Determinate truth in fuzzy plurivaluationism

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Abstract: Degree-theoretical accounts of vagueness usually equate determinate truth with truth to degree 1. However, by rendering determinate truth as a sharp notion, this identification disregards the phenomenon of higher-order vagueness. In this paper I propose a more adequate degree-theoretical model of determinate truth as a vague notion, based on the logical indistinguishability of truth degrees and measured by the length of the shortest formal argument separating a given degree from 1.

Keywords: Vagueness, Sorites paradox, Truth degree, Fuzzy logic, Higher-order vagueness, Determinate truth, Fuzzy plurivaluationism

1 Vagueness and fuzzy plurivaluationism

The phenomenon of vagueness is usually characterized by several interrelated attributes: susceptibility to the sorites paradox, the existence of borderline cases, and higher-order vagueness (cf., e.g., Keefe, 2000, pp. 6–9; Smith, 2009, pp. 1–2). Vague predicates can thus be loosely defined as such predicates P that:

1. A *sorites series* for P can be constructed: i.e., a series of objects $x_1, \ldots, x_N$ such that each two adjacent objects x_i, x_{i+1} differ only negligibly in respects relevant to P (making it implausible that they would differ in being P), and yet x_0 is clearly P and x_N is clearly not P.

2. *Borderline cases* of P exist: i.e., there are objects x that are neither determinately P nor determinately not P.

3. P manifests *higher-order vagueness:* i.e., the property of being borderline P is itself vague, and so has borderline cases.

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Vague predicates abound in natural language; prototypical examples include such properties as *tall, red, warm, large,* etc.

There are several approaches to modelling vagueness: for an overview see, e.g., Williamson (1994) or Keefe (2000). This paper deals with a particular problem related to the *fuzzy plurivaluationistic* approach to vagueness proposed by Smith (2009, ch. 6.1). Before introducing the problem, let us first illustrate fuzzy plurivaluationism and its closest relatives by describing how they model the vague predicate *tall.* For simplicity let us assume that (i) in a given context (e.g., that of Central European men), the only feature relevant for the predicate *tall* is the person's height; we also understand that (ii) it is a part of the meaning of *tall* that if a person x is tall and the height of a person y is no less than that of x, then y is tall as well; and that in the given context, (iii) people of height 150 cm are not tall, while (iv) people of height 200 cm are tall. We shall call conditions (i)–(iv) the *meaning postulates* of the predicate *tall.*

1.1 The classical semantics of *tall*

Since the standard semantics of classical logic construes all predicates as bivalent, there must be a sharp breaking point between the heights of people who are *tall* and those who are *not tall* (as depicted in Figure 1).

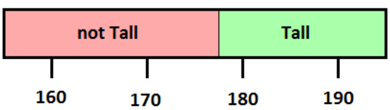

Figure 1: The classical semantics of *tall*

While employing classical bivalent semantics for modelling vague predicates such as *tall* is often satisfactory and has the undeniable advantage of simplicity, there are some drawbacks to it as well:

1. It is not clear where the breaking point should occur: in fact, no particular position of a sharp breaking point is supported by either the linguistic practice or any reasonable explication of the meaning of *tall*.

2. The breaking point creates an implausible discontinuity: it does not conform with the meaning of *tall* that an imperceptible change in height (e.g., of 0.1 cm) should make a difference.

3. On the other hand, denying the discontinuity generates the sorites paradox.
4. There are no borderline cases of *tall* in the bivalent account, while in fact there clearly are some.

Even though these disadvantages may in many situations be harmless, they indicate that the classical modelling of vague predicates is rather crude, and sometimes (e.g., in soritical settings) inadequate. One approach to improving the model is supervaluationism.

1.2 The supervaluationistic account of *tall*

Placing a particular breaking point between *tall* and *not tall* in the classical semantics can be viewed as a *precisification* of the meaning of *tall.* The basic idea of *supervaluationism* (Fine, 1975) is to consider not just one, but *all* possible precisifications of the meaning of *tall.*[2] The supervaluationistic semantics of *tall* thus consists of all classical models of *tall* that satisfy its meaning postulates:[3]

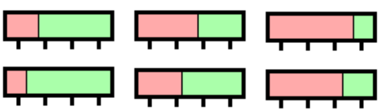

Figure 2: The supervaluationistic semantics of *tall*

This clearly addresses objection 1 of Section 1.1: the breaking point is no longer arbitrarily fixed, as we are considering all of its possible positions. Supervaluationism furthermore distinguishes between truth under a particular precisification and *supertruth,* or truth under all admissible precisifications. Since truth can be an artefact of precisification, it is, naturally, just supertruth that matters. This solves the sorites paradox: its inductive premise (if Px_i then Px_{i+1}) is not supertrue, as there is an admissible precisification that puts the breaking point between x_i and x_{i+1}. The solution avoids introducing a point of discontinuity (objection 2), as no statement

[2]The version of supervaluationistic semantics presented here is called *plurivaluationism* by Smith (2009); however, Smith's distinction between supervaluationism and plurivaluationism is immaterial for our purposes.

[3]In the context of supervaluationism, meaning postulates are called *penumbral connections* (as coined by Fine, 1975).

“The change occurs at height h” is supertrue. Another virtue of supervaluationism is that it retains classical logic, as it is concerned with truth under all *classical* precisifications of vague predicates.

One issue which remains unresolved by supervaluationism is the *jolt problem* (as termed by Smith, 2009): even though none of the statements “The change occurs at height h” is supertrue, the *existence* of a sharp breaking point (a ‘jolt’) is still supertrue, as each admissible classical precisification (see Figure 2) does indeed contain a jolt. This establishes the supertruth of such implausible statements as “There are no borderline cases of *tall*” and “There is no gradual transition between *tall* and *not tall*” in supervaluationistic models.

A different approach to vagueness that aims specifically at modelling the gradual transition between P and $\neg P$ is the degree-theoretic (or *fuzzy*) semantics (e.g., Williamson, 1994, ch. 4; Keefe, 2000, ch. 4).

1.3 The fuzzy semantics of *tall*

In order to accommodate the gradual transition between P and $\neg P$, fuzzy models of vagueness admit intermediary *degrees of truth.* Most often, the degrees of truth are represented by real numbers from the unit interval $[0, 1]$, where the degree 1 represents full truth and the degree 0 full falsity. A vague predicate P is then modelled by a *function*[4] $\|P\|\colon X \to [0, 1]$ which assigns to each object x from a domain X a number $\|Px\| \in [0, 1]$ representing the degree to which x has the property P.

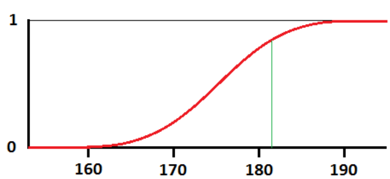

Figure 3: A fuzzy semantics of *tall*

A fuzzy model for our paradigmatic predicate *tall* is depicted in Figure 3. In this model, the membership function $\|\mathit{tall}\|$ assigns degrees from $[0, 1]$

[4]In fuzzy set theory, $\|P\|$ is called the *membership function* of P or the *fuzzy set* delimited by P. Thus in fuzzy models, vague predicates are represented by $[0, 1]$-valued ‘fuzzy’ sets instead of ordinary two-valued ‘crisp’ sets. Crisp sets can be identified with special membership functions that only assign the degrees 0 and 1.

to people according to their height (in cm on the horizontal axis). In this model, if John is, for instance, 182 cm tall, then $\|tall\|(\mathrm{John}) = 0.84$; i.e., the statement "John is tall" is assigned the truth degree 0.84. Observe that the model honours the meaning postulates (i)–(iv) for *tall* (see page 2) by (i) making $\|tall\|$ functional in height, (ii) assigning larger truth degrees to taller people, (iii) assigning the degree 0 to people of height 150 cm, and (iv) assigning the degree 1 to people of height 200 cm.

By admitting the degrees of truth, fuzzy models capture the graduality of vague predicates and remove the jolt between P and $\neg P$. The sorites paradox can now be solved by letting the truth degree of Px_i gradually decrease along the sorites series, which makes the inductive premise almost (though not quite) true. Nevertheless, fuzzy semantics still suffers from several problems. Let us mention just two of them:

1. A fuzzy model specifies a particular membership function $\|P\|$, which assigns exact truth degrees to objects; e.g., $\|tall\|(\mathrm{John}) = 0.84$. However, such precision seems unwarranted: there are many functions compatible with the meaning postulates of *tall,* and nothing in language appears to determine whether John's tallness is 0.84 or 0.83. In general it seems rather incongruous to model a vague predicate with unclear and imprecise boundaries by means of a completely precise real-valued function. This objection to fuzzy semantics is known as the problem of *artificial precision* (e.g., Williamson, 1994, p. 127; Smith, 2009, p. 277).

2. Relatedly, assigning precise degrees on a linearly ordered scale such as the real unit interval makes the degrees of all properties comparable. However, many properties are qualitatively so different that there is hardly any way to compare their intensities; for instance, it is largely meaningless to say that John is more *tall* than *young.* Let us call this the *linearity objection* to fuzzy semantics (e.g., Smith, 2009, p. 293; Williamson, 1994, p. 128).

A remedy to these problems for fuzzy semantics is to do the supervaluationist trick in the fuzzy setting (as proposed by Smith, 2009, ch. 6). The rationale for this move is the very reason behind the problem of artificial precision: namely that neither the meaning postulates nor any other linguistic facts determine membership functions uniquely. Rather, the meaning postulates (or more generally, the meaning-determining facts) only establish *constraints*[5] on membership functions in fuzzy models. This approach, which combines the merits of supervaluationism and fuzzy semantics, has been called *fuzzy plurivaluationism* by Smith (2009, p. 9).

1.4 The fuzzy plurivaluationistic semantics of *tall*

Following the above motivation, fuzzy plurivaluationistic semantics assigns to vague predicates, in general, not just a single fuzzy model, but *all* fuzzy models satisfying the constraints given by the meaning-determining facts. In the particular case of the predicate *tall,* the fuzzy plurivaluationistic semantics consists of all membership functions that satisfy its meaning postulates:

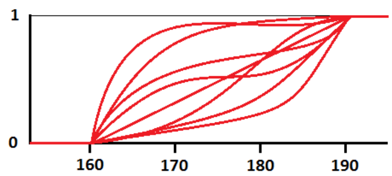

Figure 4: The fuzzy plurivaluationistic semantics of *tall*

Let us call the class of admissible fuzzy models of a vague predicate its *fuzzy plurivaluation.* Just like supervaluationism, fuzzy plurivaluationism understands particular models in the plurivaluation as *admissible precisifications* of the meaning of the vague predicate; only this time, the precisifications are gradual. Consequently, it is again just supertruth, or truth in all admissible fuzzy models, that matters, as truth in a particular fuzzy model may be an artefact of the precisification (i.e., of the particular choice of the membership function in that model).

By combining the merits of supervaluationism and fuzzy semantics, fuzzy plurivaluationism addresses several problems of either account:

- Like fuzzy models, it avoids the jolt by letting the truth degree gradually decrease from 1 to 0 along the sorites series.[6]

[5] or "fuzzy penumbral connections", to use the supervaluationistic terminology

[6] In fact, fuzzy models and fuzzy plurivaluationism make it possible to require, as another meaning postulate for *tall,* that (v) very small changes in height should only result in very small changes in the truth (degree) of *tall.* This requirement would only admit (Lipschitz) continuous membership functions for *tall,* making the absence of a jolt and the presence of borderline cases supertrue. Cf. Smith's principle of Closeness for vague predicates (2009, ch. 3.3–5).

- Like supervaluationism, it removes the arbitrariness of a particular model by considering all admissible models. This addresses the problem of arbitrary precision in fuzzy semantics: since John's tallness has different degrees in different admissible models, no statement assigning to it a precise degree (such as "John is tall to degree 0.84") is supertrue. The linearity objection is answered in a similar manner: since the degree of "John is tall" exceeds that of "John is young" only in *some* models, neither the statement "John is more *tall* than *young*" nor its converse is supertrue.

From the model-theoretic point of view, the fuzzy plurivaluation for a set of vague predicates can be explained as the class of models of a theory that formalizes the meaning postulates of the vague predicates involved (Běhounek, 2014).[7] Supertruths in the plurivaluation then coincide with *logical consequences* of this theory; i.e., (super)true statements about vague predicates are exactly the logical consequences of their meaning postulates.

An apparent problem is that taken *prima facie,* the meaning postulates of most vague predicates are inconsistent, as they typically entail the sorites paradox. One option, then, is to translate them into the language of membership functions, as we did with the meaning postulate (ii) for *tall,* interpreting it as the monotonicity condition for the membership function.[8] Another option, which additionally offers a recipe for the translation, is to use a non-classical logic designed specifically for fuzzy models—a fuzzy logic.

2 A logic for fuzzy plurivaluationism

Fuzzy logics are logics tailored to fuzzy models of Section 1.3. There is a whole family of fuzzy logics, corresponding to different possible meaning postulates for fuzzy connectives. Only some of them are suitable for modelling the logical aspects of the sorites paradox, and so for modelling vagueness; of those, perhaps the most prominent is the infinite-valued logic of Łukasiewicz (e.g., Hájek, 1998, ch. 3). For simplicity, we shall restrict our attention to this logic and particularly to its standard $[0, 1]$-valued semantics.

[7] The explanation is based on understanding the meaning postulates as (regularized) characterizations of meanings, abstracted from (irregular) linguistic meaning-determining facts (Běhounek, 2011). It is assumed that they can be formalized in a suitable logic.

[8] Similarly, the continuity (or Closeness) postulate (v) of footnote 6 can be viewed as a degree-theoretical reinterpretation of Wright's principle of Tolerance, making it consistent with the postulates (i)–(iv) for *tall.* Cf. Smith's discussion of Closeness vs. Tolerance (2009, ch. 3.5).

In the standard $[0,1]$-valued semantics, the propositional connectives of Łukasiewicz logic combine the degrees from $[0,1]$ in the following manner:

$$\|\varphi \wedge \psi\| = \min(\|\varphi\|, \|\psi\|) \tag{1}$$
$$\|\varphi \vee \psi\| = \max(\|\varphi\|, \|\psi\|) \tag{2}$$
$$\|\neg\varphi\| = 1 - \|\varphi\| \tag{3}$$
$$\|\varphi \mathbin{\&} \psi\| = \max(\|\varphi\| + \|\psi\| - 1, 0) \tag{4}$$
$$\|\varphi \to \psi\| = \min(1 - \|\varphi\| + \|\psi\|, 1). \tag{5}$$

Note that like other contraction-free logics, Łukasiewicz logic possesses *two* conjunctive connectives, $\wedge$ and $\&$.[9] It is the non-idempotent $\&$ that represents iterative cumulation of premises (as in the sorites paradox), since

$$\begin{aligned}\big\|\varphi_1 \to (\varphi_2 \to \cdots (\varphi_{n-1} \to \varphi_n) \cdots)\big\| &= \\ &= \big\|(\varphi_1 \mathbin{\&} \varphi_2 \mathbin{\&} \ldots \mathbin{\&} \varphi_{n-1}) \to \varphi_n\big\|.\end{aligned} \tag{6}$$

The following connective, often added to Łukasiewicz logic, indicates the full truth of its argument:

$$\|\triangle\varphi\| = \begin{cases} 1 & \text{if } \|\varphi\| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

First-order models for Łukasiewicz logic are defined analogously to classical first-order models, the only difference being the interpretation of predicate symbols: just like in fuzzy models of Section 1.3, each n-ary predicate P is interpreted by a $[0,1]$-valued membership function $\|P\| \colon M^n \to [0,1]$, where M is the universe of the model, instead of a two-valued characteristic function as in classical models. The quantifiers are interpreted as follows:

$$\|(\forall x)\varphi\| = \inf_{a \in M} \|\varphi(a)\|, \qquad \|(\exists x)\varphi\| = \sup_{a \in M} \|\varphi(a)\|.$$

A sentence is considered true in a model if it is evaluated to degree 1. As usual, tautologicity is defined as truth in all models; and the logical consequences of a theory T are defined as the sentences true in all models of T (i.e., all models in which all formulae from T are true).[10]

[9] Distinguishing between them actually answers most common objections raised against truth-functional fuzzy logic, including those offered by Williamson (1994, pp. 118, 136–8) and Keefe (2000, pp. 96–98).

[10] The notions of tautologicity and finitary entailment in Łukasiewicz logic turn out to be finitely axiomatizable (e.g., Hájek, 1998), while infinitary entailment requires an infinitary rule of inference or, alternatively, relaxing somewhat the standard semantics of Łukasiewicz logic to fit its finitary approximation.

Notice that the notion of *logical consequence* in Łukasiewicz logic exactly matches the (fuzzy plurivaluationistic) notion of *supertruth* in the class of models of a theory. Thus, if a theory T formalizes the meaning postulates of some vague predicates, then its logical consequences in Łukasiewicz logic are precisely the supertruths about these predicates. In this sense, (Łukasiewicz) fuzzy logic can be viewed as the logical background of fuzzy plurivaluationism. From this perspective, fuzzy plurivaluationistic modelling of vague predicates amounts to expressing their meaning postulates as a theory in fuzzy logic;[11] the class of its models then constitutes the fuzzy plurivaluation, and the logic's consequence relation captures the notion of supertruth.

This approach to modelling vague predicates has been taken, e.g., by Hájek and Novák (2003). We shall illustrate it by a (simplified) formalization of the predicate *large natural number.*

A minimalistic set of meaning postulates for the predicate *large* on natural numbers can be listed as follows: (i) zero is not large; (ii) numbers larger than large numbers are large, too; (iii) there are large numbers. These postulates can be straightforwardly formalized in Łukasiewicz logic as follows:[12]

$$\neg\,\mathrm{Large}(0) \tag{7}$$

$$(\forall m,n)(m \geq n \mathbin{\&} \mathrm{Large}(n) \rightarrow \mathrm{Large}(m)) \tag{8}$$

$$(\exists n)\,\mathrm{Large}(n) \tag{9}$$

Notice that in accordance with fuzzy plurivaluationism, these axioms do not specify a unique membership function for Large, but only constrain the class of admissible fuzzy models of Large. In particular, by the standard semantics of Łukasiewicz logic, the class of fuzzy models admitted by these axioms consists of all membership functions $\|\,\mathrm{Large}\,\| = \ell\colon \mathbb{N} \rightarrow [0,1]$ (where $\mathbb{N}$ is a model of crisp Peano Arithmetic) such that (i) $\ell(0) = 0$, (ii) ℓ is non-decreasing, and (iii) $\sup_n \ell(n) = 1$. It can be observed that already this minimal set of meaning postulates ensures that many intuitively plausible statements (e.g., that there are large prime numbers) are supertrue,

[11] The formalization of meaning postulates in fuzzy logic can moreover be done in a straightforward manner, since the vague predicates will automatically get interpreted by gradual $[0,1]$-valued membership functions as intended.

[12] Since Large is a predicate on natural numbers, the meaning postulates for (non-vague) natural numbers need be formalized in Łukasiewicz logic, too. This can be done, e.g., by means of Crisp Peano Arithmetic, which consists of the classical axioms of Peano Arithmetic together with the axioms enforcing the bivalence of its primitive predicates, e.g., $(\forall m,n)((m \leq n) \vee \neg(m \leq n))$.

while many intuitively implausible statements (e.g., that there is the least large number) are not. Various additional meaning postulates can be added in order to model the vague predicate *large* more accurately;[13] the axioms (7)–(9), nevertheless, suffice for our illustrative purposes.

3 The jolt problem for determinate truth

Recall the jolt problem for supervaluationism (Section 1.2): in the supervaluationistic model of a sorites series for a vague predicate P, the proposition

$$(\exists n)(Px_n \wedge \neg Px_{n+1}) \tag{10}$$

comes out supertrue. Fuzzy models of Section 1.3 solve the problem by letting the truth degree of Px_n decrease gradually with increasing n. Consequently, the jolt sentence (10) is no longer valid in the fuzzy plurivaluation for P, and so not (super)true of P.

However, the problem reappears for determinate truth, or the truth of P to the full degree 1: in fuzzy models of the sorites series for P, the jolt sentence for the predicate *determinately P,* which can be formalized in Łukasiewicz logic as

$$(\exists n)(\triangle Px_n \wedge \neg\triangle Px_{n+1}),$$

is still supertrue, despite the fact that the boundary between borderline and determinate cases of vague predicates is typically vague as well (that is, vague predicates typically manifest higher-order vagueness; see page 1).

One option is to admit that fuzzy plurivaluationism cannot model higher-order vagueness any better (without jolts) than supervaluationism; after all, for bivalent predicates such as *determinately P,* fuzzy models reduce to bivalent ones, so for these predicates fuzzy plurivaluationism reduces to classical supervaluationism. Another route, which we intend to pursue here, is to acknowledge that due to higher-order vagueness, the predicate *determinately P* is vague as well, and therefore should not be represented in fuzzy models by a bivalent membership function. This, however, means to contest the common identification of determinate truth with truth to degree 1 in

[13] E.g., postulates ensuring the existence of borderline cases or enforcing Lipschitz-style conditions on the membership function (cf. footnote 6). These postulates are expressible in Łukasiewicz logic, too (e.g., the former by the axiom $\neg(\forall n)(\mathrm{Large}(n) \vee \neg\,\mathrm{Large}(n))$); they would make the theory classically inconsistent, but still consistent in Łukasiewicz logic.

degree-theoretical accounts of vagueness, and search for a more adequate representation of determinate truth in fuzzy models. We will follow this route by considering the inferential rôle of determinate truth in arguments involving vagueness.

4 Indistinguishability of close truth degrees

Notice that by (1)–(5), the basic connectives of Łukasiewicz logic (without the connective $\triangle$, whose legitimacy we contest) are Lipschitz continuous, with the Lipschitz constant 1. Therefore, if the difference between the truth degrees $\|p\|$ and $\|q\|$ is at most ε, then for any formula φ of Łukasiewicz logic (without $\triangle$) containing at most n occurrences of basic propositional connectives,[14] the difference between the degrees $\|\varphi(p)\|$ and $\|\varphi(q)\|$ is at most $n \cdot \varepsilon$. Consequently, truth degrees differing by at most ε cannot be distinguished (i.e., one made true and the other false) by any formula shorter than $1/\varepsilon$ connectives; or by any argument that can be formalized in Łukasiewicz logic by a formula shorter than $1/\varepsilon$. In other words, very close truth degrees can only be distinguished by very long arguments.

In particular, truth degrees that are very close to 1 can be distinguished from 1 only by very long arguments, such as a sufficiently long sorites argument. For instance, consider the degree 0.9: it can indeed be distinguished from 1 by a 10-step sorites argument;[15] but due to the Lipschitz property of Łukasiewicz connectives, 0.9 and 1 cannot be distinguished by any shorter argument. Similarly, the truth degree 0.99 can only be distinguished from 1 by a (sorites)[16] argument of length 100; and the degree $0.999\,999\,999$ requires a sorites argument over a series of one thousand million elements to distinguish it from 1.

Thus, degrees too close to 1 cannot be distinguished from 1 by arguments of ordinary lengths. Moreover, in arguments of ordinary lengths, such degrees make no observable difference from 1: for instance, even if used 100 times in any argument that can be formalized by basic connectives of Łukasiewicz logic, using the degree $0.999\,999\,999$ instead of 1 changes

[14] Since negation preserves the distance of degrees, counting just binary basic connectives or atomic subformulae would yield a tighter estimate; however, the accuracy of this upper estimate will not be essential for our considerations.

[15] Each application of the 0.9-true inductive premise decreases the truth degree of the conclusion by 0.1, therefore 10 applications are needed to reach 0; cf. (4) and (6) in Section 2.

[16] Since the degree of iterated conjunction decreases most rapidly of all basic Łukasiewicz connectives, the sorites argument is actually optimal for distinguishing between 1 and $1 - \varepsilon$.

the resulting truth value only by $0.000\,000\,1$, or one ten-millionth. Only in very long arguments such as the sorites of enough length, their difference from 1 may become apparent. Degrees extremely close to 1, such as $1-10^{-100}$, thus behave just like 1 in any argument of feasible length.

This suggests that determinate truth should not be be represented just by the degree 1, but should include also the degrees observationally indistinguishable from 1. Even though such degrees can in principle be distinguished from 1 by very (often, unfeasibly) long arguments, in practice they behave just like determinate truth, and should therefore be regarded as representing determinate truth (almost) as well as the degree 1.

Obviously, the borderline between the degrees that can be regarded as indistinguishable from 1 and those that cannot, is not sharp: the indistinguishability of degrees depends on the length of arguments that can distinguish them. For instance, the degree 0.999 is much better distinguishable from 1 than $0.999\,999$ (as the latter requires a sorites argument of length one million, while the former just one thousand, in order to be distinguished from 1), even though in ordinary arguments of lengths up to a few hundred connectives they both behave like fully true. This in fact conforms very well with the observation that motivated these considerations—namely that because of higher-order vagueness, determinate truth must be a gradual notion, or else the jolt problem and the sorites paradox reappear for *determinately P*. The minimal length of a distinguishing argument then provides a natural measure of practical indistinguishability between a given degree and 1, and so the suitability of that degree for representing determinate truth.

5 Determinate truth redefined

The above considerations lead us to a revised definition of determinate truth in fuzzy models. The commonly accepted definition that *determinately φ* (henceforth denoted by $\mathrm{Det}\,\varphi$) is true in fuzzy models if and only if $\|\varphi\| = 1$ turns out to be unsuitable, since it does not accommodate higher-order vagueness (specifically, being bivalent, it is subject to the jolt problem just like classical supervaluationism). Our analysis corroborates that determinate truth is a gradual notion, and suggests that the truth degree of *determinately φ* is suitably measured by the length of the shortest argument that can distinguish the truth of φ from 1: the *larger* the length, the less $\|\varphi\|$ is distinguishable from 1 and so, the more φ is determinately true. Since, moreover, the quickest way to distinguish $\|\varphi\|$ from 1 in standard Łukasiewicz logic is by means of iterated conjunction (cf. footnote 16), our analysis identifies the degree of *determinately* φ with the degree of largeness of the minimal number n of conjuncts making the iterated conjunction $\varphi^n \equiv_{\text{def}} \varphi \,\&\, \ldots \,\&\, \varphi$ (n times) false. This characterization is expressed by the following definition:

$$\operatorname{Det}\varphi \equiv_{\text{def}} \operatorname{Large}\bigl(\min\{n \mid \|\varphi^n\| = 0\}\bigr). \tag{11}$$

In the setting of standard Łukasiewicz logic, the minimal distinguishing n can be explicitly calculated. By (4) we obtain:

$$\|\varphi^n\| = 1 - \min\bigl(n \cdot (1 - \|\varphi\|), 1\bigr),$$

whereby the definition (11) is equivalent to:

$$\operatorname{Det}\varphi \equiv \operatorname{Large}\lceil 1/(1 - \|\varphi\|)\rceil. \tag{12}$$

Thus, if $\|\varphi\| = 1 - \varepsilon$, then $\|\operatorname{Det}\varphi\| = \bigl\|\operatorname{Large}\lceil 1/\varepsilon\rceil\bigr\|$. (For $\|\varphi\| = 1$ we set $\|\operatorname{Det}\varphi\| = 1$ as the limit case.)

It can be observed that definition (11) involves the vague predicate Large on natural numbers. Consequently, our notion of determinate truth is vague as well. Recall that in fuzzy plurivaluationism, we do not specify a particular membership function for Large; rather, its semantics is delimited by its (formalized) meaning postulates. A set of such meaning postulates for Large has been presented in Section 2; the corresponding axioms (7)–(9) are, thus, part of the meaning of the vague notion of determinate truth. In other words, there is a *penumbral connection* between the vague notions of *determinate* truth and *large* natural number, because determinacy depends on how large a sorites argument is needed to ascertain that a determinate-looking case is in fact borderline.

It can be readily seen that modelling determinate truth by $\operatorname{Det}\varphi$ (rather than $\triangle\varphi$) succeeds in removing the jolt problem for *determinately P,* as the jolt sentence

$$(\exists n)(\operatorname{Det} Px_n \wedge \neg \operatorname{Det} Px_{n+1})$$

is no longer supertrue. This follows easily from the observation that the jolt in Large, i.e.,

$$(\exists m)(\operatorname{Large}(m+1) \wedge \neg \operatorname{Large}(m))$$

is not supertrue in the fuzzy plurivaluation of Large, i.e., the class of fuzzy models satisfying the meaning postulates (7)–(9).

6 Conclusions

We have seen that our redefinition of determinate truth solves several problems faced by its traditional identification with truth to degree 1. Unlike the connective $\triangle$, the operator Det accounts for higher-order vagueness by admitting borderline determinate cases; being gradual, it removes the jolt problem for determinate cases; and it is based on a fairly natural concept of (graded) logical indistinguishability of truth degrees in standard Łukasiewicz logic. Consonant with fuzzy plurivaluationism, the membership function of Det is not uniquely determined, but depends on the fuzzy plurivaluation for the vague predicate Large on natural numbers.

These facts indicate that determinate truth is better modelled by Det than $\triangle$, and that truth to degree 1 is an artefact of a similar kind as the precise truth degrees contested by the objection of artificial precision. In fuzzy plurivaluationism, this artefact is analogously eliminated by restriction to supertrue statements on Det, or (cf. Section 2) the logical consequences of its meaning postulates embodied in (11). The connective $\triangle$, on the other hand, directly represents the very artefact, and so should be avoided in the logical analysis of vague predicates.

Admittedly, our treatment of determinate truth in this paper has been somewhat simplified, and several details still need to be elaborated. For instance, the following consideration may require a modification of definition (11): in our exposition, we have not considered the possibility of introducing defined propositional symbols in the course of the distinguishing argument. If this is allowed, then the descent towards falsity can be much faster; for instance, if $\|p_0\| = 0.999$, then we can distinguish it from 1 in just 10 steps by letting $p_1 \equiv_{\text{def}} p_0 \mathbin{\&} p_0$, $p_2 \equiv_{\text{def}} p_1 \mathbin{\&} p_1$, etc.: then already $\|p_{10}\| = 0$. While this accelerates the distinction between $\|p_0\|$ and 1 exponentially,[17] the main point still stands: for each truth degree close to 1 there is a minimal number of steps required for distinguishing it from 1, and for degrees very close to 1 (such as $1 - 10^{-100\,000}$), the number of steps can be infeasibly large, making the degree practically indistinguishable from 1, and thereby allowing it to represent determinate truth in ordinary arguments (almost) as well as does the degree 1. Essentially, the only modification to (11) required by this acceleration is replacing $\|\varphi^n\|$ by $\|\varphi^{2^n}\|$ and adjusting the explicit calculation (12) accordingly.

[17] The corresponding 'accelerated sorites argument' could be formulated as follows: one grain makes no difference (premise, apply twice); therefore two grains make no difference; analogously, if two grains make no difference, then four grains make no difference; etc.

A further detail that requires elaboration is a more complete account of the meaning postulates for Large, as hinted in Section 2 (esp. footnote 13). Another missing item is a formalization of Det in Łukasiewicz logic: note that definition (11) refers to the semantic value of φ, which cannot be straightforwardly formalized by first-order means. Nevertheless, it can be shown, but is omitted here for reasons of space, that Det can be formalized in Łukasiewicz logic of a higher order (which has enough expressive power to internalize truth values of formulae). Yet another topic for future study is a generalization of Det to some other fuzzy logics besides standard Łukasiewicz.[18] All of these topics are left for future work.

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[18] Not all fuzzy logics are suitable for modelling vagueness, though: for instance, idempotent elements of & between 0 and 1 prevent solving the sorites paradox within such fuzzy semantics.