\!\!\leftrightarrow} A$
$y = \underline{0}$	$\{x\underline{0} \mid (\forall z)(Rxz \rightarrow \boldsymbol{A}z\underline{0})\}$	$=$	$R \lhd \boldsymbol{A}$	$\ldots$	$\lhd$-*pre-image*	$R^{\leftarrow\!\!\lhd} A$
$z = \underline{0}$	$\{xy \mid (\forall \underline{0})(\boldsymbol{A}x\underline{0} \rightarrow \boldsymbol{B}^{\mathrm{T}}\underline{0}y)\}$	$=$	$\boldsymbol{A} \lhd \boldsymbol{B}^{\mathrm{T}}$	$\ldots$	*Cartesian* $\lhd$-*product*	$A \times_{\lhd} B$
$x, y = \underline{0}$	$\{\underline{00} \mid (\forall z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \rightarrow \boldsymbol{B}z\underline{0})\}$	$=$	$\boldsymbol{A}^{\mathrm{T}} \lhd \boldsymbol{B}$	$\ldots$	*inclusion*	$A \subseteq B$
$x, z = \underline{0}$	$\{\underline{0}y \mid (\forall \underline{0})(\boldsymbol{\alpha}^{\mathrm{T}}\underline{00} \rightarrow \boldsymbol{A}^{\mathrm{T}}\underline{0}y)\}^{\mathrm{T}} = (\boldsymbol{\alpha}^{\mathrm{T}} \lhd \boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}}$	$=$	$\boldsymbol{A} \rhd \boldsymbol{\alpha}$	$\ldots$	*left* α-*resize*	$\alpha_{\rightarrow} A$
$y, z = \underline{0}$	$\{x\underline{0} \mid (\forall \underline{0})(\boldsymbol{A}x\underline{0} \rightarrow \boldsymbol{\alpha}\underline{00})\}$	$=$	$\boldsymbol{A} \lhd \boldsymbol{\alpha}$	$\ldots$	*right* α-*resize*	$A_{\rightarrow}\alpha$
$x, y, z = \underline{0}$	$\{\underline{00} \mid (\forall \underline{0})(\boldsymbol{\alpha}\underline{00} \rightarrow \boldsymbol{\beta}\underline{00})\}$	$=$	$\boldsymbol{\alpha} \lhd \boldsymbol{\beta}$	$\ldots$	*implication*	$\alpha \rightarrow \beta$

$\lhd$-*range*	$\mathrm{Rng}^{\lhd} R$	$=$	$R^{\lhd\!\!\leftrightarrow} \mathrm{V}$	$\ldots$	$R^{\mathrm{T}} \rhd \mathbf{V}$
plinth	$\mathrm{Plt}\, A$	$=$	$\mathrm{V} \subseteq A$	$\ldots$	$\mathbf{V}^{\mathrm{T}} \lhd \boldsymbol{A}$

Table 3: Operations derived from the BK-subproduct

	$\{xy \mid (\forall z)(Rxz \leftarrow Szy)\}$	$=$	$R \rhd S$	$\ldots$	$\rhd$-*product*	$R \rhd S$
$x = \underline{0}$	$\{\underline{0}y \mid (\forall z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \leftarrow Rzy)\}^{\mathrm{T}} = (\boldsymbol{A}^{\mathrm{T}} \rhd R)^{\mathrm{T}}$	$=$	$R^{\mathrm{T}} \lhd \boldsymbol{A}$	$\ldots$	$\rhd$-*image*	$R^{\rhd\!\!\rightarrow} A$
$y = \underline{0}$	$\{x\underline{0} \mid (\forall z)(Rxz \leftarrow \boldsymbol{A}z\underline{0})\}$	$=$	$R \rhd \boldsymbol{A}$	$\ldots$	$\rhd$-*pre-image*	$R^{\leftrightarrow\!\!\rhd} A$
$z = \underline{0}$	$\{xy \mid (\forall \underline{0})(\boldsymbol{A}x\underline{0} \leftarrow \boldsymbol{B}^{\mathrm{T}}\underline{0}y)\}$	$=$	$\boldsymbol{A} \rhd \boldsymbol{B}^{\mathrm{T}}$	$\ldots$	*Cartesian* $\rhd$-*product*	$A \times_{\rhd} B$
$x, y = \underline{0}$	$\{\underline{00} \mid (\forall z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \leftarrow \boldsymbol{B}z\underline{0})\}$	$=$	$\boldsymbol{A}^{\mathrm{T}} \rhd \boldsymbol{B}$	$\ldots$	*converse inclusion*	$A \supseteq B$
$x, z = \underline{0}$	$\{\underline{0}y \mid (\forall \underline{0})(\boldsymbol{\alpha}^{\mathrm{T}}\underline{00} \leftarrow \boldsymbol{A}^{\mathrm{T}}\underline{0}y)\}^{\mathrm{T}} = (\boldsymbol{\alpha}^{\mathrm{T}} \rhd \boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}}$	$=$	$\boldsymbol{A} \lhd \boldsymbol{\alpha}$	$\ldots$	*right* α-*resize*	$A_{\rightarrow}\alpha$
$y, z = \underline{0}$	$\{x\underline{0} \mid (\forall \underline{0})(\boldsymbol{A}x\underline{0} \leftarrow \boldsymbol{\alpha}\underline{00})\}$	$=$	$\boldsymbol{A} \rhd \boldsymbol{\alpha}$	$\ldots$	*left* α-*resize*	$\alpha_{\rightarrow} A$
$x, y, z = \underline{0}$	$\{\underline{00} \mid (\forall \underline{0})(\boldsymbol{\alpha}\underline{00} \leftarrow \boldsymbol{\beta}\underline{00})\}$	$=$	$\boldsymbol{\alpha} \rhd \boldsymbol{\beta}$	$\ldots$	*converse implication*	$\alpha \leftarrow \beta$

$\rhd$-*domain*	$\mathrm{Dom}^{\rhd} R$	$=$	$R^{\leftrightarrow\!\!\rhd} \mathrm{V}$	$\ldots$	$R \rhd \mathbf{V}$

Table 4: Operations derived from the BK-superproduct

	$\{xy \mid (\forall z)(Rxz \leftrightarrow Szy)\}$	$=$	$R \,\square\, S$	$\ldots$	$\square$-*product*	$R \,\square\, S$
$x = \underline{0}$	$\{\underline{0}y \mid (\forall z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \leftrightarrow Rzy)\}^{\mathrm{T}} = (\boldsymbol{A}^{\mathrm{T}} \,\square\, R)^{\mathrm{T}}$	$=$	$R^{\mathrm{T}} \,\square\, \boldsymbol{A}$	$\ldots$	$\square$-*image*	$R^{\square\!\!\rightarrow} A$
$y = \underline{0}$	$\{x\underline{0} \mid (\forall z)(Rxz \leftrightarrow \boldsymbol{A}z\underline{0})\}$	$=$	$R \,\square\, \boldsymbol{A}$	$\ldots$	$\square$-*pre-image*	$R^{\leftarrow\!\!\square} A$
$z = \underline{0}$	$\{xy \mid (\forall \underline{0})(\boldsymbol{A}x\underline{0} \leftrightarrow \boldsymbol{B}^{\mathrm{T}}\underline{0}y)\}$	$=$	$\boldsymbol{A} \,\square\, \boldsymbol{B}^{\mathrm{T}}$	$\ldots$	*Cartesian* $\square$-*product*	$A \times_{\square} B$
$x, y = \underline{0}$	$\{\underline{00} \mid (\forall z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \leftrightarrow \boldsymbol{B}z\underline{0})\}$	$=$	$\boldsymbol{A}^{\mathrm{T}} \,\square\, \boldsymbol{B}$	$\ldots$	*weak bi-inclusion*	$A \approx B$
$x, z = \underline{0}$	$\{\underline{0}y \mid (\forall \underline{0})(\boldsymbol{\alpha}^{\mathrm{T}}\underline{00} \leftrightarrow \boldsymbol{A}^{\mathrm{T}}\underline{0}y)\}^{\mathrm{T}} = (\boldsymbol{\alpha}^{\mathrm{T}} \,\square\, \boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}}$	$=$	$\boldsymbol{A} \,\square\, \boldsymbol{\alpha}$	$\ldots$	*left-right* α-*resize*	$\alpha_{\leftrightarrow} A$
$y, z = \underline{0}$	$\{x\underline{0} \mid (\forall \underline{0})(\boldsymbol{A}x\underline{0} \leftrightarrow \boldsymbol{\alpha}\underline{00})\}$	$=$	$\boldsymbol{A} \,\square\, \boldsymbol{\alpha}$	$\ldots$	*left-right* α-*resize*	$\alpha_{\leftrightarrow} A$
$x, y, z = \underline{0}$	$\{\underline{00} \mid (\forall \underline{0})(\boldsymbol{\alpha}\underline{00} \leftrightarrow \boldsymbol{\beta}\underline{00})\}$	$=$	$\boldsymbol{\alpha} \,\square\, \boldsymbol{\beta}$	$\ldots$	*equivalence*	$\alpha \leftrightarrow \beta$

Table 5: Operations derived from the BK-squareproduct

Remark 5.5 Notice that some of the analogues of notions based on sup-T-compositions are omitted from Tables 3–5 due to their triviality. The BK-subdomain $\mathrm{Dom}^{\lhd} R = R^{\leftarrow\!\!\lhd}\mathrm{V}$, i.e., $R \lhd \mathrm{V}$, is always equal to V (similarly for $\rhd$-range) and the superproduct analogue of height or plinth always equals $\underline{1}$. Therefore, by Theorem 5.3(7), the squareproduct analogue of Dom is in fact $\mathrm{Dom}^{\rhd}$, the squareproduct analogue of Rng is $\mathrm{Rng}^{\lhd} R$, and the squareproduct analogue of plinth is just plinth.

Remark 5.6 Unlike in sup-T-compositions, where the behavior of 0 w.r.t. & ensured the right type (in the sense of Convention 3.6) of the result of products for subclasses of $\mathrm{V} \times \underline{1}$ and $\underline{1} \times \underline{1}$

(e.g., that $R \circ \boldsymbol{A} \subseteq^{\triangle} \mathrm{V} \times \underline{1}$), in BK-products this is not automatic (since $0 \to 0$ is 1 rather than 0). For BK-compositions, the right type of the result has to be explicitly controlled by intersecting it with $\mathrm{V} \times \underline{1}$ or $\underline{1} \times \underline{1}$, according to the types of operands: for instance, the correct definition of $R^{\lhd\!\leftrightarrow} A$ is $(R^{\mathrm{T}} \lhd \boldsymbol{A}) \cap (\mathrm{V} \times \underline{1})$ rather than just $R^{\mathrm{T}} \lhd \boldsymbol{A}$, and for $\operatorname{Plt} A$ it is $(\mathbf{V}^{\mathrm{T}} \lhd \boldsymbol{A}) \cap (\underline{1} \times \underline{1})$ rather than just $\mathbf{V}^{\mathrm{T}} \lhd \boldsymbol{A}$. We omit the intersection in the definitions, since the right type is already indicated by Convention 3.6 and the properties studied in this paper are obviously preserved by the intersection controlling the type; thus the values of BK-compositions outside their target domain $\mathrm{V} \times \underline{1}$ or $\underline{1} \times \underline{1}$ can safely be ignored. A similar adjustment (by defining Rxy as 1 rather than 0 outside the domain $X \times Y$ of R) has to be made when using BK-compositions of heterogeneous rather than homogeneous relations (cf. Remark 4.10).

Corollary 5.7 *By Theorem 5.3(1) and the definitions of Tables 3 and 4 we have the following interdefinability between derived BK-notions:*

$$\begin{array}{rclcl}
A \times_{\rhd} B & = & A \rhd B^{\mathrm{T}} = (B \lhd A^{\mathrm{T}})^{\mathrm{T}} & = & (B \times_{\lhd} A)^{\mathrm{T}} \\
R^{\leftrightarrow\!\rhd} A & = & R \rhd A = R^{\mathrm{TT}} \rhd A & = & R^{\mathrm{T}\,\lhd\!\leftrightarrow} A \\
\operatorname{Dom}^{\rhd} R & = & R \rhd \mathrm{V} = R^{\mathrm{TT}} \rhd \mathrm{V} & = & \operatorname{Rng}^{\lhd} R^{\mathrm{T}}
\end{array}$$

Corollary 5.8 *By Theorem 5.3(7), the squareproduct notions are definable in terms of the corresponding subproduct and superproduct notions by means of min-intersection (or min-conjunction):*

$$\begin{array}{rcl}
R^{\Box\!\rightarrow} A & = & (R^{\lhd\!\leftrightarrow} A) \cap_{\wedge} (R^{\rhd\!\rightarrow} A) \\
R^{\leftarrow\!\Box} A & = & (R^{\leftarrow\!\lhd} A) \cap_{\wedge} (R^{\leftrightarrow\!\rhd} A) \\
A \times_{\Box} B & = & (A \times_{\lhd} B) \cap_{\wedge} (A \times_{\rhd} B) \\
A \approx B & = & (A \subseteq B) \wedge (B \subseteq A) \\
\alpha_{\leftrightarrow} A & = & (\alpha_{\rightarrow} A) \cap_{\wedge} (A_{\rightarrow} \alpha) \\
\alpha \leftrightarrow \beta & = & (\alpha \to \beta) \wedge (\beta \to \alpha)
\end{array}$$

The importance of the ten sup-T-based operations studied in the previous section is beyond doubt. The following examples show that the BK-related notions abound in fuzzy mathematics as well. Thus the present section can also be viewed as the systematization of these miscellaneous notions and their properties.

Example 5.9 The operation of *subproduct preimage* $R^{\leftarrow\!\lhd} A$ appears frequently in the theory of fuzzy relations [27, 18, 19]. In [26], $R^{\leftarrow\!\lhd}$ is called the *right backward strong powerset operator* of R; the operation is denoted by $\downarrow$ in [27]. It is also a quantifier construction in fuzzy description logic [31], where it is written as $(\forall R.A)$. Further graded properties of this operation besides those studied here can be found in [27, 11]. The *superproduct image* $R^{\rhd\!\rightarrow}$ is studied, e.g., in [26] where it is called the *right forward strong powerset operator* of R.

Example 5.10 In the theory of fuzzy orderings, the *subproduct image* $R^{\lhd\!\leftrightarrow} A$ and *superproduct preimage* $R^{\leftrightarrow\!\rhd} A$ denote the fuzzy set of all upper resp. lower bounds of the fuzzy set A w.r.t. a fuzzy ordering R (also called the upper and lower *cone* of A w.r.t. R). The operations $R^{\lhd\!\leftrightarrow}$ and $R^{\leftrightarrow\!\rhd}$, respectively, are called the *exclusive image* and *exclusive preimage* in [5] and the *left forward* and *left backward strong powerset operator of* R in [26]. The operation ${}^{\lhd\!\leftrightarrow}$ has also appeared in [22] and has been used for fuzzy inference in [20].

Example 5.11 For some applications of *BK-products* $\lhd, \rhd, \Box$ themselves see [35, 36]. Besides the practically oriented applications, their theoretical importance comes from the fact that many other relational notions can be expressed by means of BK-products. For example, fuzzy preorders can be characterized in terms of BK-products [7] by

$$\begin{array}{rcl}
\operatorname{Refl} R & \leftrightarrow & R^{\mathrm{T}} \lhd R \subseteq R \\
\operatorname{Trans} R & \leftrightarrow & R \subseteq R^{\mathrm{T}} \lhd R
\end{array}$$

The operation $R^{\mathrm{T}} \lhd R$ and its dual $R \lhd R^{\mathrm{T}}$ are sometimes called the *left* resp. *right trace* of R and are of their own importance [24, 11].

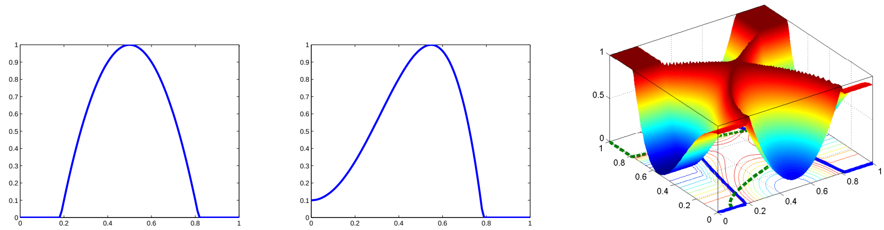

Figure 2: Fuzzy sets A and B and their Cartesian squareproduct $A \times_{\Box} B$ under the Łukasiewicz t-norm

Example 5.12 The *Cartesian products* $\times, \times_{\lhd}, \times_{\rhd}, \times_{\Box}$ are used to model sets of fuzzy rules:

$$\begin{array}{rccl}
\{(x \text{ is } \mathrm{A}_i) & \texttt{and} & (y \text{ is } \mathrm{B}_i)\}_{i\in I} & \ldots \; \bigcup_{i\in I}(A_i \times B_i) \\
\{\texttt{if} \;\; (x \text{ is } \mathrm{A}_i) & \texttt{then} & (y \text{ is } \mathrm{B}_i)\}_{i\in I} & \ldots \; \bigcap_{i\in I}(A_i \times_{\lhd} B_i) \\
\{(x \text{ is } \mathrm{A}_i) & \texttt{whenever} & (y \text{ is } \mathrm{B}_i)\}_{i\in I} & \ldots \; \bigcap_{i\in I}(A_i \times_{\rhd} B_i) \\
\{(x \text{ is } \mathrm{A}_i) & \texttt{iff} & (y \text{ is } \mathrm{B}_i)\}_{i\in I} & \ldots \; \bigcap_{i\in I}(A_i \times_{\Box} B_i)
\end{array}$$

The first three operations are used in many applications of fuzzy control theory, even though $\times$ is often misinterpreted as "implication" [38] rather than the Cartesian product based on strong *conjunction*. The Cartesian squareproduct $\times_{\Box}$ is rather neglected in the fuzzy literature, even though in many approximation problems it is more appropriate than $\times_{\lhd}$ and $\times_{\rhd}$, as it captures fuzzy equivalence between input and output fuzzy sets, expressing that "x is A to a similar degree as y is B" (see Figure 2).

Example 5.13 The *α-resizes* $\alpha A, \alpha_{\rightarrow} A, A_{\rightarrow}\alpha, \alpha_{\leftrightarrow} A$ occur in fuzzy control applications. There are two competing approaches to approximate inference over a knowledge base formalized as a set of fuzzy rules. The classical approach is *FATI* (first aggregate then infer). The *FITA* (first infer then aggregate) method of activation degrees was first used by Holmblad and Ostergaard [33] in a fuzzy control algorithm for a cement kiln. It can briefly be described as follows [28]:

For each actual input fuzzy set A and each input–output data pair (A_k, B_k) one determines a modification B^*_k of the "local" output B_k, and aggregates the modified "local" outputs into one global output: $B^* = \bigcup_{i\in I} B^*_i$. The particular choice by Holmblad and Ostergaard for B^*_k was $B^*_k(y) = \mathrm{Hgt}(A \cap_{\wedge} A_k) \cdot B_k(y)$, which is in fact the $\mathrm{Hgt}(A \cap_{\wedge} A_k)$-resize of B_k under the product t-norm.

To take another example, if Zadeh's compositional rule of inference is applied to a knowledge formalized by $\times$, which in our formalism reads $\left(\bigcup_{i\in I}(A_i \times B_i)\right) {}^{\rightarrow} A$, it can be simplified by using α-resizes in virtue of the identity

$$\left(\bigcup_{i\in I}(A_i \times B_i)\right) {}^{\rightarrow} A = \bigcup_{i\in I}(A \parallel A_i)B_i$$

which follows from Corollaries 4.13 and 4.14. Analogously, the authors of [39] speak about the consequent dilatation rule proposed in [37], where the degrees of subsethood $A \subseteq A_i$ for $i \in I$ are used to compute the final output which is in our notation written as $B^* = \bigcap_{i\in I}(A \subseteq A_i)_{\rightarrow} B_i$ (cf. the appropriate identities from Corollaries 5.16 and 5.17).

The main argument in favor of practical applications of α-resizes is the speed of computations. It is much faster to resize and then aggregate than to use the FATI approach because the values for the resizes are computed only once and then used multiple times.

Example 5.14 In theoretical investigation of fuzzy relations, α-resizes appear for instance in the following contexts: closedness under S_K-intersections for a set K of designated truth values [18, Def. 7.4] is equivalent [18, Th. 7.6] to closedness under intersections of "K-shifted" sets $(\alpha_{\rightarrow} A)$; furthermore, $A_{\rightarrow}\alpha$ and αA have been used to characterize a system of closed sets of a similarity space in [18, Th. 7.62]; the system of all extensional fuzzy sets can be characterized by means of $\alpha_{\to}A$, $A_{\to}\alpha$ and αA [34, Th. 3.2]; the α-properties of binary fuzzy relations studied in [6] are related to α-resizes of a relation [6, Th. 4.24]; etc.

The above list of applications of inf-R-compositional notions is by no means exhaustive. Like with the notions based on the sup-T-composition, the point of our construction is the possibility of applying Theorem 5.3 and Corollary 5.4 to all notions defined in Tables 3–5. Thus we are given the following corollaries entirely for free (Remarks 4.8–4.12 apply to these corollaries as well):

Corollary 5.15 *In consequence of Theorem 5.3(2) and Corollary 5.4(2),* FCT *proves:*

$$
\begin{array}{rcl@{\qquad}rcl}
R_1 \subseteq R_2 & \to & R_1 {}^{\triangleleft\!\!\!\rightarrow} A \subseteq R_2 {}^{\triangleleft\!\!\!\rightarrow} A & A_1 \subseteq A_2 & \to & R {}^{\triangleleft\!\!\!\rightarrow} A_2 \subseteq R {}^{\triangleleft\!\!\!\rightarrow} A_1 \\
R_1 \subseteq R_2 & \to & R_2 {}^{\leftarrow\!\!\!\triangleleft} A \subseteq R_1 {}^{\leftarrow\!\!\!\triangleleft} A & A_1 \subseteq A_2 & \to & R {}^{\leftarrow\!\!\!\triangleleft} A_1 \subseteq R {}^{\leftarrow\!\!\!\triangleleft} A_2 \\
A_1 \subseteq A_2 & \to & A_2 \times_{\triangleleft} B \subseteq A_1 \times_{\triangleleft} B & B_1 \subseteq B_2 & \to & A \times_{\triangleleft} B_1 \subseteq A \times_{\triangleleft} B_2 \\
A_1 \subseteq A_2 & \to & (A_2 \subseteq B \to A_1 \subseteq B) & A_1 \subseteq A_2 & \to & (B \subseteq A_1 \to B \subseteq A_2) \\
A_1 \subseteq A_2 & \to & \alpha_{\to}A_1 \subseteq \alpha_{\to}A_2 & (\alpha_1 \to \alpha_2) & \to & (\alpha_2)_{\to}A \subseteq (\alpha_1)_{\to}A \\
A_1 \subseteq A_2 & \to & (A_2)_{\to}\alpha \subseteq (A_1)_{\to}\alpha & (\alpha_1 \to \alpha_2) & \to & (A_{\to}\alpha_1 \to A_{\to}\alpha_2) \\
(\alpha_1 \to \alpha_2) & \to & [(\alpha_2 \to \beta) \to (\alpha_1 \to \beta)] & (\beta_1 \to \beta_2) & \to & [(\alpha \to \beta_1) \to (\alpha \to \beta_2)]
\end{array}
$$

$$
\begin{array}{rcl}
R_1 \subseteq R_2 & \to & \operatorname{Rng}^{\triangleleft} R_1 \subseteq \operatorname{Rng}^{\triangleleft} R_2 \\
A_1 \subseteq A_2 & \to & (\operatorname{Plt} A_1 \to \operatorname{Plt} A_2)
\end{array}
$$

$$
\begin{array}{rcl@{\qquad}rcl}
R_1 \subseteq R_2 & \to & R_2 {}^{\triangleright\!\!\!\rightarrow} A \subseteq R_1 {}^{\triangleright\!\!\!\rightarrow} A & A_1 \subseteq A_2 & \to & R {}^{\triangleright\!\!\!\rightarrow} A_1 \subseteq R {}^{\triangleright\!\!\!\rightarrow} A_2 \\
R_1 \subseteq R_2 & \to & R_1 {}^{\leftarrow\!\!\!\triangleright} A \subseteq R_2 {}^{\leftarrow\!\!\!\triangleright} A & A_1 \subseteq A_2 & \to & R {}^{\leftarrow\!\!\!\triangleright} A_2 \subseteq R {}^{\leftarrow\!\!\!\triangleright} A_1 \\
A_1 \subseteq A_2 & \to & A_1 \times_{\triangleright} B \subseteq A_2 \times_{\triangleright} B & B_1 \subseteq B_2 & \to & A \times_{\triangleright} B_2 \subseteq A \times_{\triangleright} B_1
\end{array}
$$

$$
R_1 \subseteq R_2 \quad \to \quad \operatorname{Dom}^{\triangleright} R_1 \subseteq \operatorname{Dom}^{\triangleright} R_2
$$

Corollary 5.16 *By Theorem 5.3(3, 4) and Corollary 5.4(3, 4),* FCT *proves:*

$$
\begin{array}{rcl@{\qquad}rcl}
\bigcap_{R\in\mathcal{A}}(R {}^{\triangleleft\!\!\!\rightarrow} A) & = & (\bigcap_{R\in\mathcal{A}} R) {}^{\triangleleft\!\!\!\rightarrow} A & \bigcap_{A\in\mathcal{A}}(R {}^{\triangleleft\!\!\!\rightarrow} A) & = & R {}^{\triangleleft\!\!\!\rightarrow} \bigcup_{A\in\mathcal{A}} A \\
\bigcap_{R\in\mathcal{A}}(R {}^{\leftarrow\!\!\!\triangleleft} A) & = & (\bigcup_{R\in\mathcal{A}} R) {}^{\leftarrow\!\!\!\triangleleft} A & \bigcap_{A\in\mathcal{A}}(R {}^{\leftarrow\!\!\!\triangleleft} A) & = & R {}^{\leftarrow\!\!\!\triangleleft} \bigcap_{A\in\mathcal{A}} A \\
\bigcap_{A\in\mathcal{A}}(A \times_{\triangleleft} B) & = & (\bigcup_{A\in\mathcal{A}} A) \times_{\triangleleft} B & \bigcap_{B\in\mathcal{A}}(A \times_{\triangleleft} B) & = & A \times_{\triangleleft} \bigcap_{B\in\mathcal{A}} B \\
(\forall A \in \mathcal{A})(A \subseteq B) & \leftrightarrow & (\bigcup_{A\in\mathcal{A}} A) \subseteq B & (\forall B \in \mathcal{A})(A \subseteq B) & \leftrightarrow & A \subseteq \bigcap_{B\in\mathcal{A}} B \\
\bigcap_{\alpha\in\mathcal{A}}(\alpha_{\to}A) & = & (\bigvee_{\alpha\in\mathcal{A}} \alpha)_{\to}A & \bigcap_{A\in\mathcal{A}}(\alpha_{\to}A) & = & \alpha_{\to} \bigcap_{A\in\mathcal{A}} A \\
\bigcap_{A\in\mathcal{A}}(A_{\to}\alpha) & = & (\bigcup_{A\in\mathcal{A}} A)_{\to}\alpha & \bigcap_{\alpha\in\mathrm{A}}(A_{\to}\alpha) & = & A_{\to} \bigwedge_{\alpha\in\mathcal{A}} \alpha \\
\bigwedge_{\alpha\in\mathcal{A}}(\alpha \to \beta) & \leftrightarrow & (\bigvee_{\alpha\in\mathcal{A}}) \to \beta & \bigwedge_{\beta\in\mathcal{A}}(\alpha \to \beta) & \leftrightarrow & (\alpha \to \bigwedge_{\beta\in\mathcal{A}} \beta)
\end{array}
$$

$$
\begin{array}{rcl}
\bigcap_{R\in\mathcal{A}} \operatorname{Rng}^{\triangleleft} R & = & \operatorname{Rng}^{\triangleleft} \bigcap_{R\in\mathcal{A}} R \\
(\forall A \in \mathcal{A})(\operatorname{Plt} A) & \leftrightarrow & \operatorname{Plt} \bigcap_{A\in\mathcal{A}} A
\end{array}
$$

$$
\begin{array}{rcl@{\qquad}rcl}
\bigcap_{R\in\mathcal{A}}(R {}^{\triangleright\!\!\!\rightarrow} A) & = & (\bigcup_{R\in\mathcal{A}} R) {}^{\triangleright\!\!\!\rightarrow} A & \bigcap_{A\in\mathcal{A}}(R {}^{\triangleright\!\!\!\rightarrow} A) & = & R {}^{\triangleright\!\!\!\rightarrow} \bigcap_{A\in\mathcal{A}} A \\
\bigcap_{R\in\mathcal{A}}(R {}^{\leftarrow\!\!\!\triangleright} A) & = & (\bigcap_{R\in\mathcal{A}} R) {}^{\leftarrow\!\!\!\triangleright} A & \bigcap_{A\in\mathcal{A}}(R {}^{\leftarrow\!\!\!\triangleright} A) & = & R {}^{\leftarrow\!\!\!\triangleright} \bigcup_{A\in\mathcal{A}} A \\
\bigcap_{A\in\mathcal{A}}(A \times_{\triangleright} B) & = & (\bigcap_{A\in\mathcal{A}} A) \times_{\triangleright} B & \bigcap_{B\in\mathcal{A}}(A \times_{\triangleright} B) & = & A \times_{\triangleright} \bigcup_{B\in\mathcal{A}} B
\end{array}
$$

$$
\bigcap_{R\in\mathcal{A}} \operatorname{Dom}^{\triangleright} R \quad = \quad \operatorname{Dom}^{\triangleright} \bigcap_{R\in\mathcal{A}} R
$$

$$
\begin{array}{rcl@{\qquad}rcl}
\bigcup_{R\in\mathcal{A}}(R {}^{\triangleleft\!\!\!\rightarrow} A) & \subseteq & (\bigcup_{R\in\mathcal{A}} R) {}^{\triangleleft\!\!\!\rightarrow} A & \bigcup_{A\in\mathcal{A}}(R {}^{\triangleleft\!\!\!\rightarrow} A) & \subseteq & R {}^{\triangleleft\!\!\!\rightarrow} \bigcap_{A\in\mathcal{A}} A \\
\bigcup_{R\in\mathcal{A}}(R {}^{\leftarrow\!\!\!\triangleleft} A) & \subseteq & (\bigcap_{R\in\mathcal{A}} R) {}^{\leftarrow\!\!\!\triangleleft} A & \bigcup_{A\in\mathcal{A}}(R {}^{\leftarrow\!\!\!\triangleleft} A) & \subseteq & R {}^{\leftarrow\!\!\!\triangleleft} \bigcup_{A\in\mathcal{A}} A \\
\bigcup_{A\in\mathcal{A}}(A \times_{\triangleleft} B) & \subseteq & (\bigcap_{A\in\mathcal{A}} A) \times_{\triangleleft} B & \bigcup_{B\in\mathcal{A}}(A \times_{\triangleleft} B) & \subseteq & A \times_{\triangleleft} \bigcup_{B\in\mathcal{A}} B \\
(\exists A \in \mathcal{A})(A \subseteq B) & \to & (\bigcap_{A\in\mathcal{A}} A) \subseteq B & (\exists B \in \mathcal{A})(A \subseteq B) & \to & A \subseteq \bigcup_{B\in\mathcal{A}} B \\
\bigcup_{\alpha\in\mathcal{A}}(\alpha_{\to}A) & \subseteq & (\bigwedge_{\alpha\in\mathcal{A}} \alpha)_{\to}A & \bigcup_{A\in\mathcal{A}}(\alpha_{\to}A) & \subseteq & \alpha_{\to} \bigcup_{A\in\mathcal{A}} A \\
\bigcup_{A\in\mathcal{A}}(A_{\to}\alpha) & \subseteq & (\bigcap_{A\in\mathcal{A}} A)_{\to}\alpha & \bigcup_{\alpha\in\mathrm{A}}(A_{\to}\alpha) & \subseteq & A_{\to} \bigvee_{\alpha\in\mathcal{A}} \alpha \\
\bigvee_{\alpha\in\mathcal{A}}(\alpha \to \beta) & \to & ((\bigwedge_{\alpha\in\mathcal{A}} \alpha) \to \beta) & \bigvee_{\beta\in\mathcal{A}}(\alpha \to \beta) & \to & (\alpha \to \bigvee_{\beta\in\mathcal{A}} \alpha)
\end{array}
$$

$$
\begin{array}{rcl}
\bigcup_{R\in\mathcal{A}} \operatorname{Rng}^{\lhd} R & \subseteq & \operatorname{Rng}^{\lhd} \bigcup_{R\in\mathcal{A}} R \\
(\exists A \in \mathcal{A})(\operatorname{Plt} A) & \rightarrow & \operatorname{Plt} \bigcup_{A\in\mathcal{A}} A
\end{array}
$$

$$
\begin{array}{rclcrcl}
\bigcup_{R\in\mathcal{A}}(R^{\rhd\!\rightarrow} A) & \subseteq & \left(\bigcap_{R\in\mathcal{A}} R\right)^{\rhd\!\rightarrow} A & & \bigcup_{A\in\mathcal{A}}(R^{\rhd\!\rightarrow} A) & \subseteq & R^{\rhd\!\rightarrow} \bigcup_{A\in\mathcal{A}} A \\
\bigcup_{R\in\mathcal{A}}(R^{\leftrightarrow\!\rhd} A) & \subseteq & \left(\bigcup_{R\in\mathcal{A}} R\right)^{\leftrightarrow\!\rhd} A & & \bigcup_{A\in\mathcal{A}}(R^{\leftrightarrow\!\rhd} A) & \subseteq & R^{\leftrightarrow\!\rhd} \bigcap_{A\in\mathcal{A}} A \\
\bigcup_{A\in\mathcal{A}}(A \times_{\rhd} B) & \subseteq & \left(\bigcup_{A\in\mathcal{A}} A\right) \times_{\rhd} B & & \bigcup_{B\in\mathcal{A}}(A \times_{\rhd} B) & \subseteq & A \times_{\rhd} \bigcap_{B\in\mathcal{A}} B
\end{array}
$$

$$
\bigcup_{R\in\mathcal{A}} \operatorname{Dom}^{\rhd} R \quad \subseteq \quad \operatorname{Dom}^{\rhd} \bigcup_{R\in\mathcal{A}} R
$$

The converse implications and inclusions have crisp counter-examples.

Proof: We only need to prove the claim about converse inclusions and implications, as the rest are direct corollaries of the indicated theorems. The existence of crisp counter-examples follows from the fact that neither of implications in the proof of Theorem 5.3(4) is in general convertible in classical logic. In the case of $\times_{\lhd}$, $\times_{\rhd}$, ${}_{\rightarrow}$, and $\rightarrow$, for which the quantification over z in formulae (11)–(12) in the proof is void, the only crisp counter-examples are with $\mathcal{A} = \emptyset$. For non-empty $\mathcal{A}$, the latter converses hold in those extensions of MTL in which the law of double negation $\neg\neg\varphi \rightarrow \varphi$ is valid (i.e., the extensions of IMTL, e.g., Łukasiewicz logic), since the second implications in formulae (11)–(12) are convertible under double negation (but not generally in MTL). QED

Corollary 5.17 *By Theorem 5.3(5, 6) and Corollary 5.4(5),* FCT *proves, i.a., the following identities:*

$$
\begin{array}{rclcl}
(R^{\leftarrow} A) \times_{\lhd} B & = & R \lhd (A \times_{\lhd} B) & \textit{by} & (R \circ A) \lhd B^{\mathrm{T}} = R \lhd (A \lhd B^{\mathrm{T}}) \\
(\operatorname{Dom} R) \times_{\lhd} A & = & R \lhd (\mathrm{V} \times_{\lhd} A) & & (R \circ \mathrm{V}) \lhd A^{\mathrm{T}} = R \lhd (\mathrm{V} \lhd A^{\mathrm{T}}) \\
(\alpha A) \times_{\lhd} B & = & A \times_{\lhd} (\alpha_{\rightarrow} B) & & (\alpha \circ A) \lhd B^{\mathrm{T}} = A \lhd (\alpha \lhd B^{\mathrm{T}}) \\
A \times_{\lhd} (R^{\rhd\!\rightarrow} B) & = & (A \times_{\lhd} B) \rhd R & & A \lhd (R^{\mathrm{T}} \lhd B)^{\mathrm{T}} = A \lhd (B^{\mathrm{T}} \rhd R) = (A \lhd B^{\mathrm{T}}) \rhd R \\
A \times_{\lhd} (R^{\lhd\!\leftrightarrow} B) & = & (A \times B) \lhd R & & A \lhd (R^{\mathrm{T}} \rhd B)^{\mathrm{T}} = A \lhd (B^{\mathrm{T}} \lhd R) = (A \circ B^{\mathrm{T}}) \lhd R \\
A \times_{\lhd} (\operatorname{Rng}^{\lhd} R) & = & (A \times \mathrm{V}) \lhd R & & A \lhd (R^{\mathrm{T}} \rhd \mathrm{V})^{\mathrm{T}} = A \lhd (\mathrm{V}^{\mathrm{T}} \lhd R) = (A \circ \mathrm{V}^{\mathrm{T}}) \lhd R \\
A \times_{\lhd} (B_{\rightarrow} \alpha) & = & (A_{\rightarrow} \alpha) \times_{\rhd} B & & A \lhd (B \lhd \alpha)^{\mathrm{T}} = A \lhd (\alpha \rhd B^{\mathrm{T}}) = (A \lhd \alpha) \rhd B^{\mathrm{T}}
\end{array}
$$

$$
\begin{array}{rclcl}
(R \circ S)^{\leftarrow\!\lhd} A & = & R^{\leftarrow\!\lhd}(S^{\leftarrow\!\lhd} A) & \textit{by} & (R \circ S) \lhd A = R \lhd (S \lhd A) \\
(R \lhd S)^{\lhd\!\leftrightarrow} A & = & S^{\lhd\!\leftrightarrow}(R^{\rightarrow} A) & & (R \lhd S)^{\mathrm{T}} \rhd A = (S^{\mathrm{T}} \rhd R^{\mathrm{T}}) \rhd A = S^{\mathrm{T}} \rhd (R^{\mathrm{T}} \circ A) \\
(R \rhd S)^{\lhd\!\leftrightarrow} A & = & S^{\rhd\!\rightarrow}(R^{\lhd\!\leftrightarrow} A) & & (R \rhd S)^{\mathrm{T}} \rhd A = (S^{\mathrm{T}} \lhd R^{\mathrm{T}}) \rhd A = S^{\mathrm{T}} \lhd (R^{\mathrm{T}} \rhd A) \\
(A \times B)^{\leftarrow\!\lhd} C & = & A_{\rightarrow}(B \subseteq C) & & (A \circ B^{\mathrm{T}}) \lhd C = A \lhd (B^{\mathrm{T}} \lhd C) \\
(A \times_{\lhd} B)^{\lhd\!\leftrightarrow} C & = & (A \parallel C)_{\rightarrow} B & & (A \lhd B^{\mathrm{T}})^{\mathrm{T}} \rhd C = (B \rhd A^{\mathrm{T}}) \rhd C = B \rhd (A^{\mathrm{T}} \circ C) \\
(A \times_{\rhd} B)^{\lhd\!\leftrightarrow} C & = & B_{\rightarrow}(C \subseteq A) & & (A \rhd B^{\mathrm{T}})^{\mathrm{T}} \rhd C = (B \lhd A^{\mathrm{T}}) \rhd C = B \lhd (A^{\mathrm{T}} \rhd C) \\
R^{\leftarrow\!\lhd}(A_{\rightarrow} \alpha) & = & (R^{\leftarrow} A)_{\rightarrow} \alpha & & R \lhd (A \lhd \alpha) = (R \circ A) \lhd \alpha \\
\alpha_{\rightarrow}(\operatorname{Rng}^{\lhd} R) & = & R^{\lhd\!\leftrightarrow}(\alpha \mathrm{V}) & & \alpha \lhd (R^{\mathrm{T}} \rhd \mathrm{V}) = (R^{\mathrm{T}} \rhd \mathrm{V}) \rhd \alpha = R^{\mathrm{T}} \rhd (\alpha \circ \mathrm{V}) \\
\alpha_{\rightarrow}(R^{\lhd\!\leftrightarrow} A) & = & R^{\lhd\!\leftrightarrow}(\alpha A) & & \alpha \lhd (R^{\mathrm{T}} \rhd A) = (R^{\mathrm{T}} \rhd A) \rhd \alpha = R^{\mathrm{T}} \rhd (\alpha \circ A) \\
\alpha_{\rightarrow}(R^{\leftarrow\!\lhd} A) & = & R^{\leftarrow\!\lhd}(\alpha_{\rightarrow} A) & & \alpha \lhd (R \lhd A) = R \lhd (\alpha \lhd A) \\
\alpha_{\rightarrow}(\beta_{\rightarrow} A) & = & (\alpha \mathbin{\&} \beta)_{\rightarrow} A & & \alpha \lhd (\beta \lhd A) = (\alpha \circ \beta) \lhd A \\
\alpha_{\rightarrow}(A_{\rightarrow} \beta) & = & A_{\rightarrow}(\alpha \rightarrow \beta) & & \alpha \lhd (A \lhd \beta) = A \lhd (\alpha \lhd \beta) \\
A_{\rightarrow}(\alpha \rightarrow \beta) & = & (\alpha A)_{\rightarrow} \beta & & A \lhd (\alpha \lhd \beta) = (A \circ \alpha) \lhd \beta \\
\operatorname{Rng}^{\lhd}(R \lhd S) & = & S^{\lhd\!\leftrightarrow}(\operatorname{Rng} R) & & (R \lhd S)^{\mathrm{T}} \rhd \mathrm{V} = (S^{\mathrm{T}} \rhd R^{\mathrm{T}}) \rhd \mathrm{V} = S^{\mathrm{T}} \rhd (R^{\mathrm{T}} \circ \mathrm{V}) \\
\operatorname{Rng}^{\lhd}(R \rhd S) & = & S^{\rhd\!\rightarrow}(\operatorname{Rng}^{\lhd} R) & & (R \rhd S)^{\mathrm{T}} \rhd \mathrm{V} = (S^{\mathrm{T}} \lhd R^{\mathrm{T}}) \rhd \mathrm{V} = S^{\mathrm{T}} \lhd (R^{\mathrm{T}} \rhd \mathrm{V}) \\
\operatorname{Rng}^{\lhd}(A \times_{\lhd} B) & = & (\operatorname{Hgt} A)_{\rightarrow} B & & (A \lhd B^{\mathrm{T}})^{\mathrm{T}} \rhd \mathrm{V} = (B \rhd A^{\mathrm{T}}) \rhd \mathrm{V} = B \rhd (A^{\mathrm{T}} \circ \mathrm{V}) \\
\operatorname{Rng}^{\lhd}(A \times_{\rhd} B) & = & B_{\rightarrow}(\operatorname{Plt} A) & & (A \rhd B^{\mathrm{T}})^{\mathrm{T}} \rhd \mathrm{V} = (B \lhd A^{\mathrm{T}}) \rhd \mathrm{V} = B \lhd (A^{\mathrm{T}} \rhd \mathrm{V})
\end{array}
$$

$$
\begin{array}{rcl@{\quad}l}
A \subseteq (R^{\leftarrow\!\lhd} B) & = & (R^{\rightarrow} A) \subseteq B & \textit{by} \quad A^{\mathrm{T}} \lhd (R \lhd B) = (A^{\mathrm{T}} \circ R) \lhd B = (R^{\mathrm{T}} \circ A)^{\mathrm{T}} \lhd B \\
A \subseteq (R^{\lhd\!\leftrightarrow} B) & = & B \subseteq (R^{\leftrightarrow\!\rhd} A) & \quad A^{\mathrm{T}} \lhd (R^{\mathrm{T}} \rhd B) = (A^{\mathrm{T}} \lhd R^{\mathrm{T}}) \rhd B = (R \rhd A)^{\mathrm{T}} \rhd B \\
(\alpha A) \subseteq B & = & \alpha \to (A \subseteq B) & \quad (\alpha \circ A)^{\mathrm{T}} \lhd B = (\alpha \circ A^{\mathrm{T}}) \lhd B = \alpha \lhd (A^{\mathrm{T}} \lhd B) \\
A \subseteq (\alpha_{\to} B) & = & \alpha \to (A \subseteq B) & \quad A^{\mathrm{T}} \lhd (\alpha \lhd B) = \alpha \lhd (A^{\mathrm{T}} \lhd B) \\
A \subseteq (B_{\to} \alpha) & = & (A \parallel B) \to \alpha & \quad A^{\mathrm{T}} \lhd (B \lhd \alpha) = (A^{\mathrm{T}} \circ B) \lhd \alpha \\
\alpha \to (\beta \to \gamma) & = & (\alpha \,\&\, \beta) \to \gamma & \quad \alpha \lhd (\beta \lhd \gamma) = (\alpha \circ \beta) \lhd \gamma \\
\alpha \to (\beta \to \gamma) & = & \beta \to (\alpha \to \gamma) & \quad \alpha \lhd (\beta \lhd \gamma) = (\alpha \circ \beta) \lhd \gamma = (\beta \circ \alpha) \lhd \gamma = \beta \lhd (\alpha \lhd \gamma) \\
\mathrm{Plt}(R^{\lhd\!\leftrightarrow} A) & = & A \subseteq (\mathrm{Dom}^{\rhd} R) & \quad \mathrm{V}^{\mathrm{T}} \lhd (R^{\mathrm{T}} \rhd A) = (\mathrm{V}^{\mathrm{T}} \lhd R^{\mathrm{T}}) \rhd A = (R \rhd \mathrm{V})^{\mathrm{T}} \rhd A \\
\mathrm{Plt}(R^{\leftarrow\!\lhd} A) & = & (\mathrm{Rng}\, R) \subseteq A & \quad \mathrm{V}^{\mathrm{T}} \lhd (R \lhd A) = (\mathrm{V}^{\mathrm{T}} \circ R) \lhd A = (R^{\mathrm{T}} \circ \mathrm{V})^{\mathrm{T}} \lhd A \\
\mathrm{Plt}(\alpha_{\to} A) & = & \alpha \to \mathrm{Plt}\, A & \quad \mathrm{V}^{\mathrm{T}} \lhd (\alpha \lhd A) = \alpha \lhd (\mathrm{V}^{\mathrm{T}} \lhd A) \\
\mathrm{Plt}(\alpha_{\to} A) & = & \alpha \mathrm{V} \subseteq A & \quad \mathrm{V}^{\mathrm{T}} \lhd (\alpha \lhd A) = (\mathrm{V}^{\mathrm{T}} \circ \alpha) \lhd A = (\alpha \circ \mathrm{V})^{\mathrm{T}} \lhd A \\
\mathrm{Plt}(A_{\to} \alpha) & = & (\mathrm{Hgt}\, A) \to \alpha & \quad \mathrm{V}^{\mathrm{T}} \lhd (A \lhd \alpha) = (\mathrm{V}^{\mathrm{T}} \circ A) \lhd \alpha \\
\mathrm{Plt}(\mathrm{Rng}^{\lhd} R) & = & \mathrm{Plt}(\mathrm{Dom}^{\rhd} R) & \quad \mathrm{V}^{\mathrm{T}} \lhd (R^{\mathrm{T}} \rhd \mathrm{V}) = (\mathrm{V}^{\mathrm{T}} \lhd R^{\mathrm{T}}) \rhd \mathrm{V} = (R \rhd \mathrm{V})^{\mathrm{T}} \rhd \mathrm{V}
\end{array}
$$

Remark 5.18 Some of the identities of Corollary 5.17 express important theorems on fuzzy relations. For instance, the identity $(A \subseteq (R^{\leftarrow\!\lhd} B)) \leftrightarrow ((R^{\rightarrow} A) \subseteq B)$ entails the equivalence of two characterizations of the property of *extensionality* of a fuzzy class A w.r.t. a fuzzy relation R defined as $\mathrm{Ext}_R\, A \equiv_{\mathrm{df}} (\forall xy)(Rxy \,\&\, Ax \to Ay)$, since the latter can be expressed as $(R^{\rightarrow} A) \subseteq A$. The next identity $(A \subseteq (R^{\lhd\!\leftrightarrow} B)) \leftrightarrow (B \subseteq (R^{\leftrightarrow\!\rhd} A))$ expresses a graded theorem on fuzzy preorders (cf. Example 5.10) that all elements of A are upper bounds of B iff all elements of B are lower bounds of A. These theorems are well-known in the non-graded setting; here we get their graded variants (i.e., also for partially valid inclusions) for free.

Corollary 5.19 *Furthermore, by Corollary 5.7,* FCT *proves the following identities dual to Corollary 5.17 for superproduct notions:*

$$
\begin{array}{rcl@{\qquad}rcl}
(R \circ S)^{\rhd\!\rightarrow} A & = & S^{\rhd\!\rightarrow} (R^{\rhd\!\rightarrow} A) & A \times_{\rhd} (R^{\rightarrow} B) & = & (A \times_{\rhd} B) \rhd R \\
(R \rhd S)^{\leftrightarrow\!\rhd} A & = & R^{\leftrightarrow\!\rhd} (S^{\leftarrow} A) & A \times_{\rhd} (\mathrm{Rng}\, R) & = & (A \times_{\rhd} \mathrm{V}) \rhd R \\
(R \lhd S)^{\leftrightarrow\!\rhd} A & = & R^{\leftarrow\!\lhd} (S^{\leftrightarrow\!\rhd} A) & A \times_{\rhd} (\alpha B) & = & (\alpha_{\to} A) \times_{\rhd} B \\
(A \times B)^{\rhd\!\rightarrow} C & = & B_{\to} (A \subseteq C) & (R^{\leftarrow\!\lhd} A) \times_{\rhd} B & = & R \lhd (A \times_{\rhd} B) \\
(A \times_{\lhd} B)^{\leftrightarrow\!\rhd} C & = & A_{\to} (C \subseteq B) & (R^{\leftrightarrow\!\rhd} A) \times_{\rhd} B & = & R \rhd (A \times B) \\
(A \times_{\rhd} B)^{\leftrightarrow\!\rhd} C & = & (B \parallel C)_{\to} A & (\mathrm{Dom}^{\rhd} R) \times_{\rhd} B & = & R \rhd (\mathrm{V} \times B) \\
R^{\rhd\!\rightarrow} (A_{\to} \alpha) & = & (R^{\rightarrow} A)_{\to} \alpha & & & \\
\alpha_{\to} (R^{\rhd\!\rightarrow} A) & = & R^{\rhd\!\rightarrow} (\alpha_{\to} A) & A \subseteq (R^{\rhd\!\rightarrow} B) & = & (R^{\leftarrow} A) \subseteq B \\
\alpha_{\to} (R^{\leftrightarrow\!\rhd} A) & = & R^{\leftrightarrow\!\rhd} (\alpha A) & \mathrm{Plt}(R^{\rhd\!\rightarrow} A) & = & (\mathrm{Dom}\, R) \subseteq A \\
\alpha_{\to} (\mathrm{Dom}^{\rhd} R) & = & R^{\leftrightarrow\!\rhd} (\alpha \mathrm{V}) & \mathrm{Plt}(R^{\leftrightarrow\!\rhd} A) & = & A \subseteq (\mathrm{Rng}^{\lhd} R) \\
\mathrm{Dom}^{\rhd} (R \lhd S) & = & R^{\leftarrow\!\lhd} (\mathrm{Dom}^{\rhd} S) & & & \\
\mathrm{Dom}^{\rhd} (R \rhd S) & = & R^{\leftrightarrow\!\rhd} (\mathrm{Dom}\, S) & & & \\
\mathrm{Dom}^{\rhd} (A \times_{\lhd} B) & = & A_{\to} (\mathrm{Plt}\, B) & & & \\
\mathrm{Dom}^{\rhd} (A \times_{\rhd} B) & = & (\mathrm{Hgt}\, B)_{\to} A & & &
\end{array}
$$

Although not used in the previous corollaries, the following lemma is needed for some more complex identities between BK-based terms:

Lemma 5.20 FCT *proves:*

1. $\mathbf{V} \lhd \boldsymbol{A}^{\mathrm{T}} = \mathbf{V} \circ \boldsymbol{A}^{\mathrm{T}}, \quad \mathbf{V} \lhd \boldsymbol{\alpha} = \mathbf{V} \circ \boldsymbol{\alpha}$

2. $\boldsymbol{A} \lhd \underline{\mathbf{1}} = \mathbf{V}, \quad \boldsymbol{A} \rhd \underline{\mathbf{1}} = \boldsymbol{A}, \quad \boldsymbol{\alpha} \lhd \underline{\mathbf{1}} = \underline{\mathbf{1}}, \quad \boldsymbol{\alpha} \rhd \underline{\mathbf{1}} = \boldsymbol{\alpha}$

Proof: $\mathbf{V} \lhd \boldsymbol{A}^{\mathrm{T}} = \{xy \mid \mathbf{V}x\underline{0} \to \boldsymbol{A}\underline{0}y\} = \{xy \mid \boldsymbol{A}\underline{0}y\} = \{xy \mid \mathbf{V}x\underline{0} \,\&\, \boldsymbol{A}\underline{0}y\} = \mathbf{V} \circ \boldsymbol{A}^{\mathrm{T}}$, and analogously for the other identities. QED

Example 5.21 The following identities are among corollaries of Lemma 5.20:

$$
\begin{array}{rll}
R \overset{\hookleftarrow\!\lhd}{} \mathrm{V} = \mathrm{V} & \textit{by} & R \lhd \mathrm{V} = R \lhd (\mathrm{V} \lhd \underline{1}) = (R \circ \mathrm{V}) \lhd \underline{1} \\
(A \subseteq \mathrm{V}) = \underline{1} & & A^{\mathrm{T}} \lhd \mathrm{V} = A^{\mathrm{T}} \lhd (A \lhd \underline{1}) = (A^{\mathrm{T}} \circ A) \lhd \underline{1} \\
\alpha_{\rightarrow} \mathrm{V} = \mathrm{V} & & \mathrm{V} \rhd \alpha = (\mathrm{V} \lhd \underline{1}) \rhd \alpha = \mathrm{V} \lhd (\underline{1} \rhd \alpha) = \mathrm{V} \lhd \underline{1} \\
(\mathrm{V} \times \mathrm{V}) \overset{\leftarrow\!\rhd}{} A = \mathrm{V} & & (\mathrm{V} \circ \mathrm{V}^{\mathrm{T}}) \rhd A = (\mathrm{V} \lhd \mathrm{V}^{\mathrm{T}}) \rhd A = \mathrm{V} \lhd (\mathrm{V}^{\mathrm{T}} \rhd A) = \mathrm{V} \lhd (A^{\mathrm{T}} \lhd \mathrm{V})^{\mathrm{T}} = \mathrm{V} \lhd \underline{1}
\end{array}
$$

Remark 5.22 The corollaries in this and the previous section show that a fairly large fragment of the elementary theory of fuzzy relations can be reduced to identities provable by several simple equational rules, namely those of Propositions 2.14(1) and 3.7, Theorems 4.2(1,5) and 5.3(1,5,6), and Lemmata 4.15 and 5.20. These rules can be viewed as axioms of an equational calculus for proving identities between fuzzy relational operations. It seems to be an open problem if there are elementary theorems on fuzzy relations expressible as identities in the language of $\circ$, $^{\mathrm{T}}$, V, $\underline{1}$, BK-products, and the notions listed in Tables 1–4, which are not provable from these equational rules (possibly extended by some missing identities), though provable in FCT (and, for that matter, if there are any such identities in which the elementary theories of fuzzy and *crisp* relations differ).

Remark 5.23 Sup-T-compositions and BK-products operate on binary fuzzy relations, i.e., fuzzy classes of ordered pairs of elements xy. The inner structure of these elements x, y can be arbitrary: if they are, for instance, themselves ordered pairs x_1x_2 and y_1y_2, then relational products are in fact operating on ordered quadruples. Composition-based notions with class operands (e.g., $\subseteq$) are thus applicable to binary fuzzy relations as well. In this way, inclusion of fuzzy relations $R \subseteq S$ can be regarded as the BK-product $(R')^{\mathrm{T}} \lhd S'$, where for a binary relation R and *quaternary* relations P, Q we define

$$
\begin{array}{rll}
P \lhd Q & =_{\mathrm{df}} & \{x_1x_2y_1y_2 \mid (\forall z_1z_2)(Px_1x_2z_1z_2 \rightarrow Qz_1z_2y_1y_2)\} \\
P^{\mathrm{T}} & =_{\mathrm{df}} & \{y_1y_2x_1x_2 \mid Px_1x_2y_1y_2\} \\
R' & =_{\mathrm{df}} & \{xy\underline{00} \mid Rxy\}
\end{array}
$$

The corollaries shown above thus apply to inclusion, compatibility, Cartesian products, etc., not only of unary fuzzy classes, but also fuzzy relations of arbitrary arities. In this way, many further notions of the theory of fuzzy relations are reducible to sup-T- and BK-compositions: e.g., symmetry of a fuzzy relation R is expressible as $(R')^{\mathrm{T}} \lhd (R^{\mathrm{T}})'$; cf. also Example 5.11 for transitivity and reflexivity and Remark 5.18 for extensionality. The machinery demonstrated above thus can be used also for proving properties of such relational notions.

6 Conclusions

We have shown a method for mass proofs of theorems of certain forms in the theory of fuzzy relations. Its soundness is based on the notion of relative interpretation between theories over fuzzy logics, which allows a representation of fuzzy classes and formal truth values as certain kinds of fuzzy relations. This expands the applicability of simple properties of sup-T-compositions and BK-products of fuzzy relations to a larger language (of more than 30 operations) which includes many important concepts of the theory of fuzzy sets and fuzzy relations. Consequently, a large number of theorems of the latter theory are reduced to corollaries of a few simple properties of relational products, thus becoming verifiable by simple equational computations.

Among all possible kinds of fuzzy relational compositions, in this paper we have restricted our attention only to the sup-T-composition and BK-products, because they generate the most interesting families of derived notions, which occur most often in fuzzy mathematics. Similar investigation of notions based on other kinds of relational products is a topic left for future work.

Besides the practical consequences (e.g., for automated proofs of relational theorems) the results show that using a suitable formal apparatus provided by first-order and higher-order fuzzy logic enables exploitation of formal syntactic methods that can trivialize a large part of fuzzy mathematics. Together with the metatheorems of [12, §3.4] on fuzzy class operations, the methods presented here effectively reduce elementary fuzzy set theory and a large part of fuzzy relational theory to calculations in propositional fuzzy logic and simple relational algebra. Moreover they show that for a certain class of results, the fuzziness of fuzzy relations does not present an additional difficulty to the usual theory of *crisp* relations: it can be observed that Theorems 4.2 and 5.3, upon which all of the corollaries are based, hold equally for fuzzy and crisp relations. Thus a large part of the theory of crisp relations generalizes straightforwardly to fuzzy relations if a suitable framework of formal fuzzy logic is employed.

References

[1] Wyllis Bandler and Ladislav J. Kohout. Mathematical relations, their products and generalized morphisms. Technical Report EES-MMS-REL 77-3, Man–Machine Systems Laboratory, Department of Electrical Engineering, University of Essex, Essex, Colchester, 1977.

[2] Wyllis Bandler and Ladislav J. Kohout. Fuzzy relational products and fuzzy implication operators. In *International Workshop of Fuzzy Reasoning Theory and Applications*, London, 1978. Queen Mary College, University of London.

[3] Wyllis Bandler and Ladislav J. Kohout. Fuzzy power sets and fuzzy implication operators. *Fuzzy Sets and Systems*, 4:183–190, 1980.

[4] Wyllis Bandler and Ladislav J. Kohout. Fast fuzzy relational algorithms. In A. Ballester, D. Cardus, and E. Trillas, editors, *Proceedings of the 2nd World Conference on Mathematics at the Service of Man*, pages 123–131, Las Palmas, Spain, 1982. Universidad Politechnica de Las Palmas.

[5] Wyllis Bandler and Ladislav J. Kohout. On the general theory of relational morphisms. *International Journal of General Systems*, 13(1):47–66, 1986.

[6] Wyllis Bandler and Ladislav J. Kohout. Special properties, closures and interiors of crisp and fuzzy relations. *Fuzzy Sets and Systems*, 26:317–331, 1988.

[7] Wyllis Bandler and Ladislav J. Kohout. On the universality of the triangle superproduct and the square product of relations. *International Journal of General Systems*, 25:399–403, 1997.

[8] Libor Běhounek. Extensionality in graded properties of fuzzy relations. In *Proceedings of 11th IPMU Conference*, pages 1604–1611, Paris, 2006. Edition EDK.

[9] Libor Běhounek. Relative interpretations over first-order fuzzy logics. In František Hakl, editor, *Doktorandský den '06*, pages 1–6, Prague, 2006. ICS AS CR / Matfyzpress. Available at http://www.cs.cas.cz/hakl/doktorandsky-den/history.html.

[10] Libor Běhounek. On the difference between traditional and deductive fuzzy logic. *Fuzzy Sets and Systems*, 159(10):1153–1164, 2008.

[11] Libor Běhounek, Ulrich Bodenhofer, and Petr Cintula. Relations in Fuzzy Class Theory: Initial steps. *Fuzzy Sets and Systems*, 159(14):1729–1772, 2008.

[12] Libor Běhounek and Petr Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.

[13] Libor Běhounek and Petr Cintula. Fuzzy class theory as foundations for fuzzy mathematics. In *Fuzzy Logic, Soft Computing and Computational Intelligence: 11th IFSA World Congress*, volume 2, pages 1233–1238, Beijing, 2005. Tsinghua University Press/Springer.

[14] Libor Běhounek and Petr Cintula. General logical formalism for fuzzy mathematics: Methodology and apparatus. In *Fuzzy Logic, Soft Computing and Computational Intelligence: 11th IFSA World Congress*, volume 2, pages 1227–1232, Beijing, 2005. Tsinghua University Press/Springer.

[15] Libor Běhounek and Petr Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.

[16] Libor Běhounek and Petr Cintula. Fuzzy Class Theory: A primer v1.0. Technical Report V-939, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2006. Available at www.cs.cas.cz/research/library/reports_900.shtml.

[17] Libor Běhounek and Petr Cintula. Features of mathematical theories in formal fuzzy logic. In Patricia Melin, Oscar Castillo, Lluis T. Aguilar, Janusz Kacprzyk, and Witold Pedrycz, editors, *Foundations of Fuzzy Logic and Soft Computing*, volume 4529 of *Lecture Notes in Artificial Intelligence*, pages 523–532. Springer, Berlin etc., 2007.

[18] Radim Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles*, volume 20 of *IFSR International Series on Systems Science and Engineering*. Kluwer Academic/Plenum Press, New York, 2002.

[19] Ulrich Bodenhofer, Martine De Cock, and Etienne E. Kerre. Openings and closures of fuzzy preorderings: Theoretical basics and applications to fuzzy rule-based systems. *International Journal of General Systems*, 32(4):343–360, 2003.

[20] Martina Daňková. On approximate reasoning with graded rules. *Fuzzy Sets and Systems*, 158:652–673, 2007.

[21] Bernard De Baets and Etienne Kerre. Fuzzy relational compositions. *Fuzzy Sets and Systems*, 60(1):109–120, 1993.

[22] Martine De Cock and Etienne E. Kerre. Fuzzy modifiers based on fuzzy relations. *Information Sciences*, 160:173–199, 2004.

[23] Francesc Esteva and Lluís Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.

[24] János Fodor. Traces of fuzzy binary relations. *Fuzzy Sets and Systems*, 50(3):331–341, 1992.

[25] János Fodor. Contrapositive symmetry of fuzzy implications. *Fuzzy Sets and Systems*, 69(2):141–156, 1995.

[26] Anna Frascella and Cosimo Guido. Transporting many-valued sets along many-valued relations. *Fuzzy Sets and Systems*, 159(1):1–22, 2008.

[27] Siegfried Gottwald. *A Treatise on Many-Valued Logics*, volume 9 of *Studies in Logic and Computation*. Research Studies Press, Baldock, 2001.

[28] Siegfried Gottwald. Some general considerations on the evaluation of fuzzy rule systems. In *Proceedings of Joint EUSFLAT–LFA Conference*, pages 639–644, Barcelona, 2005.

[29] Siegfried Gottwald. Universes of fuzzy sets and axiomatizations of fuzzy set theory. Part I: Model-based and axiomatic approaches. *Studia Logica*, 82(2):211–144, 2006.

[30] Petr Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordercht, 1998.

[31] Petr Hájek. Making fuzzy description logic more general. *Fuzzy Sets and Systems*, 154(1):1–15, 2005.

[32] Petr Hájek and Petr Cintula. Triangular norm predicate fuzzy logics. To appear in the book Fuzzy Logic and Related Structures: Proceedings of Linz Seminar 2005 (Siegfried Gottwald, Petr Hájek, Erich Peter Klement eds.), 2007.

[33] L. P. Holmblad and J. J. Ostergaard. Control of a cement kiln by fuzzy logic. In Madan M. Gupta and Elie Sanchez, editors, *Fuzzy Information and Decision Processes*, pages 389–399. Elsevier, Amsterdam, 1982.

[34] F. Klawonn and J.L. Castro. Similarity in fuzzy reasoning. *Mathware and Soft Computing*, 3(2):197–228, 1995.

[35] Ladislav J. Kohout and Wyllis Bandler. Fuzzy relational products in knowledge engineering. In Vilém Novák (et al.), editor, *Fuzzy Approach to Reasoning and Decision Making*, pages 51–66. Academia / Kluwer, Prague / Dordrecht, 1992.

[36] Ladislav J. Kohout and E. Kim. The role of BK-products of relations in soft computing. *Soft Computing*, 6:92–115, 2002.

[37] P. Magrez and Ph. Smets. Fuzzy modus ponens: A new model suitable for applications in knowledge-based systems. *International Journal of Intelligent Systems*, 4:181–200, 1989.

[38] E. H. Mamdani and S. Assilian. An experiment in linguistic synthesis with a fuzzy logic controller. *International Journal of Man–Machine Studies*, 7:1–13, 1975.

[39] Nehad N. Morsi and A. A. Fahmy. On generalized modus ponens with multiple rules and a residuated implication. *Fuzzy Sets and Systems*, 129:267–274, 2002.

[40] Nehad N. Morsi, W. Lotfallah, and M.S. El-Zekey. The logic of tied implications, part 2: Syntax. *Fuzzy Sets and Systems*, 157:2030–2057, 2006.

[41] Gaisi Takeuti and Satoko Titani. Fuzzy logic and fuzzy set theory. *Archive for Mathematical Logic*, 32:1–32, 1992.

[42] Lotfi A. Zadeh. Similarity relations and fuzzy orderings. *Information Sciences*, 3:177–200, 1971.