Fuzzy Logics Interpreted as Logics of Resources

Libor Běhounek*

Girard's linear logic (1987) is often interpreted as the logic of resources, while formal fuzzy logics (see esp. Hájek, 1998) are usually understood as logics of partial truth. I will argue that deductive fuzzy logics can be interpreted in terms of resources as well, and that under most circumstances they actually capture resource-aware reasoning more accurately than linear logic. The resource-based interpretation then provides an alternative motivation for formal fuzzy logics, and gives an explanation of the meaning of their intermediary truth values that can be justified more easily than their traditional motivation based on partial truth.

1 Linear and substructural logics

Recall that linear logic and its variants are representatives of basic substructural logics (see, e.g., Restall, 2000; Paoli, 2002; Ono, 2003), i.e., logics that result from discarding some of the structural rules from the Gentzen-style calculi LK and LJ for classical and intuitionistic logic. In particular, *linear* logic **LL** discards the rules of contraction (C)

$$\frac{\Gamma, A, A, \Delta \Longrightarrow \Sigma}{\Gamma, A, \Delta \Longrightarrow \Sigma} \qquad \frac{\Gamma \Longrightarrow \Sigma, A, A, \Pi}{\Gamma \Longrightarrow \Sigma, A, \Pi}$$

and weakening (W)

$$\frac{\Gamma \Longrightarrow \Sigma}{A, \Gamma \Longrightarrow \Sigma} \qquad \frac{\Gamma \Longrightarrow \Sigma}{\Gamma \Longrightarrow \Sigma, A}$$

from the calculus LK for classical logic. *Intuitionistic* linear logic **ILL** discards the same rules (C, W) from the calculus LJ for intuitionistic logic. *Affine* linear logic **ALL** and *intuitionistic affine* linear logic **IALL**

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discard only the rule of contraction (C) from the calculi LK and LJ, respectively, but retain the rule of weakening (W).

Recall further that substructural logics work in general with two conjunctions: the *lattice conjunction* $\wedge$ (also called weak, additive, or extensional conjunction) and *fusion* & (also called group, strong, multiplicative, or intensional conjunction). Similarly there are in general two disjunctions (lattice $\vee$ and strong) as well as two implications, two negations, etc., but the latter split connectives will not play a significant role in our account, as we shall mainly deal with intuitionistic substructural logics (which lack strong disjunction) and commutative fusion (then both implications coincide). Since in such substructural logics implication internalizes the sequent sign $\Longrightarrow$ and & the comma on the left-hand side of sequents (cf. Ono, 2003), the validity of the sequent $A_1, \ldots, A_n \Longrightarrow B$ is equivalent to the validity of the formula $A_1 \mathbin{\&} \ldots \mathbin{\&} A_n \to B$. Consequently, the rule of contraction corresponds to the validity of $A \to A \mathbin{\&} A$ and the rule of weakening to the validity of $A \mathbin{\&} B \to A$.

The algebraic semantics of substructural logics is that of *residuated lattices* (see, e.g., Jipsen & Tsinakis, 2002; Ono, 2003; Galatos, Jipsen, Kowalski, & Ono, 2007), i.e., lattices endowed with an additional monoidal operation $*$ (representing &) monotone w.r.t. the lattice order $\leq$, and its two *residuals* $/, \backslash$ (representing implications) that satisfy the residuation law

$$x * y \leq z \quad \text{iff} \quad y \leq x \backslash z \quad \text{iff} \quad x \leq z / y.$$

If $*$ is commutative, the two residuals $/, \backslash$ coincide and are usually denoted by $\Rightarrow$. The set of designated elements is $\{x \mid x \geq 1\}$, where 1 is the neutral element of the monoidal operation $*$. If convenient, residuated lattices may be expanded (to Ono's **FL**-algebras) by a constant 0 for falsity, which makes it possible to define negation as $x \Rightarrow 0$.

The term *substructural logics* will in this paper denote logics of classes of residuated lattices, following the stipulative definition by Ono (2003). In particular, (affine) intuitionistic linear logic is the logic of all (bounded integral) commutative residuated lattices,[1] and (affine) linear logic is the logic of those that furthermore satisfy the law of double negation.

2 Linear logic as the logic of resources

The reason why linear logic has been regarded as the logic of resources is illustrated by Girard's (1995) well-known 'Marlboro–Camels' example:

[1] A residuated lattice is called *commutative* if its monoidal operation $*$ is commutative; it is called *bounded integral* if $0 \leq x \leq 1$ for all elements x. We shall usually work with commutative residuated lattices only.

Consider the propositions

$$\begin{aligned} D &= \text{“I pay \$1”} \\ M &= \text{“I get a pack of Marlboro”} \\ C &= \text{“I get a pack of Camels”}. \end{aligned}$$

Then the sequent

$$D \to M, D \to C \Longrightarrow D \to M \,\&\, C$$

expressing the inference

If I pay $1, I get a pack of Marlboro
If I pay $1, I get a pack of Camels
∴ If I pay $1, I get a pack of Marlboro and I get a pack of Camels

is derivable by the rules of classical as well as intuitionistic logic. The inference is, however, viewed as counter-intuitive, if the conclusion is straightforwardly understood as getting *both* packs. The disputable sequent is not derivable in linear logic, though: linear logic only derives the sequent

$$D \to M, D \to C \Longrightarrow D \,\&\, D \to M \,\&\, C$$

which under a similar interpretation captures the fact that I need to pay *two* dollars to get both packs of cigarettes.

In this sense, linear logic is said to regard formulae as ‘resources’, which are ‘spent’ when used as premises of implications (in the Marlboro–Camels example, the premise D is spent by being detached from $D \to M$ to obtain M, and cannot be used again for $D \to C$ to obtain $M \,\&\, C$). More formally, since premises cannot in linear logic be contracted (due to the lack of the rule (C)), they act as tokens for ‘resources’ needed to support the conclusion: a sequent is valid in linear logic only if it has the needed *amounts* of premises required for arriving at the conclusion.[2] In other words, linear logic ‘counts’ premises of sequents as if they represented resources needed for ‘buying’ the conclusion (where different propositional letters would represent different *types* of resources, while their occurrences in the sequent would represent *tokens* or units of that type).

Nevertheless, this feature of linear logic is due solely to the absence of the rule (C) of contraction, and therefore is common to *all* contraction-free substructural logics. It is not clear why exactly linear logic should

[2] *Exactly* the needed amounts in **LL** or **ILL**; *at least* the needed amounts in their affine versions (it being an effect of weakening that we need not spend all premises). In logics with both (C) and (W), e.g., classical or intuitionistic logic, each premise required for arriving at the conclusion only needs to be present at least once.

be more adequate as a logic of resources than any other contraction-free logic. Rather, it is to be expected that different contraction-free logics will correspond to different assumptions on the structure of resources. In the following sections I will argue that linear logics are in fact adequate only for very general structures of resources, while under most common circumstances, stronger logics are appropriate.

3 The structure of resources

As a first task, we need to refine our conception of resources. Since we aim at an *informal* semantic explanation of certain logics, instead of giving a formal definition we shall just list a few examples indicating what kind of resources we have in mind, and specify the mathematical properties they are assumed to satisfy.

Our notion of a resource will be rather broad: it can include any kind of things that can be counted or measured, that can be acquired and expended, or used for any purpose. Among the resources we consider are, e.g.: money (costs, prices, debts, etc.); goods (packs of cigarettes, clothes, cars, etc.); industrial materials (chemicals, natural raw materials, machine components, etc.); cooking ingredients (flour, salt, potatoes, etc.); computer resources (disk space, computation time, etc.); penalties (which can be regarded as a kind of costs incurred); sets, multisets, or sequences (tuples or vectors) of the above; etc.

It can be observed that all of these (as well as many other) kinds of resources exhibit the structure of a residuated lattice. In particular, there is:

- A partial order $\preceq$ comparing the amounts of the resources. For instance, 300 g of flour is more than 200 g of flour; two pens and three pencils are more than one pen and three pencils; etc. For the sake of compatibility with further definitions, we shall understand $x \preceq y$ as "the resource x is *larger* than or equal to y". The order need not be linear, as for instance two pens are not comparable with three pencils (if different items are counted separately). However, it can be assumed that $\preceq$ is a *lattice order,* as this is true for all prototypical cases: by definition, it amounts to supposing that for any two resources x, y (for instance: $x = 2$ pens and 3 pencils; $y = 1$ pen and 4 pencils), there is the least resource that is at least as large as both (in this case, 2 pens and 4 pencils) and the largest resource that is at most as large as both (here, 1 pen and 3 pencils). Even though there may exist resources that do not satisfy this assumption, we leave them aside in our considerations.

- A monoidal operation $*$ of *composition* (or fusion) of resources. For example, 300 g of flour and 200 g of flour is 500 g of flour; 2 pens and 3 pencils plus 1 pen and 3 pencils are 3 pens and 6 pencils; etc. Putting the resources together can be assumed to be associative (i.e., we presume that the total sum does not depend on the order of summation). The kinds of resources we consider always have a neutral element e, the *empty resource,* which does not change the amount when added to another resource: e.g., 0 g of flour; 0 pens and 0 pencils; etc. Even though composition of resources need not be commutative (consider, e.g., the order of adding ingredients when cooking), for the sake of simplicity of exposition we shall only consider commutative $*$ here (generalization to non-commutative $*$ is always straightforward).

- Finally, resources of all typical kinds can be 'subtracted' or 'evened up', i.e., their composition has the *residual* operation $\Rightarrow$ expressing the remainder, or the difference of amounts: $x \Rightarrow y$ is the least resource to be added to x in order to get a resource at least as large as y.[3] For example, if $x = 200$ g of flour and $y = 300$ g of flour, then $x \Rightarrow y$ is 100 g of flour, as one needs to add 100 g of flour to 200 g of flour to get at least 300 g; while if $x = 2$ pens and 3 pencils, and $y = 1$ pen and 3 pencils, then $x \Rightarrow y$ is 0 pens and 0 pencils (i.e., the empty resource e), as we need not add anything to x to get at least y.

All kinds of resources we consider thus have the structure of a (commutative) residuated lattice $\mathrm{L} = (L, \wedge, \vee, *, \Rightarrow, e)$. Particular kinds of resources can have additional properties: for example, most usual kinds of resources satisfy the so-called divisibility condition $x*(x \Rightarrow y) = x \wedge y$.

Since we aim at a simple resource-based interpretation of existing logical calculi rather than development of an expressively rich logic of resources for computer science, we do not consider such phenomena as, e.g., resource dynamics or possible non-totality of $*$ (which are modeled by such systems as the logic of bunched implications, computation logics, or synchronous and asynchronous calculi—see, e.g., Pym & Tofts, 2006 for references), but only reconstruct and refine the assumptions on resources that are adopted by linear logic.

[3] I.e., $x \Rightarrow y = \sup\{z \mid z * x \preceq y\}$, which is an equivalent formulation of the residuation law in complete lattices. For incomplete lattices, a more cautious formulation based on Dedekind–MacNeille cuts is due, namely $\{z \mid z * x \preceq y\} = \{z \mid z \preceq x \Rightarrow y\}$, which is a general equivalent of the residuation law.

4 Formulae as resources

There are at least two possible representations of resource-based semantics of substructural logics. One of them takes resources (i.e., elements of the residuated lattice L described in Section 3) directly as semantic values assigned to propositional formulae. Recall that a logical calculus can have interpretations other than propositional: cf., e.g., the interpretation of the Lambek calculus as the categorial grammar (where the semantic values of formulae are grammatical categories), or the Curry–Howard combinatorial interpretation of the implicational fragment of intuitionistic logic (where formulae are interpreted as types and proofs as programs). In a similar vein, we can interpret the algebraic semantics of substructural logics under the "formulae-as-resources" paradigm as follows:

- The semantic value of a formula φ is a *resource* $\|\varphi\| \in \mathrm{L}$.
- The Tarski condition $\|1\| = e$ of the algebraic semantics interprets the formula 1 as the empty resource (or 'being for free').
- Similarly, the clause $\|\varphi \mathbin{\&} \psi\| = \|\varphi\| * \|\psi\|$ says that conjunction represents the *fusion of resources.*
- The value of implication, $\|\varphi \to \psi\| = \|\varphi\| \Rightarrow \|\psi\|$, is the resource needed to get at least $\|\psi\|$, given the resource $\|\varphi\|$.
- Finally, the lattice connectives $\wedge, \vee$ represent the meet and join of resources (with respect to the size order $\preceq$ of resources).

The formula φ is regarded as valid under a given evaluation iff $e \preceq \|\varphi\|$, i.e., iff it represents a resource that is for free or even cheaper.

5 Resources as possible worlds

Another way how to interpret substructural logics in terms of resources (cf. Pym & Tofts, 2006) is to regard the structure L of resources as a Kripke frame $(L, \succeq)$ endowed with a monoidal structure $(*, e)$. Unlike in the "formulae-as-resources" paradigm, formulae are here interpreted as propositions, and resources only serve as indices that may (or may not) validate them. The forcing relation $r \Vdash \varphi$, "the resource $r \in \mathrm{L}$ *supports* the formula φ", is required to satisfy the following conditions:

- $e \Vdash 1$,
- $r \Vdash \varphi \mathbin{\&} \psi$ iff $\exists s, t \in \mathrm{L}\colon\ r \preceq s * t$ and $s \Vdash \varphi$ and $t \Vdash \psi$,

- $r \Vdash \varphi \to \psi$ iff $\forall s \in \mathrm{L}$: if $s \Vdash \varphi$, then $r * s \Vdash \psi$,
- $r \Vdash \varphi \wedge \psi$ iff $r \Vdash \varphi$ and $r \Vdash \psi$ ("shared resources"—contrast the clause for &),
- $r \Vdash \varphi \vee \psi$ iff $\exists s, t \in \mathrm{L}$: $r \preceq s \vee t$ and ($s \Vdash \varphi$ or $s \Vdash \psi$) and ($t \Vdash \varphi$ or $t \Vdash \psi$),

and the condition of persistence (if $r \preceq s$ and $s \Vdash \varphi$, then $r \Vdash \varphi$), expressing that "larger resources suffice as well". The formula φ is defined to be valid under $\Vdash$ iff $e \Vdash \varphi$, i.e., iff supported even by the empty resource.[4]

6 The role of tautologies

In the above semantics, tautologies w.r.t. a class $\mathcal{K}$ of (commutative) residuated lattices are defined as the formulae φ that get a value $\|\varphi\| \succeq e$ under all evaluations of propositional letters in any residuated lattice $\mathrm{L} \in \mathcal{K}$ (resp. are supported by e under all $\Vdash$ in every Kripke frame $\mathrm{L} \in \mathcal{K}$). The tautologies of substructural logics thus represent combinations of resources that are always "for free or cheaper".

More importantly, since all residuated lattices validate

$$e \preceq r \Rightarrow s \quad \text{iff} \quad r \preceq s,$$

tautologies of the form $\varphi \to \psi$ internalize sound rules of resource transformations that "preserve expenses" (in the sense of $\preceq$). Inference in substructural logics can thus be understood as inference *salvis expensis,* in a similar manner as inference *salva veritate* in classical logic.[5]

Classes of residuated lattices admitted as possible structures of resources then determine particular logics of resources in the above sense. In particular, by the known completeness theorem, **ILL** is the logic of all commutative residuated lattices, and so it is an adequate logic if just the general structure of a commutative residuated lattice is assumed for admissible kinds of resources. Its variants **IALL**, **ALL**, and **LL** restrict the structure of resources to narrower classes of commutative residuated lattices, and other substructural logics correspond to further specific classes of residuated lattices of resources.[6]

[4] As this is not the aim of this paper, we omit the details on the correspondence between the Kripke-style and algebraic semantics of substructural logics. For more information see (Ono & Komori, 1985).

[5] Note that the general validity of $\|\varphi\| \preceq \|\psi\|$ defines the *local* consequence relation (expressed, i.a., by sequents in Section 1), while Hilbert-style calculi for substructural logics usually capture the *global* consequence relation "$e \preceq \|\psi\|$ whenever $e \preceq \|\varphi\|$".

[6] For example, classical logic can be interpreted as the logic distinguishing just two sizes of resources: empty $e = \|1\|$ and non-empty $f = \|0\| \prec e$.

In the following sections I will argue that most typical kinds of resources satisfy the so-called *prelinearity condition,* and so are in fact governed by deductive fuzzy logics rather than linear logics.

7 Deductive fuzzy logics

Deductive fuzzy logics can be delimited as logics of (classes of) *linearly* ordered residuated lattices (Běhounek & Cintula, 2006; Běhounek, 2008). Among the extensions of **ILL** they can be characterized as those that satisfy the axiom of *prelinearity* (Pre): $((A \to B) \wedge 1) \vee ((B \to A) \wedge 1)$, or in the presence of weakening, equivalently $(A \to B) \vee (B \to A)$.

Let us call residuated lattices for which a substructural logic $\boldsymbol{L}$ is sound, $\boldsymbol{L}$*-algebras.* The prelinearity axiom ensures that a deductive fuzzy logic $\boldsymbol{L}$ is sound and complete, not only w.r.t. the class of *all* $\boldsymbol{L}$-algebras (the *general completeness theorem*), but also w.r.t. the class of all *linear* $\boldsymbol{L}$-algebras (the *linear completeness theorem*). The linear completeness theorem characterizes deductive fuzzy logics among substructural logics; the finitary ones are moreover characterized by the *linear subdirect decomposition property,* which says that each $\boldsymbol{L}$-algebra is a subdirect product[7] of linear $\boldsymbol{L}$-algebras. (See Cintula, 2006 for details.)

Besides the general and linear completeness theorems, most important deductive fuzzy logics furthermore enjoy the *standard completeness theorem,* i.e., the completeness w.r.t. a set of (selected) $\boldsymbol{L}$-algebras on the unit interval $[0, 1]$ of reals (with the usual ordering $\leq$), called the *standard* $\boldsymbol{L}$-algebras. Since $\boldsymbol{L}$-algebras on $[0, 1]$ are fully determined by the monoidal operation $*$, standard-complete deductive fuzzy logics can be defined as logics of (sets of) such monoidal operations $*$ on $[0, 1]$. For example,

- *Łukasiewicz logic* **Ł** is the logic of the Łukasiewicz t-norm $x * y = (x + y - 1) \vee 0$,
- *Gödel–Dummett logic* **G** is the logic of the minimum, i.e., of $x * y = x \wedge y$,
- *Product fuzzy logic* $\boldsymbol{\Pi}$ is the logic of the ordinary product of reals, $x * y = x \cdot y$,
- Hájek's *basic fuzzy logic* **BL** is the logic of all continuous t-norms,[8]
- *Monoidal t-norm logic* **MTL** is the logic of all left-continuous t-norms,

[7] I.e., a subalgebra of the direct product with all projections total.

[8] A commutative associative monotone binary operation on $[0, 1]$ with a neutral element $e \in [0, 1]$ is called a *uninorm.* A *t-norm* is a uninorm with $e = 1$.

- *Uninorm logic* **UL** is the logic of all left-continuous uninorms, etc.

For more information on these logics see (Hájek, 1998; Esteva & Godo, 2001; Metcalfe & Montagna, 2007).

The weakest deductive fuzzy logic extending a substructural logic $\boldsymbol{L}$ is often[9] obtained by adding the prelinearity axiom (Pre) to $\boldsymbol{L}$: for instance,

$$\mathbf{ILL} + (\mathrm{Pre}) = \mathbf{UL}$$
$$\mathbf{IALL} + (\mathrm{Pre}) = \mathbf{MTL}$$

are the weakest deductive fuzzy logics extending intuitionistic linear logics, or the logics of linear commutative (bounded integral) residuated lattices. (For **LL** and **ILL**, the double negation law is to be added to **UL** resp. **MTL**.)

8 Fuzzy logics as logics of costs

Since deductive fuzzy logics are logics of (special classes of) residuated lattices, they can be interpreted as logics of resources in the same way as other substructural logics. Specifically, by the linear completeness theorem (see Section 7), deductive fuzzy logics are sound and complete w.r.t. particular classes of *linear* residuated lattices, and so they are adequate for resources that are linearly ordered by $\preceq$. In other words, deductive fuzzy logics are those logics of resources in which we can assume that all resources are *comparable.*

Prototypical linearly ordered resources are *costs,* that is, resources converted to money. Even though resources in general need not be comparable (cf. the examples in Section 3), their costs (if specified) can always be compared, as money (of a single currency) forms a linear scale.[10] Besides money, there are many other kinds of resources that are linearly ordered, e.g., gallons of fuel, computation time, operational memory, etc. Irrespective of their nature, we shall call all linearly ordered resources *costs,* to distinguish them from resources that are not linearly ordered. For convenience, costs with values in the interval $[0, +\infty]$, e.g., monetary prices (where 0 is "gratis" and $+\infty$ may represent the price of unattainable goods), will be called *prices.*

Deductive fuzzy logics can thus be regarded as *logics of costs,* in the same sense as linear logics are regarded as logics of resources. Different ways of adding up costs—given by the fusion operation—yield different deductive fuzzy logics. The most typical examples are given below:

[9] Always if modus ponens is the only derivation rule of $\boldsymbol{L}$ (Cintula, 2006).
[10] This idea is due to Petr Cintula (pers. comm.).

- If prices are summed by ordinary addition, we obtain the *product logic* $\mathbf{\Pi}$, since the residuated lattice $[0, +\infty]$ with the fusion $+$ and the lattice order $\geq$ is isomorphic (via the function $p \mapsto 2^{-p}$) to the standard product algebra $[0, 1]$ with the fusion $\cdot$ and the lattice order $\leq$. Note that in the standard product algebra, 0 represents the infinite cost and 1 the null cost. If the infinite cost is not considered, the standard product algebra without 0 (called the standard *cancellative hoop*) and its logic **CHL** (*cancellative hoop logic,* see Esteva, Godo, Hájek, & Montagna, 2003) are obtained.[11]

- If prices are bounded by a value $a \in (0, +\infty)$ and summed by bounded addition truncated at a, we obtain the *Łukasiewicz logic* **Ł**, since the residuated lattice $[0, a]$ with bounded addition and $\geq$ is isomorphic via $p \mapsto (a - p)/a$ to the standard [0,1] algebra for Łukasiewicz logic. The bound a (corresponding to 0 in the standard algebra) appears naturally if, e.g., a fixed maximum price is set, if there is a maximal possible cost in the given setting, or if the price a is in the given context unaffordable.

- If prices are combined by the maximum, *Gödel logic* **G** (or its hoop variant) is obtained (by the same isomorphism $p \mapsto 2^{-p}$ as in the case of addition). The maximum may seem a strange operation for summation of prices, but it occurs naturally whenever the costs can be shared by the summands. For example, if temporary results can be erased before the computation proceeds, the memory needed for temporary results is only the maximum (rather than sum) of their sizes.

Logics of other particular t-norms are obtained by using variously distorted 'addition' of prices. For instance, the logic of an ordinal sum of the three basic t-norms corresponds to using different summation rules (of the three described above) in different intervals of prices. The logic **MTL** is obtained if all monotone commutative associative left-continuous operations with the zero price acting as the neutral element are admitted as 'addition' of prices; similarly for **BL** and continuous such operations, etc. The logic **UL** and other uninorm logics only differ by permitting also negative prices, which express gains rather than costs.

[11] If the costs come in packages (e.g., if one has to buy a whole pack of cigarettes even if one needs only a few), the algebra is in general just a **ΠMTL**-chain instead of a product algebra, and the resulting logic in general only extends the logic **ΠMTL** (Esteva & Godo, 2001) or its hoop variant. A similar effect of packaging, which destroys the divisibility of the algebra (see Section 3), can be observed in other algebras of costs as well. (This observation is based on remarks by Rostislav Horčík and Petr Cintula.)

9 Fuzzy logics as logics of resources

In spite of the linear completeness theorem, which makes it possible to regard deductive fuzzy logics as logics of linearly ordered costs, algebras for deductive fuzzy logics need not be linear (consider, e.g., their direct products). By the general completeness theorem (see Section 7), a deductive fuzzy logic $\boldsymbol{L}$ is also sound and complete w.r.t. the class of all $\boldsymbol{L}$-algebras: thus $\boldsymbol{L}$ can also be interpreted as the the logic of all kinds of resources that form the structure of a (possibly non-linear) $\boldsymbol{L}$-algebra.

Let us restrict our attention to finitary deductive fuzzy logics only, as they include all prototypical cases; for the sake of brevity, let us call them just *fuzzy logics* further on. By the linear subdirect decomposition theorem (see Section 7), any $\boldsymbol{L}$-algebra for a fuzzy logic $\boldsymbol{L}$ can be decomposed into a subdirect product of linear $\boldsymbol{L}$-algebras. Fuzzy logics can thus be characterized as logics of such resources that either are linearly ordered, or can at least be decomposed into linearly ordered components. In other words, a sound and complete resource-based semantics of fuzzy logics need not be just that of costs, but also that of resources representable as *tuples* (possibly infinitary) of costs.

It can be observed that many kinds of non-linear resources can actually be represented as tuples of linearly ordered values. For example, ingredients for making pizza and those for making spaghetti are not subsets of each other, thus cooking ingredients do not form a linearly ordered residuated lattice.[12] Nevertheless, they can be decomposed into (potentially infinitely many) linearly ordered components, as the *amounts* of each individual item on an ingredient list are always comparable; and indeed it can be checked that the prelinearity axiom is valid in this residuated lattice.[13]

In fact, most typical resources (including those mentioned in Section 3) *are* indeed decomposable in this way into linear components. Even many resources for which such a decomposition is not known (e.g., human intelligence) can at least be believed to be linearly decomposable (into some unknown and very fine linear components). It is actually rather hard to find a kind of resources that demonstrably cannot be so decomposed.

[12] The elements of the residuated lattice of all possible ingredient lists (such as can be found in recipe books) are tuples of quantities of particular ingredient types (e.g., [300 g of flour, 2 tomatoes, 2 lt of oil], zero amounts omitted). The tuples are naturally ordered by inclusion (i.e., pointwise by component sizes), and fusion represents adding up amounts of each ingredient.

[13] Since the fusion of amounts is (unbounded) addition in each component and infinite amounts do not occur, by extending the considerations of Section 8 the residuated lattice can actually be identified as a cancellative hoop, and the logic of cooking ingredients as the cancellative hoop logic **CHL**.

Thus we can conclude that all typical kinds of resources are linearly decomposable, and therefore they satisfy the axiom of prelinearity, which is not valid in linear logic nor in its affine or intuitionistic variants; consequently, they are actually governed by *deductive fuzzy logics* rather than linear logics. Linear logics are thus only adequate for a very general structure of resources, which admits even the rare kinds of resources that are not decomposable into linearly ordered components. As regards most usual kind of resources, linear logic is too weak for them, as it does not validate the law of prelinearity they obey. Assuming commutativity of fusion, the weakest logic adequate for typical resources is the uninorm logic **UL** (or **MTL** is weakening is assumed, i.e., if the empty resource is the smallest). Specific structures of typical resources are governed by even stronger fuzzy logics—in particular, product logic $\mathbf{\Pi}$ if resources are combined by addition in each linear component, Łukasiewicz logic **Ł** if the addition is bounded, and Gödel logic **G** in the case of shared resources (i.e., if they componentwise combine by the maximum).

Thus it turns out that despite the common opinion, it is actually fuzzy logics, rather than linear logics, that could be categorized as typical logics of resources.[14] The interpretation in terms of resources and costs moreover provides an alternative motivation for deductive fuzzy logics and an explanation of the meaning of their intermediary truth-values that can in some respects be more easily justified than the standard account based on degrees of partial truth.

Libor Běhounek
Institute of Computer Science
Academy of Sciences of the Czech Republic
Pod Vodárenskou věží 2, 182 07 Prague 8, Czech Republic
behounek@cs.cas.cz

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[14] The price paid for the more accurate account is a more complex proof theory, as prelinearity destroys the good proof-theoretical properties of linear logics.

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