On the difference between traditional and deductive fuzzy logic

Libor Běhounek

Institute of Computer Science, Academy of Sciences of the Czech Republic
Pod Vodárenskou věží 2, 182 07 Prague 8, Czech Republic
`behounek@cs.cas.cz, www.cs.cas.cz/hp`

Abstract

In three case studies on notions of fuzzy logic and fuzzy set theory (Dubois–Prade's gradual elements, the entropy of a fuzzy set, and aggregation operators), the paper exemplifies methodological differences between traditional and deductive fuzzy logic. While traditional fuzzy logic admits various interpretations of membership degrees, deductive fuzzy logic always interprets them as degrees of truth preserved under inference. The latter fact imposes several constraints on systems of deductive fuzzy logic, which need not be followed by mainstream fuzzy logic. That makes deductive fuzzy logic a specific area of research that can be characterized both methodologically (by constraints on meaningful definitions) and formally (as a specific class of logical systems). An analysis of the relationship between deductive and traditional fuzzy logic is offered.

Key words: Deductive fuzzy logic, fuzzy elements, gradual sets, entropy of fuzzy sets, aggregation, membership degrees, methodology of fuzzy mathematics.
1991 MSC: 03B52, 03E72, 68T37

Introduction

Lotfi Zadeh [41] has made the distinction between fuzzy logic in broad sense (FLb) and fuzzy logic in narrow sense (FLn). FLn is based on certain many-valued logics, but its agenda differs from that of formal logic: it deals with such concepts as linguistic variable, fuzzy if–then rule, defuzzification, interpolative reasoning, etc.; and FLb roughly coincides with the broad theory and applications of fuzzy sets.

In this paper we shall focus on a sub-area of FLn that studies or uses *formal deductive systems* of fuzzy logic. Prototypical examples of such systems are those centered around Hájek's basic fuzzy logic BL of continuous t-norms [22], including for instance Łukasiewicz, Gödel, and product logics [22], the logics MTL [20], ŁΠ [21], etc., both propositional and first- or higher-order [22,35,4]. The area also covers those parts of fuzzy mathematics (i.e., of FLb) which are built as deductive axiomatic theories based on these formal fuzzy logics (cf. [38,25,26,6,4,5,9], etc.). To avoid a conflict of terms, we shall call this area *deductive fuzzy logic* (FLd). Other parts of FLn and FLb will in the present paper be labeled *traditional fuzzy logic* (FLt), as the latter has a much longer tradition than the relatively newer FLd.

The aim of this paper is to point out and analyze certain fundamental differences between FLd and FLt. The differences are illustrated in three case studies, regarding respectively:

(1) Dubois and Prade's notion of fuzzy element
(2) The notion of entropy of fuzzy sets
(3) Aggregation of fuzzy data

Since the paper is methodological rather than technical, I omit most technical details and focus on the analysis of the principles behind the approaches of FLt and FLd. I assume that the reader has a basic knowledge of some formal system of fuzzy logic, for instance Hájek's logic BL of continuous t-norms [22]. Here I only briefly recapitulate some characteristic features of deductive fuzzy logics, which will be of importance for further considerations:

- Deductive fuzzy logic is a kind of (many-valued) *logic.* [1] Therefore, like other kinds of logics, it primarily studies preservation of some quality ("truth") of propositions under inference. In the particular case of formal *fuzzy* logic, the quality is *partial truth,* i.e., the *degrees* of truth. [2] Thus, deductive fuzzy

[1] Fuzzy logic understood as part of the theory of many-valued logics is sometimes called *mathematical, symbolic,* or *formal* fuzzy logic, as it employs the methods of mathematical (symbolic, formal) logic [23]. Deductive fuzzy logic in our sense is a proper part of mathematical fuzzy logic: it will be shown that in addition to being formal systems of (mathematical, or symbolic) fuzzy logic, deductive fuzzy logics should satisfy certain principles in order to be suitable for graded logical deduction.
[2] Note that throughout the paper, "preservation of partial truth" or "graded inference" refers to the so-called *local* consequence relation. The more commonly studied *global* consequence relation expresses the preservation of *full* truth between fuzzy propositions. The global consequence relation is defined as follows: ψ globally follows from $\varphi_1, \ldots, \varphi_n$ iff the following holds: whenever all $\varphi_1, \ldots, \varphi_n$ are fully true (i.e., of truth degree 1), so is ψ. The local consequence relation, on the other hand, is defined by means of partial truth: ψ locally follows from $\varphi_1, \ldots, \varphi_n$ iff the truth degree of ψ is at least as large as the aggregation (by strong conjunction) of the truth degrees of all $\varphi_1, \ldots, \varphi_n$.
Even though it is the global consequence relation which is most often studied in current mathematical fuzzy logic, local consequence is important for actual reasoning

logic interprets membership degrees exclusively as degrees of truth of the membership predication. In this it differs from the rest of traditional fuzzy logic, which admits various interpretations of membership degrees [17,16].
- As a kind of *formal* or *symbolic* logic, FLd strictly distinguishes syntax from semantics. In syntax, deductive fuzzy logic works with some fixed language composed of propositional connectives, quantifiers, predicate and function symbols, and variables. The symbols (and formulae built up from these symbols) are then interpreted in semantical models, which are composed of usual fuzzy sets and fuzzy relations of FLt. In this way the formulae of the symbolic language formally describe actual fuzzy sets.
- FLd is based on the axiomatic method and works in the formal deductive way. Valid statements about fuzzy sets are derived in an axiomatic theory, through iterated application of sound rules of a particular system of deductive fuzzy logic. Since FLd employs non-classical many-valued logic, formal theories in FLd can have some peculiar features [9], which are not met in standard axiomatic theories of FLt or classical mathematics.
- Most systems of FLd impose specific constraints on some of its components. For example, most systems of formal fuzzy logic require that conjunction be realized as a left-continuous t-norm, and are much less interested in other conjunctive operators studied in FLt. A partial explanation of this selectiveness of FLd will be elaborated in the present paper. It can be shown that such restrictions largely follow from the features of FLd listed above (viz. the interpretation of degrees in terms of truth, the study of partial truth preservation, formal deducibility, etc.).

A further explanation and illustration of these points, as well as an analysis of the difference between FLd and FLt which results from the above features of FLd, will be given in the following sections. I will argue that FLd is a rather sharply delimited area of FLt, and that the agenda of FLd differs significantly from that of FLt. Therefore, to avoid confusion in fuzzy set theory, we should clearly distinguish between their respective areas of competence.

---

in formal fuzzy logic, as it can be used even when premises are only partially true. Notice that usual systems of deductive fuzzy logic axiomatize the global (rather than local) consequence relation; however, the relation of local consequence between $\varphi_1, \ldots, \varphi_n$ and ψ can in these logics be defined as the (global) validity of $(\varphi_1 \mathbin{\&} \ldots \mathbin{\&} \varphi_n) \to \psi$, where & is strong conjunction and $\to$ implication. As argued in Case Study 3 below, the requirements on good behavior of local consequence and its interplay with & and $\to$ form the constitutive features of deductive fuzzy logics.

Case study 1: Dubois and Prade's gradual elements

In [18], Dubois and Prade have introduced the notions of gradual element and gradual set by the following definitions:

Definition 1 *Let S be a set and L a complete lattice with top 1 and bottom 0. A* fuzzy *(or* gradual*) element e in S is identified with a (partial) assignment function $a_e\colon L \setminus \{0\} \to S$.*

Definition 2 *A* gradual subset *G in S is identified with its assignment function $a_G\colon L \setminus \{0\} \to 2^S$. If S is fixed, we may simply speak of* gradual sets.

A prototypical example of a fuzzy element is the fuzzy middle-point of a fuzzy interval A, which assigns the middle point of the α-level of A to each $\alpha \in L\setminus\{0\}$. Notice that the assignment function of a gradual element need not be monotone nor injective (cf. the middle points of certain asymmetric fuzzy intervals). Fuzzy elements of this kind are met in many real-life situations (e.g., the average salary of older people). Gradual elements and gradual sets are claimed by the authors to be a missing primitive concept in fuzzy set theory.

The authors proceed to define the fuzzy set induced by a gradual set, the membership of a gradual element in a fuzzy set, etc. As these notions are not important for our present case, I refer the reader to the original article [18]. We shall only notice that the operations proposed for gradual sets are defined cutwise (with possible rearrangements of cuts in the case of complementation).

The declared motivation for introducing gradual elements is to distinguish *impreciseness* (i.e., intervals) from *fuzziness* (i.e., gradual change from 0 to 1). As implicit in [18,19], a general guideline for definitions of fuzzy notions should be the following principle (we shall call it the *principle of cuts*):

> **Principle of cuts:** *The α-cuts of a fuzzy notion FX should be instances of the corresponding crisp notion X.*

I.e., the fuzzy version FX of a crisp notion X should be defined in such a way that the α-cuts of FX's are X's. Thus the fuzzy counterpart of the notion of element is exactly the fuzzy element of Definition 1, that of the notion of set is the gradual set of Definition 2, etc.

The definitions of gradual sets and gradual elements are clearly sound and the notions will probably prove to be of considerable importance for FLt. Let us see if they can be represented in FLd as well. A more detailed analysis of this question has been done in [13]; here we extract its important parts:

Apparently there are no direct counterparts of gradual elements or sets among the primitive concepts of current propositional or first-order fuzzy logics. Nevertheless, it can be shown that gradual elements and gradual sets are representable in *higher-order fuzzy logic* [4,7] or *simple fuzzy type theory* [35,4]. For technical details of the representation see [13]; here we only sketch the construction:

(1) By the comprehension axioms of higher-order fuzzy logic, the notions of crisp kernel, fuzzy subset, fuzzy powerset, and crisp function are definable in higher-order fuzzy logic (see [4], [13] or a freely available primer [7] for details).
(2) By a standard construction (cf. [38]), an internalization of truth degrees is definable in higher-order fuzzy logic (see [13] or [10] for the details of the construction and some meta-mathematical provisos). The lattice that represents truth degrees within the theory is defined as $\mathrm{L} = \mathrm{Ker}(\mathrm{Pow}(\{a\}))$, i.e., the kernel of the powerset of the crisp singleton of any element a of the universe of discourse. (In fuzzy type theory of [35], this step can be omitted, since the set of truth values is a primitive concept there.)
(3) Since Definitions 1 and 2 need no further ingredients beyond those listed in items (1)–(2), crisp functions from L to the domain of discourse or its powerset represent respectively the notions of gradual element and gradual set in higher-order fuzzy logic. By similar means, all other notions defined in [18] can be defined in higher-order fuzzy logic as well (see [13]).

In particular, the definitions of gradual elements and gradual sets in the standard framework of higher-order logic (or *fuzzy class theory* [4,7]) run as follows:

Definition 3 *A fuzzy element of S (in higher-order fuzzy logic) is any (second-order) class $\mathcal{E}$ such that*

$$\mathrm{Crisp}\,\mathcal{E} \mathbin{\&} \Delta(\mathrm{Dom}\,\mathcal{E} \subseteq \mathrm{L} \setminus \{\emptyset\}) \mathbin{\&} \Delta(\mathrm{Rng}\,\mathcal{E} \subseteq S) \mathbin{\&} \mathrm{Fnc}\,\mathcal{E}.$$

Definition 4 *A gradual subset of S in higher-order fuzzy logic is any (second-order) class $\mathcal{G}$ such that*

$$\mathrm{Crisp}\,\mathcal{G} \mathbin{\&} \Delta(\mathrm{Dom}\,\mathcal{E} \subseteq \mathrm{L} \setminus \{\emptyset\}) \mathbin{\&} \Delta(\mathrm{Rng}\,\mathcal{E} \subseteq \mathrm{Ker}\,\mathrm{Pow}\,S) \mathbin{\&} \mathrm{Fnc}\,\mathcal{G}.$$

In this way, the FLt notions of gradual element and gradual set can also be defined in FLd of higher order. However, their rendering in FLd is not very satisfactory. First, the formal representatives in FLd of the simple FLt notions are rather complex—namely certain very special second-order predicates, whose relationship to traditional fuzzy sets (i.e., first-order predicates) is far from perspicuous. [3] Although this presents no obstacle to handling them

[3] It can, e.g., be observed that the FLd models of Definitions 3 and 4 do not exactly follow the principle of cuts, since the crisp elements or sets are in fact *functional*

in the formal framework of higher-order fuzzy logic, the apparatus of FLd does not much simplify working with these notions (unlike it does with traditional fuzzy sets), since they are represented by crisp functions like in their semantic treatment by FLt. Considering the fundamental role fuzzy elements are to play in Dubois and Prade's recasting of fuzzy set theory, it would certainly be desirable to have fuzzy elements and gradual sets rendered more directly in FLd—as primitive notions rather than defined complex entities, preferably of propositional or first-order rather than higher-order level. These demands, however, encounter the following deep-rooted difficulty:

The new notions represent the horizontal (cut-wise) view of a fuzzy set (construed as a system of cuts), while usual fuzzy set theory represents fuzzy sets vertically (by membership degrees of its elements). Predicates in first-order fuzzy logic only formalize the vertical view of fuzzy sets; and although the latter can also be represented by systems of cuts, all usual FLd systems of first-order fuzzy logic require that the cuts be *nested.* This requirement is already built in the propositional core of common formal fuzzy logics, all of which presuppose the following principle (further on, we shall call it the *principle of persistence*):

> **Principle of persistence:** *If a proposition φ is guaranteed to be (at least) α-true, then it is also guaranteed to be (at least) β-true for all $\beta \leq \alpha$.*

The principle is manifested, i.a., in the transitivity of implication, which is satisfied in all systems of FLd and is indispensable for multi-step logical deduction (more on this in Case Study 3 below). Since Dubois and Prade's gradual sets do not meet this requirement (the α-cuts need not be nested), the known systems of first-order fuzzy logic cannot represent them as fuzzy predicates. (Similarly, known systems of propositional fuzzy logic cannot represent them as fuzzy propositions.)

The reason why Dubois and Prade's notions depart so radically from the presuppositions of FLd resides in the conceptual difference between the approaches to fuzziness in FLd and FLt. In FLt, there are many possible interpretations of the meaning of membership degrees [16,17]. In particular (as stressed by Dubois and Prade in [18]), fuzzy sets may in FLt represent *imprecision* and membership degrees the *gradual change.* In FLd, however, membership degrees are only interpreted as guaranteed degrees of *truth;* and fuzzy sets in FLd represent the degree of satisfaction of truth conditions rather than interval-like imprecision. Thus in FLt, membership degrees can be understood as mere indices which parameterize the membership in a fuzzy set and which allow the gradual change from 0 to 1 ("fuzziness by fibering"). In FLd, truth degrees are what is preserved in graded inference, i.e., preserved w.r.t. the

values rather than α-cuts of the crisp functions that represent gradual elements and sets in higher-order FLd.

ordering of truth values; and this enforces the principle of persistence.

It should be noticed that the principle of cuts, which motivates the distinction between gradual elements and fuzzy sets in [18], is not itself alien to FLd. On the contrary—when following a certain FLd-sound methodology, many fuzzy counterparts of crisp notions do satisfy the principle of cut. The methodology was already sketched in [27, §5] by Höhle, then elaborated in [4, §7], and proposed as a general guideline for FLd in [6]; it consists in re-interpreting the formulae of classical crisp definitions in many-valued logic. If fuzzy notions are defined in this natural way, then the principle of cuts is often observed: the α-cuts of fuzzy sets are crisp sets, the α-cuts of fuzzy relations are crisp relations, the α-cuts of fuzzy Dedekind or MacNeille cuts [27,11,3,2] come out as crisp Dedekind–MacNeille cuts, etc. Unlike in FLt, however, in FLd the fuzzy notions have also to conform with the principle of persistence; this constrains the α-cuts to *nested* systems of the corresponding crisp objects. In the particular case of fuzzy elements, the α-cuts of an FLd fuzzy element a must not only be crisp elements (as in FLt), but also must satisfy the principle of persistence for all formulae, in particular for the formula $x = a$. The latter already necessitates that the α-cut of a equals its β-cuts for all $\beta \leq \alpha$; and since this should hold for all α, the fuzzy element a has to be constant. Thus in FLd we can only have constant fuzzy elements, which can be identified with ordinary crisp elements. Similarly, by enforcing the nesting of α-cuts, the principle of persistence reduces in FLd gradual sets to common fuzzy sets.

No doubt fuzzy elements are a natural notion, abundant in many real-life situations; therefore the above difficulties should not stop us from investigating them. There are no obstacles to investigating them in the framework of FLt. However, current FLd can only render them indirectly in a higher-order setting, since they do not conform to the principle of persistence upon which all current systems of FLd are founded. Thus even though (advanced) FLd can (clumsily) capture the new notions, they actually do not fall into its primary area; and so the way in which FLd can contribute to the investigation of these notion is rather limited.[4] This of course does not diminish the importance of the new notions for FLt and does not even exclude the usefulness of their formal counterparts in some parts of FLd. The above analysis only shows that when employing fuzzy elements in FLd, we shall have to deal with complex objects (crisp functions from the set of internalized truth values) rather than some kind of more primitive notion.

[4] One of the few advantages of studying gradual elements in formal higher-order setting might be the possibility of generalizing them easily to "fuzzy gradual elements" by dropping the condition of the crispness of the function that represents a gradual element or set in Definitions 3 and 4. The apparatus of higher-order fuzzy logic then facilitates the investigation of this higher-order fuzzy notion, which could be more difficult to study in the classical models of FLt.

A further analysis will be needed to find out if Dubois and Prade's gradual elements and sets can be treated propositionally or as a primitive first-order notion in a *radically new* system of deductive fuzzy logic. Since a direct logical rendering of gradual sets would need to drop the principle of persistence, it would have to adopt an entirely different concept of truth preservation under inference; such a radical change would consequently affect virtually all logical notions. Unfortunately, many straightforward approaches are not viable, as they would trivialize the theory. E.g., a notion of truth preservation based on the identity (rather than order) of truth degrees would reduce truth degrees to mere indices exactly in the way FLt does; however, it would trivialize the logic to classical Boolean logic. [5] From the opposite point of view, this could be an indication that by treating membership degrees as mere indices (rather than truth degrees that should be preserved under graded inference), FLt does not in fact step out of the classical framework; it is the gradual inference what makes things genuinely fuzzy from the FLd point of view, rather than just employing some set of indices like $[0, 1]$.

Case study 2: The entropy of a fuzzy set

Various definitions of the *entropy* of a fuzzy set have been proposed in traditional fuzzy mathematics, for instance:

- De Luca and Termini's [15] entropy $E_k(A) = D_k(A) + D_k(A^c)$
- Yager's [39] entropy $Y_p(A) = 1 - \ell^p(A, A^c)/(\ell^p(A, \emptyset))^p$
- Kaufmann's [29] entropy $K_p(A) = 2n^{-1/p} \cdot \ell^p(A, \overline{A})$
- Kosko's [31] entropy $R_p(A) = \ell^p(A, \overline{A})/\ell^p(A, \underline{A})$

where A is a finite $[0, 1]$-valued fuzzy set; A^c is its additive complement, $A^c(x) = 1 - A(x)$; $\overline{A}$ is defined as $\overline{A}(x) = 1$ if $A(x) \geq 0.5$, and 0 otherwise; $\underline{A} = (\overline{A})^c$; p, k are parameters, $p \geq 1$ and $k > 0$; $D_k(A) = -k \sum_i A(x_i) \log A(x_i)$; and ℓ^p is the distance between finite fuzzy sets defined as $\ell^p(A, B) = (\sum_i |A(x_i) - B(x_i)|^p)^{1/p}$.

The common feature of all such entropy measures is that they assign the minimal (zero) entropy to crisp sets, and maximal (unit) entropy to fuzzy sets with $A(x) = 0.5$ for all x in the universe of discourse. [6]

[5] The α-levels of fuzzy or gradual notions are crisp, therefore they follow the rules of classical logic, i.e., the logic of Boolean algebras. An α-level based definition of truth preservation would correspond to taking the direct product of Boolean algebras B_α for all levels $\alpha \in [0, 1]$. However, the direct product of Boolean algebras is a Boolean algebra, therefore the resulting logic would remain classical.

[6] In more detail, they satisfy de Luca and Termini's [15] axioms for entropy measures $E\colon [0, 1]^X \to [0, 1]$, namely: (i) $E(A) = 0$ iff A is crisp; (ii) $E(A) = 1$ iff $A(x) = 0.5$ for all $x \in X$; (iii) $E(A) \leq E(B)$ if for every $x \in X$ either

The definition is motivated (and the name *entropy* justified) by the idea that the membership degree 0.5 tells us the least amount of information ("nothing") about the membership of x in A. In other words, that the membership degree of 0.5 gives us the same degree of "certainty" that x belongs to A as that x does not belong to A, and so it provides us with no information (knowledge) as to *whether* x belongs to A. The membership degrees of 0 and 1, on the other hand, give us full "knowledge" or "certainty" about the membership of x in A, and thus provide us with maximal information as regards the membership of x in A. The degree of fuzziness, measured by the entropy measures, thus (in FLt) expresses the informational contents contained in the fuzziness of the fuzzy set.

In FLd, on the other hand, such concepts of entropy do not have good motivation.[7] This is because in FLd, the membership degree cannot be interpreted as the degree of *knowledge* or *certainty* of whether x belongs to A or not, but only as the degree of the (guaranteed) *truth* of the statement that x belongs to A. From the FLd point of view it is not true that $A(x) = 0.5$ gives us the least information on the membership in A. On the contrary—each membership degree gives us the same (namely, full) information about the *extent* of membership in A.

The difference between the information conveyed by membership degrees in FLt and FLd can be illustrated by the following consideration. We have the following trivial observation in all usual systems of FLd that contain a well-behaved implication connective $\Rightarrow$.

Fact 5 *If it is provable that $A(z) \Rightarrow \varphi(z)$ for all z, then for any membership degree α, if the truth degree of $A(x)$ is α, then the truth degree of $\varphi(x)$ is at least α.*

Thus in FLd, if we know that $x \in A$ is true to degree 0.5 and that all elements of A satisfy some property φ (in the sense of FLd—i.e., that $A(z) \Rightarrow \varphi(z)$ is valid for all z), then we know that x satisfies φ at least to degree 0.5. Therefore in FLd, the truth degree of 0.5 does *not* represent "no knowledge" or "equal possibility of both cases". Rather, like any other membership degree, it represents a certain *guaranteed degree of participation* of x on the properties of A. In other words, *any* membership degree α of $x \in A$ tells us in FLd that the properties entailed by the membership in A will be satisfied by x at least to degree α.

$A(x) \leq B(x) \leq 0.5$ or $A(x) \geq B(x) \geq 0.5$; and (iv) $E(A) = E(A^c)$.

[7] At least not as measures of the informational contents of fuzziness. If definable in a particular fuzzy logic, they can only serve as measures of fuzziness, without any connection to information.

From the informational point of view, in FLd (as shown by Fact 5) the membership degree 0.5 restricts the possible truth values of $\varphi(x)$, for any property φ entailed by the membership in A, to the interval $[0.5, 1]$. In this sense, the least informative membership degree should in FLd be 0 (as it does not restrict the interval at all) and the most informative degree should be 1 (as it maximally restricts the interval to the single value 1). However, 0 is also the most informative (and 1 the least informative) degree as regards the satisfaction by x of the properties of another set, namely A^c. Therefore in FLd, the informational contents of membership degrees is not determined simply by their value.

Thus from the point of view of FLd, no membership degree conveys more information than another just by its value. Therefore, no concept of entropy which assigns the least informational contents to the fuzzy set with $A(x) = 0.5$ for all x is well-motivated in FLd. Consequently we have to conclude that the notions of entropy belong to the area of FLt rather than FLd; and even though they can be defined in higher-order FLd,[8] their significance in FLd and the extent to which FLd can help investigate them is limited. This does not deny their importance and good motivation in FLt under the interpretations of membership degrees as indicated above (of knowledge, certainty, etc.); only they are not meaningful for the concept of *guaranteed truth,* which is the domain of FLd.

As stressed above, the unmotivatedness of the concept of entropy in FLd is caused by the fact that membership degrees represent in FLd the degrees of *truth* (of the statement "$x \in A$") rather than the degrees of *knowledge* or *certainty* about $x \in A$. The *uncertainty* about $x \in A$ would not in FLd be expressed by an intermediate membership degree, but rather by an *uncertain* membership degree. The first idea how to render uncertain membership degrees in FLd is, obviously, to take a crisp or fuzzy set of possible membership degrees, like in interval-valued fuzzy sets [1] or type-2 fuzzy sets [40]. However, in the framework of FLd, this idea has to be refined: a fuzzy set of membership degrees does not in FLd represent the degree of certainty or knowledge about the membership degrees, either, but only expresses the degree of *truth* of some property of membership degrees. Thus it would be necessary to introduce some modality, e.g., "it is known that", and interpret the fuzzy set of membership

[8] For instance, Yager's entropy Y_1 and Kosko's entropy R_1 can be defined in the higher-order fuzzy logic ŁΠ [4], since it contains all arithmetical ingredients necessary for their definitions: additive negation $(1 - x)$, product implication (i.e., division), and the Baaz Δ connective which ensures [4, §7] the definability of crisp finite sequences, needed for the inductive definition of sums of membership degrees. Any classically definable entropy measure is eventually definable in higher-order FLd by more sophisticated means, since classical mathematics is interpretable in standard higher-order fuzzy logics [4, §7]. (By definability we mean here definability in standard $[0, 1]$ models.)

degrees α as expressing the *truth* degree of the statement "it is known that the membership degree of $x \in A$ is α", rather than the degree of knowledge itself. This subtle difference is insignificant for atomic epistemic statements, but plays a role when considering complex epistemic statements composed by means of propositional connectives. (For more on this distinction see [24,22].) The rendering of the uncertainty of membership in a fuzzy set, which motivates the notion of entropy in FLt, is thus in FLd much more complicated than what is expressed by simple intermediary membership degrees.

Case study 3: Aggregation of fuzzy data

The exclusive interpretation of membership degrees as guaranteed degrees of truth leads to certain restrictions on admissible logical systems of FLd. Since the intended interpretation "truth at least to α" is based on an order ("at least") of truth degrees, logical systems suitable for FLd have to be among the logics of partially ordered (or at least preordered) algebras or logical matrices, i.e., among Rasiowa's implicative logics [37] or Cintula's weakly implicative logics [14].[9] The property of prelinearity, advocated in [8] as the characteristic feature of deductive *fuzzy* logics, then leads to Cintula's class of weakly implicative fuzzy logics [14]. Another condition that further constrains the class of logical systems best suitable for *deductive* fuzzy logic is the law of residuation [22,36]. As will be shown in this section, the law of residuation and related requirements present another important difference between FLd and FLt.

One of the typical tasks of applied FLt is to gather some fuzzy data $\varphi_1, \ldots, \varphi_k$, aggregate their truth values by means of some aggregation operator $\bigodot$,[10] and draw some conclusion ψ (possibly, about the action to be performed or the answer to be given) based on $\bigodot_{i=1}^{k} \varphi_i$. In symbols, to perform an inference $(\bigodot_{i=1}^{k} \varphi_i) \to \psi$, where $\to$ is a suitable implication. We have in mind, e.g., the following kinds of applications:

[9] The defining conditions of (weakly) implicative logics embody the correspondence between the full truth of implication and the (pre)ordering of truth degrees. Besides the conditions of substitution-invariant Tarski consequence (common to most systems of formal logic), weakly implicative logics require the logical validity of (i) the axiom $\varphi \to \varphi$ and the rules of (ii) modus ponens (from φ and $\varphi \to \psi$ infer ψ), (iii) transitivity of implication (from $\varphi \to \psi$ and $\psi \to \chi$ infer $\varphi \to \chi$), and (iv) congruence of all connectives w.r.t. bi-implication $\varphi \to \psi$ and $\psi \to \varphi$. Weakly implicative logics in general admit multiple degrees of full truth; Rasiowa's implicative logics forbid them by the additional rule (v) of weakening (from φ infer $\psi \to \varphi$).

[10] The term "aggregation operator" is here understood in a broad sense, without requiring any fixed set of axioms for the operator.

Example 6 In a fuzzy controller based on if-then rules, the input data φ_i are the truth values of the evaluating expressions "X_i is Y_i" given by the measured values of linguistic variables X_i; the output of a single rule is the truth value of "X is Y", inferred from φ_i's by suitable operations $\odot$ and $\rightarrow$.

Example 7 A fuzzy logic based engine for answering database queries (say, for accommodation search) may ask for the degrees of the user's preferences, i.e., the weights of such variables as price, distance, etc. Based on the aggregated weighted values of these variables for particular hotels, the engine lists the hotels in descending order by their suitability for the user.

An important observation about this kind of applications is that just one inference step is performed for each set of input data:

- When a fuzzy controller performs an action based on the fuzzy inference, the values of measured variables change, and the next inference is based on the new (changed) data.
- When listing hotels in the order of the user's preferences, the evaluation of each hotel is based on the hotel's own parameters; the evaluation of the next hotel takes new (i.e., the next hotel's) data.

In such cases, therefore, the device may work in a cycle, but each iteration processes a new set of data. The modus operandi of such applications of FLt is as depicted in Figure 1.

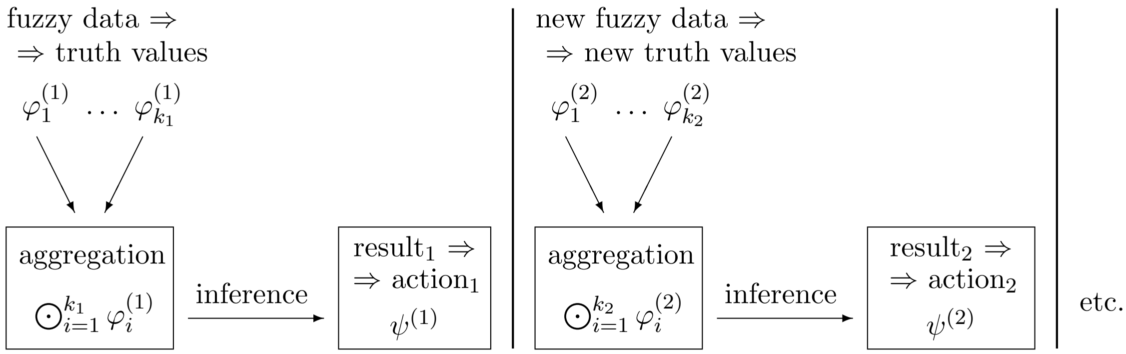

Fig. 1. Modus operandi of applied FLt

Another observation is that the data that enter the aggregation and inference are usually extra-logical (measured in the real world, read from a database etc.). In particular, they usually do not contain the operators $\odot$ and $\rightarrow$ of the inference mechanism, and so in FLt inference one usually need not consider nested implications (the formulae expressing the inference laws are "flat").

The operations used for aggregation of the input data vary widely among particular applications. Consequently, various classes of aggregation operators $\odot$ are studied in theoretical FLt, including t-norms and co-norms, uninorms, copulas, semi-copulas and quasi-copulas, various kinds of averages and means, etc. (for a brief overview see, e.g., [30, Ch. 3]).

The situation in FLd is different, as is the typical modus operandi of FLd. The formally-deductive aims of FLd require the preservation of guaranteed truth values also in successive (iterated) inferences, which are typical for multiple-step deductions. In formal derivations we often have intermediary steps and results, lemmata, partial conclusions, etc., and we want the guaranteed truth degree of a conclusion to remain coherent throughout long deductions. Therefore, a typical modus operandi of FLd is the one depicted in Figure 2.

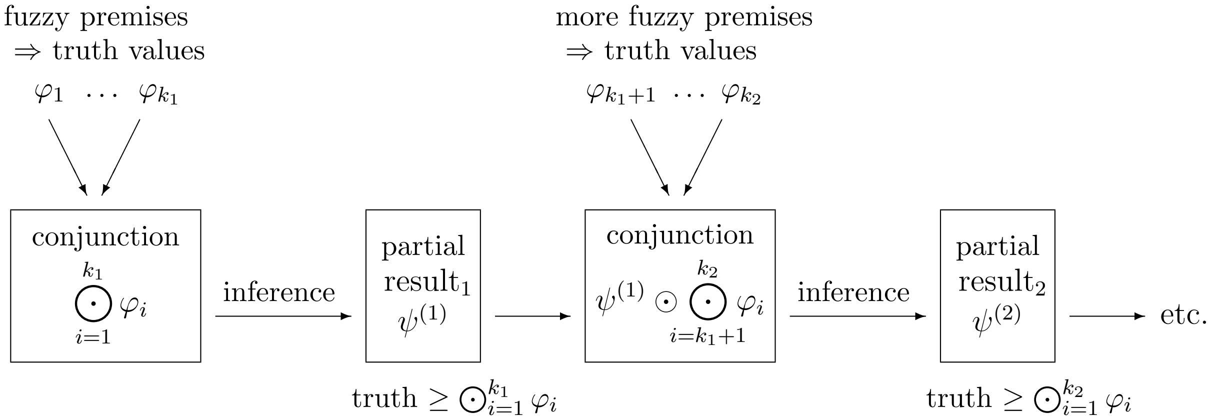

Fig. 2. Modus operandi of FLd

Observe first that in the multiple-step derivations of FLd, the premises of the first steps still play a role in the following steps, since partial results enter further deductions. Furthermore, in the formally logical setting of FLd, formulae entering deductions need not be purely extralogical and can have inner logical structure, i.e., be built up from subformulae by means of logical connectives, including those used for inference, i.e., $\odot$ and $\to$. Thus unlike in FLt, the formulae in FLd inference need not be "flat" and nested implications can occur. Implication thus plays a double role in FLd deduction: it is used for making inferences, but can also occur as a connective within a formula that enters the inference as a premise or comes out as a conclusion. Similarly conjunction is used for the aggregation of premises, but can also appear as a connective inside the premises and conclusions. If both roles of the operators are to match, they have to satisfy conditions that describe the match of the roles. Namely, whenever φ_1 is a premise of implication (inference) and $\varphi_2 \to \psi$ is its conclusion, both roles of implication will accord iff φ_1 and φ_2 together (i.e., aggregated) imply ψ (since both φ_1 and φ_2 are after all premises for ψ—one in implication-as-inference and one in implication-as-connective); and vice versa, if φ_1 and φ_2 jointly imply ψ, then φ_1 alone should imply $\varphi_2 \to \psi$ (for the same reason). Similarly, φ_1 and φ_2 aggregated should imply ψ if and only if $\varphi_1 \odot \varphi_2$ implies ψ (this corresponds to the match of both roles for conjunction). Since by the earlier considerations implication-as-inference is in FLd understood as truth-preservation (i.e., the relation $\leq$), the requirement can be formulated as the condition

$$\varphi_1 \odot \varphi_2 \leq \psi \quad \text{iff} \quad \varphi_1 \leq \varphi_2 \to \psi \tag{1}$$

The general form for an arbitrary number of premises as in Figure 2 already follows from (1). This law of *residuation* is therefore required in FLd for ensuring the coherence of the guaranteed truth thresholds in multiple-step deductions with nested implications, while it need not be required in one-step inferences with flat formulae in FLt.

The principle of residuation restricts significantly the class of conjunctive operators admissible in FLd. Together with a few reasonable additional requirements (see Remark 10 below) it confines the FLd-suitable $[0, 1]$-conjunctions & to *left-continuous t-norms* (or residuated uninorms, if we admit degrees of full truth) [20,28,33]. Other operators for fuzzy data aggregation are not meaningful in FLd, though they are both meaningful and important in FLt (as FLt need not preserve guaranteed truth degree in nested and iterated inferences).

Like in the case of fuzzy elements and the notion of entropy, many FLt conjunctive operators are still definable in systems of deductive fuzzy logic: e.g., a broad class of t-norms which are *not* left-continuous is representable in the logic ŁΠ [21,34]. Nevertheless, as in the cases above, the apparatus of FLd is most efficient for conjunctions to which the FLd systems are tailored, i.e., which respect the above constraints.

Remark 8 The constraints on admissible conjunction connectives rule out the meaningfulness of most cut-wise definitions in FLd. Since most left-continuous t-norms are not idempotent, α-cuts are in general not preserved by conjunction in most systems of FLd (except in Gödel fuzzy logic of the minimum t-norm). [11] Thus, e.g., the cut-wise definition of the intersection of fuzzy sets is from the FLd point of view only meaningful in Gödel logic; in other systems of FLd, the cut-wise intersection (which equals the minimum-intersection) does not satisfy the defining condition of intersection that the membership degree of x in $A \cap B$ be the conjunction of the membership degrees of x in A and B, i.e., that $(A \cap B)x = Ax \,\&\, Bx$.

Thus, e.g., Dubois and Prade's definitions of elementary operations on gradual sets proposed in [18] (which are cut-wise, as we noted in the first case study), can only be well-motivated in FLt. Similar considerations restrict the FLd-meaningfulness of many parts of categorial (sheaf) approach to fuzzy sets,

[11] Most FLd conjunctions thus do not satisfy the axiom often required in FLt of aggregation operators, namely that $x \odot \cdots \odot x = x$.

which often works cut-wise (i.e., fiber-wise) and thus belongs to FLt rather than FLd.

Again, this does not diminish the importance of cut-wise notions in FLt; only we should be aware that they are not well-motivated in FLd. In deductive fuzzy logic, many cut-wise notions can still be defined, and some of them do have some importance even in FLd. For instance, in all logics based on continuous t-norms, the minimum conjunction $\wedge$ and maximum disjunction $\vee$ are definable, and by means of these connectives one can define the cut-wise operations of min-intersection and max-union. However, their role in FLd systems is different than that of the notions based on usual (strong) conjunction &; in particular, min-conjunction cannot be used as a surrogate for strong conjunction, since both connectives have different meaning. Strong conjunction & represents the use of *both* conjuncts, while min-conjunction $\wedge$ represents the use of any *one* of the conjuncts, as can be seen from the following equivalences valid in BL and related systems:

$$[(\varphi_1 \mathbin{\&} \varphi_2) \to \psi] \leftrightarrow [\varphi_1 \to (\varphi_2 \to \psi)] \tag{2}$$

$$[(\varphi_1 \wedge \varphi_2) \to \psi] \leftrightarrow [(\varphi_1 \to \psi) \vee (\varphi_2 \to \psi)] \tag{3}$$

Since it is (2) that we need in iterated inference rather than (3), minimum conjunction cannot be used for aggregation of premises in FLd. Similarly, min-intersection does not represent membership in *both* fuzzy sets, but only in *any* of them, and cannot be used in contexts when both Ax and Bx are required. The following example demonstrates the methodological consequences of the distinction between the two conjunctions in FLd.

Example 9 The notion of antisymmetry of a fuzzy relation R w.r.t. a similarity E defined with min-conjunction, i.e., by $\inf_{xy}(Rxy \wedge Ryx \to Exy)$ as in [11] or similarly in [27], is not well-motivated in FLd, since in antisymmetry we clearly need *both* Rxy and Ryx to infer Exy (neither Rxy nor Ryx alone is sufficient for Exy in antisymmetric relations; cf. (3)). Thus in FLd, we have to define antisymmetry with strong conjunction, i.e., as $\inf_{xy}(Rxy \mathbin{\&} Ryx \to Exy)$, even though some theorems of [27,11] will then fail. From the *deductive* point of view, min-conjunction antisymmetry is only well-motivated in Gödel logic.

Remark 10 As mentioned at the beginning of this section, the requirements on the transmission of truth in FLd lead to the defining conditions of Rasiowa's implicative logics or Cintula's weakly implicative (fuzzy) logics. However, these conditions only ensure good behavior of *fully true* implication, which then corresponds to the order of truth degrees [14]. Inferentially sound behavior of *partially true* implication and conjunction requires further axioms, including the law of residuation (as seen above), since only then implication internalizes the transmission of partial truth and conjunction internalizes the cumulation of premises in graded inference. The latter law also makes formal systems of FLd belong to the well-known and widely studied class of substructural logics (in Ono's [36] sense, i.e., the logics of residuated lattices).

Thus from the point of view of *deductive* fuzzy logic, Cintula's class of weakly implicative fuzzy logic is still too broad. Best suitable logics for FLd are only those weakly implicative fuzzy logics that satisfy residuation and several natural requirements of the internalization of local consequence (namely the logical axioms expressing the antitony resp. isotony of implication in the first resp. second argument, and associativity and commutativity of conjunction; cf. [12]). The resulting class can be understood as the *formal* mathematical delimitation of deductive fuzzy logics. The above conditions characterize them as those weakly implicative fuzzy logics which are extensions or expansions of the logic UL of residuated uninorms [33] or, if we add the law of weakening $\varphi \to (\psi \to \varphi)$, of the logic MTL of left-continuous t-norms [20]. [12]

Again this does not imply that other weakly implicative fuzzy logics or logics used in FLt are deficient. However, only logics from the above defined class suit best to the motivation of FLd (i.e., transmission of guaranteed partial truth in multi-step deductions) and admit the construction of formal fuzzy mathematics in the sense of [6]. This is because their implication and conjunction respectively internalize the local consequence relation and the cumulation of premises: they have, in Ono's [36] words, a "deductive face". Other logics [13] lie outside the primary area of interest of FLd, though they may be of their own importance and interest in FLt.

Conclusions

The three case studies show that FLd differs from broader FLt in many aspects, including the area of competence, methods, motivation, formalism, etc. It should perhaps be admitted that symbolic fuzzy logicians on the one hand and researchers in "mainstream fuzzy logic" on the other hand do rather different things and work in two distinct, even though related, areas (with some non-empty intersection). Since after the years of usage there is no chance for changing the name "fuzzy logic" in either tradition, a suitable determinative

[12] The class is only slightly broader than Metcalfe and Montagna's class of "substructural fuzzy logics" [33], which in addition requires the completeness w.r.t. standard $[0, 1]$ semantics.

[13] Including Zadeh's original system with min-conjunction, max-disjunction, $(1-x)$-negation and S-implication, as well as Łukasiewicz logic with strong conjunction replaced by min-conjunction—a system both favored and targeted by many philosophers of vagueness.

adjective (like *symbolic, formal, mathematical,* or as proposed here, *deductive*) attached to the name of the narrower and younger of both areas is probably the best solution to possible terminological confusions.

It is sometimes complained that fuzzy logic does not have a clear methodology for defining its notions and the direction of research. FLd, as its very narrow and specific branch, however, does possess a rather clear methodology, inherited for a large part from the methodology of non-classical logics and classical foundations of mathematics [22,6]. This may be a consequence of the fact that FLd has chosen and clarified *one* of all possible interpretations of membership degrees, and now studies the properties of this single clarified concept. FLt, on the other hand, admits many interpretations of membership degrees and often tries to investigate them together, without separating them properly and without clarifying carefully which of the possible interpretations is considered. [14]

A historical parallel can be seen in the early history of classical (crisp) set theory. As noticed by Kreisel in [32], Cantor's notion of set was a mixture of at least three concepts—finite sets of individuals, subsets of some domain, and properties (unbound classes). Part of the opposition against set theory was due to its confusion of these notions of set: the *crude mixture* (as Kreisel calls it) did not possess good properties, and the paradoxes of naive set theory confirmed the bad feeling. Only after one element of the crude mixture (viz. iterated subsets) was clearly separated by Russell and Zermelo and shown to have good and rich enough properties, the notion of set could start playing its foundational role in mathematics.

Similarly the theory of fuzzy sets presents a mixture of various *different* notions of fuzzy set (truth-based, possibilistic, linguistic, frequentistic, probabilistic, etc.). While FLd has distilled one element of the mixture (namely the truth-based notion of fuzzy set), FLt often continues to investigate the crude mixture as a whole, only partially aware of the distinctions needed to be made. (Not that it never distinguishes the areas of applicability of its own notions: sometimes it does; but often it forgets to do so or is not careful enough.)

The methodological success of FLd and its advances should stimulate FLt to distinguish with similar clarity the *exact* components of the crude mixture of notions of fuzzy set. Theoretical gains from their clear separation and investi-

[14] For instance, general definitions (e.g., cut-wise) of operations on fuzzy sets are often given, regardless of what is the intended interpretation of membership functions. (This is also the case with the operations on gradual sets defined in [18].) However, suitable definitions may depend on the intended meaning of membership degrees (as also demonstrated by the unsuitability of many such definitions for FLd), since different underlying phenomena may have different properties (and thus also different demands on the behavior of suitable operations).

gation of the most promising ones would certainly be large (as was, e.g., the gain from conceptual and methodological clarification of the notion of probability); some areas of FLt besides FLd (e.g., possibility theory) already seem to be close to such clarification.

Acknowledgments

The work was supported by grant No. B100300502 of the Grant Agency of the Academy of Sciences of the Czech Republic and by the Institutional Research Plan AV0Z10300504. I profited from discussions with Petr Cintula, Martina Daňková, and Petr Hájek.

References

[1] K. T. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20:87–96, 1986.

[2] L. Běhounek. Towards a formal theory of fuzzy Dedekind reals. In *Proceedings of Joint EUSFLAT–LFA Conference*, pages 946–954, Barcelona, 2005.

[3] L. Běhounek. Two notions of fuzzy lattice completion. In V. Novák and M. Štěpnička, editors, *Proceedings of LSC IV & 4th Workshop of the ERCIM Working Group on Soft Computing*, pages 22–24, Ostrava, 2005. Research report No. 77, Institute for Research and Application of Fuzzy Modeling, University of Ostrava.

[4] L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.

[5] L. Běhounek and P. Cintula. Fuzzy class theory as foundations for fuzzy mathematics. In *Fuzzy Logic, Soft Computing and Computational Intelligence: 11th IFSA World Congress*, volume 2, pages 1233–1238, Beijing, 2005. Tsinghua University Press/Springer.

[6] L. Běhounek and P. Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.

[7] L. Běhounek and P. Cintula. Fuzzy Class Theory: A primer v1.0. Technical Report V-939, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2006. Available at www.cs.cas.cz/research/library/reports_900.shtml.

[8] L. Běhounek and P. Cintula. Fuzzy logics as the logics of chains. *Fuzzy Sets and Systems*, 157(5):604–610, 2006.

[9] L. Běhounek and P. Cintula. Features of mathematical theories in formal fuzzy logic. In P. Melin, O. Castillo, L. T. Aguilar, J. Kacprzyk, and W. Pedrycz, editors, *Foundations of Fuzzy Logic and Soft Computing*, volume 4529 of *Lecture Notes in Artificial Intelligence*, pages 523–532. Springer, Berlin etc., 2007.

[10] L. Běhounek and M. Daňková. Relational compositions in Fuzzy Class Theory. Submitted to Fuzzy Sets and Systems, 2007.

[11] R. Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles*, volume 20 of *IFSR International Series on Systems Science and Engineering*. Kluwer Academic/Plenum Press, New York, 2002.

[12] L. Běhounek. An alternative justification of the axioms of fuzzy logics. Abstract, to appear in the Bulletin of Symbolic Logic.

[13] L. Běhounek. Dubois and Prade's fuzzy elements: A challenge for formal fuzzy logic. In S. Gottwald, P. Hájek, and M. Ojeda-Aciego, editors, *Proceedings of LSC IV & 4th Workshop of the ERCIM Working Group on Soft Computing*, pages 90–95, Málaga, 2006.

[14] P. Cintula. Weakly implicative (fuzzy) logics I: Basic properties. *Archive for Mathematical Logic*, 45(6):673–704, 2006.

[15] A. de Luca and S. Termini. A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory. *Information and Control*, 20:301–312, 1972.

[16] D. Dubois, L. Godo, H. Prade, and F. Esteva. An information-based discussion of vagueness. In H. Cohen and C. Lefebvre, editors, *Handbook of Categorization in Cognitive Science*, chapter 40, pages 892–913. Elsevier, 2005.

[17] D. Dubois and H. Prade. The three semantics of fuzzy sets. *Fuzzy Sets and Systems*, 90:141–150, 1997.

[18] D. Dubois and H. Prade. Fuzzy elements in a fuzzy set. In Y. Liu, G. Chen, and M. Ying, editors, *Fuzzy Logic, Soft Computing and Computational Intelligence: 11th IFSA World Congress*, volume 1, pages 55–60, Beijing, 2005. Tsinghua University Press/Springer.

[19] D. Dubois and H. Prade. Fuzzy intervals versus fuzzy numbers: Is there a missing concept in fuzzy set theory? In S. Gottwald, P. Hájek, U. Höhle, and E. P. Klement, editors, *Fuzzy Logics and Related Structures: Abstracts of 26th Linz Seminar on Fuzzy Set Theory*, pages 45–46, Linz, 2005.

[20] F. Esteva and L. Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.

[21] F. Esteva, L. Godo, and F. Montagna. The $Ł\Pi$ and $Ł\Pi\frac{1}{2}$ logics: Two complete fuzzy systems joining Łukasiewicz and product logics. *Archive for Mathematical Logic*, 40(1):39–67, 2001.

[22] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordercht, 1998.

[23] P. Hájek. What is mathematical fuzzy logic. *Fuzzy Sets and Systems*, 157(5):597–603, 2006.

[24] P. Hájek, L. Godo, and F. Esteva. Fuzzy logic and probability. In *Proceedings of the 11th UAI Conference*, pages 237–244, Montreal, 1995.

[25] P. Hájek and Z. Haniková. A development of set theory in fuzzy logic. In M. Fitting and E. Orlowska, editors, *Beyond Two: Theory and Applications of Multiple-Valued Logic*, volume 114 of *Studies in Fuzziness and Soft Computing*, pages 273–285. Physica-Verlag, Heidelberg, 2003.

[26] P. Hájek and V. Novák. The sorites paradox and fuzzy logic. *International Journal of General Systems*, 32:373–383, 2003.

[27] U. Höhle. Fuzzy real numbers as Dedekind cuts with respect to a multiple-valued logic. *Fuzzy Sets and Systems*, 24(3):263–278, 1987.

[28] S. Jenei and F. Montagna. A proof of standard completeness for Esteva and Godo's logic MTL. *Studia Logica*, 70(2):183–192, 2002.

[29] A. Kaufmann. *Introduction to the Theory of Fuzzy Subsets*, volume I. Academic Press, New York, 1975.

[30] G. J. Klir and B. Yuan. *Fuzzy Sets and Fuzzy Logic: Theory and Applications.* Prentice Hall, 1995.

[31] B. Kosko. Fuzzy entropy and conditioning. *Information Sciences*, 40:165–174, 1986.

[32] G. Kreisel. Informal rigour and completeness proofs. In I. Lakatos, editor, *Problems in the Philosophy of Mathematics*, pages 138–157. North-Holland, Amsterdam, 1967.

[33] G. Metcalfe and F. Montagna. Substructural fuzzy logics. To appear in Journal of Symbolic Logic.

[34] F. Montagna and G. Panti. Adding structure to MV-algebras. *Journal of Pure and Applied Algebra*, 164(3):365–387, 2001.

[35] V. Novák. On fuzzy type theory. *Fuzzy Sets and Systems*, 149(2):235–273, 2004.

[36] H. Ono. Substructural logics and residuated lattices—an introduction. In V. F. Hendricks and J. Malinowski, editors, *50 Years of Studia Logica*, volume 21 of *Trends in Logic*, pages 193–228. Kluwer, Dordrecht, 2003.

[37] H. Rasiowa. *An Algebraic Approach to Non-Classical Logics.* North-Holland, Amsterdam, 1974.

[38] G. Takeuti and S. Titani. Fuzzy logic and fuzzy set theory. *Archive for Mathematical Logic*, 32:1–32, 1992.

[39] R. R. Yager. On the measure of fuzziness and negation, part I: Membership in the unit interval. *International Journal of General Systems*, 5:221–229, 1979.

[40] L. A. Zadeh. Concept of a linguistic variable and its application to approximate reasoning—I. *Information Sciences*, 8:199–249, 1975.

[41] L. A. Zadeh. Fuzzy logic and approximate reasoning. *Synthese*, 30:407–428, 1975.