Continuous Relations over Topological Spaces in Fuzzy Class Theory

Libor Běhounek[1]⋆ and Tomáš Kroupa[2]⋆⋆

[1] Institute of Computer Science
Academy of Sciences of the Czech Republic
Pod Vodárenskou věží 2, 182 07 Prague, Czech Republic
behounek@cs.cas.cz

[2] Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic
kroupa@utia.cas.cz

Fuzzy topology has benefited from a considerable attention in the past (see, e.g., [12–14]). By now, there are several established approaches to fuzzy topology: lattice-theoretical, categorial, based on membership functions, etc. Recent advances in mathematical fuzzy logic, esp. after [10], enabled a new, *logic-based* approach to fuzzy topology. This approach is part of a broader program of logic-based fuzzy mathematics described in [3], which consists in the development of axiomatic theories over higher-order fuzzy logic. It extends and elaborates the methodology sketched by Höhle in [11, §5], namely a reinterpretation of classical definitions in a suitable calculus of fuzzy logic. A similar attempt to build fuzzy topology within a logical framework appeared also in [17]. We follow this line of research in the strictly formal framework of axiomatic theories over Hájek-style deductive fuzzy logic. The application of this kind of formalism leads to a universal gradedness of definitions and theorems which is not usual under traditional approaches; on the other hand it is limited to certain methodological presuppositions [1]: thus it complements (rather than competes with) the more traditional approaches. Due to the different strength of results, also some rather elementary theorems need to be proved anew in our setting, even though many results on related notions have already been obtained in the frameworks of more traditional approaches to fuzzy topology.

Initial results in our logic-based fuzzy topology have been presented in conference papers [6, 5], where the relationship between three notions of fuzzy topology (namely those based on open or closed sets, neighborhoods, and interior operators) were studied. In the present contribution we restrict our attention to fuzzy topology based on open sets and make first steps towards the notion of continuity in this setting. Instead of the more usual notion of continuous function, we study the notion of continuous *relation* between two fuzzy topologies. This enables us to avoid making such a basic concept as continuity depend on the notion of fuzzy function, which has many competing definitions.[3]

⋆ The work of the first author was supported by the program Information Society project No. 1ET100300517 and Institutional Research Plan AV0Z10300504.

⋆⋆ The work of the second author was supported by grant No. B100300502 of GA AV ČR and grant No. 1M0572 of MŠMT ČR.

[3] We are indebted for this idea to Rostislav Horčík.

The notion of continuous relation between topological spaces has already appeared in several areas of mathematics. Continuous relations have been defined in general topology [16, 9] and also investigated by means of induced multifunctions in set-valued analysis [7]. In [15], their study was initiated in the purely formal framework of higher-order intuitionistic logic, as a part of formal (pointless) topology.

In this paper we give initial observations on the concept of continuous fuzzy relation between a pair of fuzzy topologies. We work in the framework of Fuzzy Class Theory FCT (or higher-order fuzzy logic) introduced in [2]. For the sake of generality, here we use its variant over the logic MTL_Δ of all left-continuous t-norms [8]. Besides the original paper [2], the apparatus of FCT is described in detail in the primer [4], which is freely available online. Due to space restrictions, we do not repeat the definitions here. We use standard abbreviations and notions from [4, §1.6,1.7]; furthermore we use $X \sqsubseteq Y$ for $\Delta(X \subseteq Y)$ and Id_X for $\{\langle x,x\rangle \mid x \in X\}$.

In [6], the FCT notion of open fuzzy topology has been introduced. It uses the following predicates that express the (degree of) closedness of a fuzzy class of fuzzy classes τ under $\bigcup$ and $\cap$:

$$\mathsf{ic}(\tau) \equiv_{\mathrm{df}} (\forall A, B \in \tau)(A \cap B \in \tau)$$
$$\mathsf{Uc}(\tau) \equiv_{\mathrm{df}} (\forall \sigma \subseteq \tau)\left(\bigcup \sigma \in \tau\right)$$

Definition 1. *In FCT, we define the predicate indicating the degree to which* $\tau \sqsubseteq \mathrm{Ker}\,\mathrm{Pow}\,X$ *is an* open fuzzy topology *on a crisp class X as*

$$\mathsf{OTop}(X,\tau) \equiv_{\mathrm{df}} (\emptyset \in \tau) \;\&\; (X \in \tau) \;\&\; \mathsf{ic}(\tau) \;\&\; \mathsf{Uc}(\tau)$$

A predicate expressing that A such that $A \sqsubseteq X$ *is a* neighborhood *of x in* τ *is defined as*

$$\mathrm{Nb}_\tau(x,A) \equiv_{\mathrm{df}} (\exists B \in \tau)(B \subseteq A \;\&\; x \in B)$$

Given a class of classes τ*, we define the* interior *of a class A such that* $A \sqsubseteq X$ *as*

$$\mathrm{Int}_\tau(A) =_{\mathrm{df}} \bigcup \{B \in \tau \mid B \subseteq A\}$$

Models of the predicate OTop are closest to L-fuzzy topologies of Höhle-type studied in [13]. We assume the ground set X of an open fuzzy topology to be crisp, since quantification over fuzzy domains is not yet well understood in the fully graded setting of FCT. Even though there are no technical obstacles for using fuzzy X, graded definitions over fuzzy X would need a much more careful general discussion about their meaning and motivation. Thus in this contribution we stick to crisp ground sets of fuzzy topologies.

In the sequel we assume that $R \sqsubseteq X_1 \times X_2$ and $S \sqsubseteq X_2 \times X_3$, where each X_i is a crisp class. By τ_i we denote a fuzzy class of fuzzy classes such that $\tau_i \sqsubseteq \mathrm{Ker}\,\mathrm{Pow}\,X_i$. In Definition 2 we introduce three predicates, each of them expressing a different definition of continuous relation (by open classes, by neighborhoods, and by the interior operator). It is worth mentioning that the definition of the predicate NCont resembles the one used by Sambin [15, §2.3] over intuitionistic logic.

Definition 2.

$$\mathrm{OCont}(R) \equiv_{\mathrm{df}} (\forall B \in \tau_2)(R^{\leftarrow} B \in \tau_1)$$
$$\mathrm{NCont}(R) \equiv_{\mathrm{df}} (\forall x \in X_1)(\forall B \in \tau_2)(R^{\rightarrow}\{x\} \| B \to (\exists A)(\mathrm{Nb}_{\tau_1}(x,A) \;\&\; A \subseteq R^{\leftarrow} B))$$
$$\mathrm{ICont}(R) \equiv_{\mathrm{df}} (\forall B)(R^{\leftarrow} \mathrm{Int}_{\tau_2}(B) \subseteq \mathrm{Int}_{\tau_1}(R^{\leftarrow} B))$$

The following proposition says that all of the above introduced predicates are fuzzily equivalent under rather general conditions: note that the second-order fuzzy classes τ_1 and τ_2 are only required to be closed under unions of fuzzy families of fuzzy classes. Also notice that the theorem is graded, i.e., the fuzzy equivalence holds at least to the degree of $\mathsf{Uc}(\tau_1)$ resp. $\mathsf{Uc}^2(\tau_2)$. We omit all proofs due to space restrictions.

Proposition 1. *It is provable in FCT:*

1. $\mathsf{Uc}(\tau_1) \to (\mathrm{NCont}(R) \leftrightarrow \mathrm{OCont}(R))$
2. $\mathsf{Uc}(\tau_1) \to (\mathrm{OCont}(R) \leftrightarrow \mathrm{ICont}(R))$
3. $\mathsf{Uc}^2(\tau_2) \to (\mathrm{ICont}(R) \leftrightarrow \mathrm{NCont}(R))$

As the properties OCont, ICont, and NCont are equivalent if τ_1 resp. τ_2 are sufficiently union-closed, we shall restrict our attention to the predicate OCont. The following proposition shows that continuous relations form a “fuzzy system of morphisms” between fuzzy topologies.

Proposition 2. *It is provable in FCT:*

1. $\mathrm{OCont}(\mathrm{Id}_X)$
2. $\mathrm{OCont}(R) \;\&\; \mathrm{OCont}(S) \to \mathrm{OCont}(R \circ S)$

Like in classical topology, when examining the continuity of a relation, it is sufficient to verify openness for preimages of open classes from a base.

Proposition 3. *FCT proves:*

$$\{\bigcup \nu \mid \nu \subseteq \sigma\} \subseteq \tau_2 \;\&\; \mathsf{Uc}(\tau_1) \to [\mathrm{OCont}(R) \leftrightarrow (\forall B \in \sigma)(R^{\leftarrow} B \in \tau_1)]$$

A non-trivial example of continuous relations between fuzzy topological spaces is introduced in Example 1. The predicate OCont has crisp instances, too: in particular, continuous relations studied herein are special cases of so-called lower semicontinuous multifunctions investigated in set-valued analysis [7].

Example 1. In [6], an interval fuzzy topology on domains densely ordered by a crisp relation $\leq$ has been defined as the coarsest fully $\bigcup$-closed topology that fully contains the fuzzy subbase of fuzzily open fuzzy intervals $[A,B]$. It can be proved that $\leq$ is a continuous relation w.r.t. this topology: since the fuzzy family of open fuzzy intervals is closed under $\cap$, by Proposition 3 it is sufficient to prove that $\leq^{\leftarrow}[A,B]$ is open for any open fuzzy interval $[A,B]$. It can even be shown that $\leq^{\leftarrow}[A,B]$ is an open interval of the form $[-\infty, C]$ for a right-open C.[4]

[4] Since the properties of openness and right-openness are fuzzy, the last two sentences should be read in the graded manner, i.e., as implications between the fuzzy conditions.

References

1. L. Běhounek. On the difference between traditional and formal fuzzy logic. To appear in *Fuzzy Sets and Systems*.
2. L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.
3. L. Běhounek and P. Cintula. From fuzzy logic to fuzzy mathematics: a methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.
4. L. Běhounek and P. Cintula. Fuzzy Class Theory: A Primer v1.0. Available online at `www.cs.cas.cz/research/library/reports_900.shtml`, Prague, 2006.
5. L. Běhounek and T. Kroupa. Interior-based topology in Fuzzy Class Theory. In V. Novák and M. Štěpnička, editors, *New Dimensions in Fuzzy Logic and Related Technologies. Proceedings of EUSFLAT 2007*, pages 145–151, Ostrava, 2007. University of Ostrava.
6. L. Běhounek and T. Kroupa. Topology in Fuzzy Class Theory: Basic notions. In *Proceedings of IFSA 2007*, number 4529 in Lecture Notes in Artificial Intelligence, pages 513–522, Cancun, 2007. Springer.
7. C. Castaing and M. Valadier. *Convex analysis and measurable multifunctions*. Springer-Verlag, Berlin, 1977. Lecture Notes in Mathematics, Vol. 580.
8. F. Esteva and L. Godo. Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.
9. G. Grimeisen. Continuous relations. *Math. Z.*, 127:35–44, 1972.
10. P. Hájek. *Metamathematics of fuzzy logic*. Kluwer Academic Publishers, Dordrecht, 1998. Trends in Logic—Studia Logica Library, Vol. 4.
11. U. Höhle. Fuzzy real numbers as Dedekind cuts with respect to a multiple-valued logic. *Fuzzy Sets and Systems*, 24(3):263–278, 1987.
12. U. Höhle. *Many valued topology and its applications*. Kluwer Academic Publishers, Boston, MA, 2001.
13. U. Höhle and A. Šostak. A general theory of fuzzy topological spaces. *Fuzzy Sets and Systems*, 73(1):131–149, 1995.
14. S. E. Rodabaugh. Point-set lattice-theoretic topology. *Fuzzy Sets and Systems*, 40(2):297–345, 1991.
15. G. Sambin. Some points in formal topology. *Theoret. Comput. Sci.*, 305(1-3):347–408, 2003.
16. O. Wyler. A characterization of regularity in topology. *Proc. of Amer. Math. Soc.*, 29(3):588–590, 1971.
17. M. S. Ying. Fuzzy topology based on residuated lattice-valued logic. *Acta Math. Sin. (Engl. Ser.)*, 17(1):89–102, 2001.