Logical Foundations of Fuzzy Mathematics

Logické základy fuzzy matematiky

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The form of the dissertation

The dissertation consists of 10 papers on logic-based fuzzy mathematics published in peer-reviewed international journals [2, 8, 10, 6, 7, 13], and peer-reviewed proceedings of international conferences [4, 5, 15, 14]. By the time of the submission of this thesis, the papers have been cited 26 times in peer-reviewed international journals and 15 times in edited volumes and proceedings of international conferences (excluding auto-citations and citations by co-authors). Two co-authored conference papers related to the dissertation won the Best Paper [9] and Distinguished Student Paper [14] awards (respectively at the 11th IFSA World Congress and the 5th Conference of EUSFLAT).

The papers are accompanied with a cover study (Part I of the thesis, 50 pp.), which introduces the area of logic-based fuzzy mathematics, argues for the significance of the area of research, presents the state of the art, indicates the author's contribution to the field, and comments on the papers comprising the thesis.

The present text is an extract from the dissertation for the purpose of the defence. For basic definitions of the first-order fuzzy logics $\mathrm{MTL}_\triangle$ and ŁΠ, which are not repeated here, see [25, 21, 20] or the papers included in the thesis.

Cover study (Part I of the thesis)

Fuzzy mathematics can be characterized as the study of *fuzzy structures,* i.e., mathematical structures in which the two values 0, 1 are at some points replaced by a richer system of degrees. Under the *logic-based* approach, fuzzy structures are formalized by means of *axiomatic theories* over suitable systems of *fuzzy logic,* whose rules replace the rules of classical logic in formal derivation of theorems. The main advantages of the logic-based approach are the general gradedness of defined notions, methodological clarity provided by the axiomatic method, and the applicability of a foundational architecture mimicking that of classical mathematics. Logic-based fuzzy mathematics is part of a broader area of non-classical mathematics (i.e., mathematical disciplines axiomatizable in non-classical logics), as well as a specific subfield of general fuzzy methods. Following earlier isolated developments in logic-based fuzzy set theory and arithmetic, a systematic logic-based study of fuzzy mathematics was made possible by recent advances of first-order fuzzy logic that opened the way for Henkin-style higher-order fuzzy logic (or simple fuzzy type theory), which is capable of serving as a foundational theory for logic-based fuzzy mathematics. The author's contribution to the development of logic-based fuzzy mathematics has been presented in the published papers that comprise the main body of the thesis.

Papers comprising the thesis

L. Běhounek: On the difference between traditional and deductive fuzzy logic, *Fuzzy Sets and Systems,* 2008. [6]

The paper analyzes methodological differences between traditional and logic-based fuzzy mathematics. It points out that while traditional fuzzy mathematics regards various interpretations of membership degrees (e.g., epistemic, possibilistic, frequence-based, etc.), in formal fuzzy logics truth degrees represent the quality that is preserved under the (local) consequence relation; consequently they have to comply with certain principles (expressed, e.g., by the transitivity of implication), by which traditional fuzzy mathematics is not bound. Concepts of traditional fuzzy mathematics that do not conform to the methodological presumptions of formal fuzzy logic (including, for instance, Dubois and Prade's gradual elements or the entropy of fuzzy sets) are therefore not well-motivated from the point of view of logic-based fuzzy mathematics and fall outside its scope of interest. Logic-based fuzzy mathematics thus presents a very specific field which has rather little in common with other areas of fuzzy mathematics.

Besides this clarification of the methodology and delimitation of the scope of logic-based fuzzy mathematics within general fuzzy methods, the paper also defines a class of fuzzy logics suitable for logic-based fuzzy mathematics (termed *deductive* fuzzy logics) as the intersection of the class of substructural logics in Ono's sense [28] (i.e., logics of classes of residuated lattices) and Cintula's [18] weakly implicative fuzzy logics (i.e., logics of classes of linearly ordered logical matrices, cf. also [11]). Arguments are given that this class represents minimal requirements on logics complying with the methodological assumptions of logic-based fuzzy mathematics.

L. Běhounek, P. Cintula: From fuzzy logic to fuzzy mathematics: A methodological manifesto, *Fuzzy Sets and Systems,* 2006. [10]

The position paper proposes a three-layer architecture for logic-based fuzzy mathematics, parallel to that of modern foundations of classical mathematics, with the respective layers formed of (i) a suitable system of first-order fuzzy logic, (ii) a foundational theory axiomatized in the fuzzy logic, and (iii) particular mathematical disciplines formalized within the foundational theory. Several design choices are defended for the layer of formal fuzzy logic on the grounds of formalistic methodology and the axiomatic method, incl. the priority of formal theories over particular models, abstraction from particular truth degrees, plurality of fuzzy logics (as suitable for different semantic models), definitions of fuzzy logics as axiomatic systems (rather than non-axiomatizable sets of standard tautologies), and classical syntax (as opposed to labeled systems such as the evaluated syntax of [27]). Henkin-style higher-order fuzzy logic ŁΠ developed by the authors in [8] was suggested as the foundational theory for a particular implementation of these guidelines, and a systematic development of formal fuzzy mathematics within its framework was proposed in the paper.

L. Běhounek, P. Cintula: Fuzzy class theory, *Fuzzy Sets and Systems,* 2005. [8]

In the paper, Henkin-style higher-order fuzzy logic ŁΠ is introduced as an axiomatic approximation of the theory of Zadeh's fuzzy sets and fuzzy relations of arbitrary orders. First, a theory $\text{Ł}\Pi_2$ of fuzzy classes, or Henkin-style monadic second-order fuzzy logic ŁΠ,[1] is defined as follows:

Definition 1. $\text{Ł}\Pi_2$ is a first-order theory over the multi-sorted logic ŁΠ, with the sorts of elements (lowercase variables $x, y, \dots$) and classes (uppercase variables $A, B, \dots$). Primitive symbols are the membership predicate $\in$ between elements and classes (where $x \in A$ can be shorthanded as Ax) and the (crisp) identity predicate $=$ for each sort. Besides the identity axioms of reflexivity $\xi = \xi$ and Leibniz's principle $\xi = \zeta \to \Delta(\varphi(\xi) \leftrightarrow \varphi(\zeta))$ for both sorts, the theory is axiomatized by the axiom of extensionality and the comprehension scheme:

$$(\forall x)\,\Delta(x \in X \leftrightarrow x \in Y) \to X = Y$$
$$(\exists X)(\forall x)\,\Delta(x \in X \leftrightarrow \varphi(x)), \quad X \text{ not free in } \varphi.$$

(Eliminable) comprehension terms $\{x \mid \varphi(x)\}$ can be conservatively introduced, with the axioms $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$. The intended models of $\text{Ł}\Pi_2$ are constructed by interpreting element variables as ranging over a fixed crisp universe U, class variables ranging over the system L^U of all functions from U to a linear ŁΠ-algebra L, and realizing $=$ as crisp identity and $\in$ as functional application.

Elementary theory of fuzzy classes is developed and shown to be largely reducible to propositional logic ŁΠ, by the following metadefinitions and metatheorems:

[1] The logic ŁΠ of [21] was chosen for its expressive power, as it contains definable connectives for all usual arithmetical operations as well as a large class of t-norm connectives. However, since connectives pertaining to different left-continuous t-norms are seldom used at one time, the logic MTL_Δ of [20] was more often used instead of ŁΠ in later developments of fuzzy mathematics (as argued in [6], MTL_Δ is the weakest logic suitable for such purposes).

Definition 2. For a propositional formula $\varphi(p_1, \ldots, p_n)$ define the n-ary fuzzy class operation induced by φ as

$$\mathrm{Op}_{\varphi(p_1,\ldots,p_n)}(X_1, \ldots, X_n) =_{\mathrm{df}} \{x \mid \varphi(x \in X_1, \ldots, x \in X_n)\}.$$

The definition covers usual class operations, including the empty class $\emptyset = \mathrm{Op}_0$, the universal class $\mathrm{V} = \mathrm{Op}_1$, the kernel $\mathrm{Ker}(X) = \mathrm{Op}_{\Delta p}(X)$ the α-cuts $X_\alpha = \mathrm{Op}_{\Delta(\alpha \to p)}(X)$, the $*$-complement $-_* X = \mathrm{Op}_{\neg_* p}(X)$, the $*$-intersection $X \cap_* Y = \mathrm{Op}_{p \&_* q}(X, Y)$, the $*$-union $X \cup_* Y = \mathrm{Op}_{p \veebar_* q}(X, Y)$, etc., for any ŁΠ-representable left-continuous t-norm $*$.

Definition 3. For a propositional formula $\varphi(p_1, \ldots, p_n)$ define the following relations between $X_1, \ldots, X_n$:

$$\mathrm{Rel}^{\forall}_{\varphi(p_1,\ldots,p_n)}(X_1, \ldots, X_n) \equiv_{\mathrm{df}} (\forall x)\varphi(x \in X_1, \ldots, x \in X_n)$$
$$\mathrm{Rel}^{\exists}_{\varphi(p_1,\ldots,p_n)}(X_1, \ldots, X_n) \equiv_{\mathrm{df}} (\exists x)\varphi(x \in X_1, \ldots, x \in X_n).$$

The definition covers fuzzy equalities $X \approx_* Y \equiv \mathrm{Rel}^{\forall}_{p \leftrightarrow_* q}(X, Y)$, fuzzy inclusions $X \subseteq_* Y \equiv \mathrm{Rel}^{\forall}_{p \to_* q}(X, Y)$, fuzzy compatibility relations $X \parallel_* Y \equiv \mathrm{Rel}^{\exists}_{p \&_* q}(X, Y)$, the unary properties $\mathrm{Hgt}(X) \equiv \mathrm{Rel}^{\exists}_{p}(X)$ of height, $\mathrm{Rel}^{\forall}_{p}$ of plinth, $\mathrm{Rel}^{\exists}_{\Delta p}$ of normality, $\mathrm{Rel}^{\forall}_{\Delta(p \vee \neg p)}$ of crispness, etc.

Theorem 4. *For propositional formulae* $\varphi, \psi_1, \ldots, \psi_n$,

$$\begin{aligned} \text{ŁΠ} \vdash \varphi(\psi_1, \ldots, \psi_n) \quad &\textit{iff} \quad \text{ŁΠ}_2 \vdash \mathrm{Rel}^{\forall}_{\varphi}(\mathrm{Op}_{\psi_1}(X_{1,1}, \ldots, X_{1,k_1}), \ldots, \mathrm{Op}_{\psi_n}(X_{n,1}, \ldots, X_{n,k_n})) \\ &\textit{iff} \quad \text{ŁΠ}_2 \vdash \mathrm{Rel}^{\exists}_{\varphi}(\mathrm{Op}_{\psi_1}(X_{1,1}, \ldots, X_{1,k_1}), \ldots, \mathrm{Op}_{\psi_n}(X_{n,1}, \ldots, X_{n,k_n})) \end{aligned}$$

Corollary 5. *For propositional formulae* φ, ψ,

$$\begin{aligned} &\textit{If} \quad \text{ŁΠ} \vdash \varphi \to \psi \quad \textit{then} \quad \text{ŁΠ}_2 \vdash \mathrm{Op}_{\varphi}(X_1, \ldots, X_n) \subseteq \mathrm{Op}_{\psi}(X_1, \ldots, X_n) \\ &\textit{If} \quad \text{ŁΠ} \vdash \varphi \leftrightarrow \psi \quad \textit{then} \quad \text{ŁΠ}_2 \vdash \mathrm{Op}_{\varphi}(X_1, \ldots, X_n) \approx \mathrm{Op}_{\psi}(X_1, \ldots, X_n) \\ &\textit{If} \quad \text{ŁΠ} \vdash \varphi \vee \neg\varphi \quad \textit{then} \quad \text{ŁΠ}_2 \vdash \mathrm{Crisp}(\mathrm{Op}_{\varphi}(X_1, \ldots, X_n)) \end{aligned}$$

Theorem 6. *For propositional formulae* $\varphi_i, \varphi'_i, \psi_{i,j}, \psi'_{i,j}$,

$$\text{ŁΠ} \vdash \mathop{\&_*}_{i=1}^{k} \varphi_i(\psi_{i,1}, \ldots, \psi_{i,n_i}) \to \bigwedge_{i=1}^{k'} \varphi'_i(\psi'_{i,1}, \ldots, \psi'_{i,n'_i}) \quad \textit{iff}$$

$$\textit{iff} \quad \text{ŁΠ}_2 \vdash \mathop{\&_*}_{i=1}^{k} \mathrm{Rel}^{\forall}_{\varphi_i} \left(\mathrm{Op}_{\psi_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi_{i,n_i}}(\vec{X}\,)\right) \to$$
$$\to \bigwedge_{i=1}^{k'} \mathrm{Rel}^{\forall}_{\varphi'_i} \left(\mathrm{Op}_{\psi'_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi'_{i,n'_i}}(\vec{X}\,)\right)$$

$$\textit{iff} \quad \text{ŁΠ}_2 \vdash \mathop{\&_*}_{i=1}^{k-1} \mathrm{Rel}^{\forall}_{\varphi_i} \left(\mathrm{Op}_{\psi_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi_{i,n_i}}(\vec{X}\,)\right)$$
$$\&_* \mathrm{Rel}^{\exists}_{\varphi_k} \left(\mathrm{Op}_{\psi_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi_{k,n_k}}(\vec{X}\,)\right) \to$$
$$\to \bigvee_{i=1}^{k'} \mathrm{Rel}^{\exists}_{\varphi'_i} \left(\mathrm{Op}_{\psi'_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi'_{i,n'_i}}(\vec{X}\,)\right)$$

(Meta)theorems 4 and 6 reduce a large class of elementary theorems on fuzzy classes to simple propositional calculations, as e.g., the provability in MTL_Δ of

$\Delta p \to p$	proves	$\mathrm{Ker}(X) \subseteq X$
$p \to p \vee q$	"	$X \subseteq X \cup_\wedge Y$
$0 \to p$	"	$\emptyset \subseteq X$
$(p \to q) \to (p \,\&\, r \to q \,\&\, r)$	"	$(X \subseteq Y) \to (X \cap_* Z \subseteq Y \cap_* Z)$
$(p \to r) \,\&\, (q \to r) \to (p \vee q \to r)$	"	$(X \subseteq_* Z) \,\&_*\, (Y \subseteq_* Z) \to (X \cup_\wedge Y \subseteq_* Z)$
$p \,\&\, (p \to q) \to q$	"	$\mathrm{Hgt}(X) \,\&_*\, (X \subseteq_* Y) \to \mathrm{Hgt}(Y)$,

etc., in $\text{Ł}\Pi_2$. This trivializes most theorems found in the first dozens of pages of typical textbooks on fuzzy set theory (many of them even in their stronger graded forms, as the relations $\mathrm{Rel}_\varphi^{\forall/\exists}$ need not be crisp, unlike their traditional definitions).

The extension of $\text{Ł}\Pi_2$ by the apparatus of tuples $\langle \xi_1, \ldots, \xi_k \rangle$ (or briefly $\xi_1 \ldots \xi_k$) for each sort, which equips it with the subsorts for k-tuples for each $k \in \omega$, the tuple-forming and component-extracting functions, and the usual axioms of tuple equality

$$\langle \xi_1, \ldots, \xi_k \rangle = \langle \zeta_1, \ldots, \zeta_k \rangle \to \xi_i = \zeta_i, \quad \text{for each } k \text{ and } i \leq k,$$

is sketched in the paper, which allows introducing usual fuzzy relational notions, including the operations of Cartesian product $X \times_* Y =_{\mathrm{df}} \{\langle x, y \rangle \mid x \in X \,\&_*\, y \in Y\}$, domain $\mathrm{Dom}(R) =_{\mathrm{df}} \{x \mid Rxy\}$, range, converse, etc., as well as graded properties of fuzzy relations, e.g., graded reflexivity $\mathrm{Refl}\, R \equiv_{\mathrm{df}} (\forall x) Rxx$, $*$-symmetry $\mathrm{Sym}_*\, R \equiv_{\mathrm{df}} (\forall x)(\forall y)(Rxy \to_* Ryx)$, etc.

The apparatus of $\text{Ł}\Pi_2$ is then extended to all higher orders by adding sorts of variables $X^{(n)}$ for classes of the n-th order and the membership predicates $\in^{(n)}$ between successive sorts, governed by the axioms of extensionality and comprehension of the same form as in $\text{Ł}\Pi_2$. Higher-order operations then become available, such as

fuzzy class $*$-union,	$\bigcup_* \mathcal{A} = \{x \mid (\exists A)(A \in \mathcal{A} \,\&_*\, x \in A)\}$;
fuzzy class $*$-intersection,	$\bigcap_* \mathcal{A} = \{x \mid (\forall A)(A \in \mathcal{A} \to_* x \in A)\}$;
fuzzy $*$-power class,	$\mathrm{Pow}_*\, A = \{X \mid X \subseteq_* A\}$; etc.

Since the axioms have the same form for all orders of variables, all definitions and theorems propagate to all higher orders as well. This construction yields Henkin-style logic ŁΠ of the n-th order, and the whole hierarchy for $n \in \omega$ yields a simple type theory over the logic ŁΠ (also called *Fuzzy Class Theory* or FCT). The theory can obviously be constructed over any deductive fuzzy logic and seems to be equivalent to Novák's Church-style fuzzy type theory FTT of [26] (over the logics where the latter is well-defined).

The theory was proposed as a foundational theory for logic-based fuzzy mathematics. The formalizability of usual concepts of traditional fuzzy set theory was demonstrated on Zadeh's extension principle, which is definable as a third-order class

$$\boldsymbol{Z}_* =_{\mathrm{df}} \{\langle \{R\}, \mathcal{S} \rangle \mid \mathcal{S} = \{\langle X, Y \rangle \mid (\exists x, y)(Rxy \,\&_*\, x \in X \,\&_*\, y \in Y)\}\}.$$

Furthermore, the representability in FCT of all structures of classical mathematics that can be rendered by classical first-order theories was shown, by considering a theory $\mathrm{FCT}(T)$ obtained by adding to FCT the axioms of the classical theory T and further axioms stating the crispness of all predicates of T. It was shown that each model of $\mathrm{FCT}(T)$ contains a crisp model of T, and each crisp model of T is contained in a model of $\mathrm{FCT}(T)$; consequently, $T \vdash \varphi$ in classical logic iff $\mathrm{FCT}(T) \vdash \varphi$ in first-order ŁΠ. All usual classical mathematical structures (including all structures needed for the formalization of traditional fuzzy mathematics) are thus available in the theory.

L. Běhounek, U. Bodenhofer, P. Cintula: Relations in Fuzzy Class Theory—Initial steps, *Fuzzy Sets and Systems,* 2008. [7]

The paper investigates *graded properties of fuzzy relations* (originally introduced and studied by Gottwald, [22, 23]) in FCT over the logic MTL_Δ. Since FCT is a formal theory over a many-valued (sc., fuzzy) logic, all notions defined in the theory are by default many-valued (or *graded*) as well. Graded properties defined in FCT generalize non-graded (crisp) properties studied by traditional fuzzy mathematics, as the former obtain meaningful non-zero truth values even if satisfied only imperfectly, in which case the latter are simply false. Traditional non-graded notions can usually be defined as the graded ones true to degree 1. Consequently, *graded theorems* on graded properties (of the form ψ *is at least as true as* φ) are stronger then traditional non-graded theorems (usually of the form *if* φ *is fully true then so is* ψ), as the latter are direct corollaries of the former (but not vice versa). The apparatus of FCT makes it easy to derive such graded theorems (formalized as $\varphi \to \psi$), which generalize traditional results (formalized as $\Delta\varphi \to \Delta\psi$). One of the aims of the paper was to illustrate this generalization by a number of examples—besides its primary goal to develop in FCT the basics of the formal theory of fuzzy relations, which is indispensable for all other areas of logic-based fuzzy mathematics.

First, Gottwald's results from [22, 23] on graded

$$\begin{aligned} &\text{reflexivity,} && \operatorname{Refl} R \equiv_{\mathrm{df}} (\forall x) Rxx, \\ &\text{symmetry,} && \operatorname{Sym} R \equiv_{\mathrm{df}} (\forall xy)(Rxy \to Ryx), \\ &\text{transitivity,} && \operatorname{Trans} R \equiv_{\mathrm{df}} (\forall xyz)(Rxy \mathbin{\&} Ryz \to Rxz), \text{ etc.,} \end{aligned}$$

have been reproduced in FCT (31 theorems, e.g., $\operatorname{Trans} R \& \operatorname{Trans} S \to \operatorname{Trans}(R \cap S)$ and the like).

Then, graded properties of the operations of image $R^{\uparrow} A =_{\mathrm{df}} \{y \mid (\exists x)(Ax \mathbin{\&} Rxy)\}$, dual image $R^{\downarrow} A =_{\mathrm{df}} \{x \mid (\forall y)(Rxy \to Ay)\}$, opening $R^{\uparrow}(R^{\downarrow} A)$, and closure $R^{\downarrow}(R^{\uparrow} A)$ have been investigated with 72 theorems proved in FCT, for instance the following property of openings:

$$R^{\uparrow}(R^{\downarrow} A) = \bigcup \{B \mid (B = R^{\uparrow}(R^{\downarrow} B)) \mathbin{\&} (B \subseteq A)\}.$$

Next, the graded concepts of the upper cone ${}^{\triangle}A =_{\mathrm{df}} \{x \mid (\forall a)(a \in A \to Rax)\}$ and the dual lower cone ${}^{\triangledown}A$, the fuzzy class $\operatorname{Max} A =_{\mathrm{df}} A \cap {}^{\triangle}A$ of the maxima and the dually defined class $\operatorname{Min} A$ of the minima, as well as the class $\operatorname{Sup} A =_{\mathrm{df}} \operatorname{Min} {}^{\triangle}A$ of the suprema and the dual class $\operatorname{Inf} A$ of the infima have been introduced and investigated: 28 theorems of FCT have been derived, including the lemmata needed for the MacNeille completion (such as ${}^{\triangle\triangledown\triangle}A = {}^{\triangle}A$, $\bigcap_{A \in \mathcal{A}} {}^{\triangle}A = {}^{\triangle}\big(\bigcup_{A \in \mathcal{A}} A\big)$, etc.), the $\subseteq$-monotony and $(R \cap R^{-1})$-uniqueness of the maxima and minima, or the interdefinability of the suprema and infima by $\operatorname{Sup} A = \operatorname{Inf} {}^{\triangle}A$.

Further, Valverde-style characterizations [29] of fuzzy preorders and similarity relations have been generalized to their graded versions. Fuzzy preorders, $\operatorname{Preord} R \equiv_{\mathrm{df}} \operatorname{Refl} R \mathbin{\&} \operatorname{Trans} R$, can be characterized by means of Fodor's left traces $R^{\ell} =_{\mathrm{df}} \{xy \mid (\forall z)(Rzx \to Rzy)\}$ in a graded manner as $\operatorname{Preord} R \leftrightarrow (R \approxeq R^{\ell})$, where $A \approxeq B$ is the bi-inclusion $(A \subseteq B) \mathbin{\&} (B \subseteq A)$. Estimates between the degrees of $\operatorname{Preord} R$ and the Valverde-representability $\operatorname{ValP}(R, \mathcal{A}) \equiv_{\mathrm{df}} R \approxeq \{xy \mid (\forall A \in \mathcal{A})(Ax \to Ay)\}$ have been given, namely

$$(\exists \mathcal{A})(\operatorname{ValP}^2(R, \mathcal{A}) \mathbin{\&} \mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) \longrightarrow \operatorname{Preord} R \longrightarrow (\exists \mathcal{A})(\operatorname{ValP}(R, \mathcal{A}) \mathbin{\&} \operatorname{Crisp} \mathcal{A}),$$

where φ^n abbreviates $\&_{i=1}^{n} \varphi$. Similar results have been obtained for similarity relations (i.e., symmetric fuzzy preorders, or fuzzy equivalences, $\operatorname{Sim} R \equiv_{\mathrm{df}} \operatorname{Refl} R \mathbin{\&} \operatorname{Sym} R \mathbin{\&} \operatorname{Trans} R$).

Finally, a graded version of the correspondence between fuzzy similarities and fuzzy T-partitions of De Baets and Mesiar [19] has been given. In particular, if the graded predicate $\operatorname{Part} R$ of being a T-partition is defined by conjunction of the conditions of crispness, covering, disjointness, and the normality of its elements,

$$\operatorname{Crisp} \mathcal{A} \;\;\&\;\; (\forall x)(\exists A \in \mathcal{A})\Delta Ax \;\;\&\;\; (\forall A, B \in \mathcal{A})(A \parallel B \to A \approx B) \;\;\&\;\; (\forall A \in \mathcal{A})(\exists x)\Delta Ax,$$

and the fuzzy quotient class (or the class of equivalence classes) and the fuzzy relation generated by a system of (equivalence) classes are respectively defined as

$$\mathrm{V}/R =_{\mathrm{df}} \{A \mid (\exists x)(A = \{y \mid Rxy\})\}, \quad R^{\mathcal{A}} =_{\mathrm{df}} \{xy \mid (\exists A \in \mathcal{A})(Ax \,\&\, Ay)\},$$

then the following graded correspondence theorems (and their variants) are provable in FCT:

$$\mathrm{Part}\,\mathcal{A} \to \mathrm{Sim}\,R^{\mathcal{A}}, \qquad \Delta\,\mathrm{Sim}\,R \to \Delta\,\mathrm{Part}\,\mathrm{V}/R, \qquad \mathrm{Sim}\,R \to R^{\mathrm{V}/R} \approxeq R$$
$$\mathrm{Part}\,\mathcal{A} \to (\forall A \in \mathcal{A})(\exists B \in \mathrm{V}/R^{\mathcal{A}})(A \approxeq B) \wedge (\forall B \in \mathrm{V}/R^{\mathcal{A}})(\exists A \in \mathcal{A})(A \approxeq B)$$

L. Běhounek, M. Daňková: Relational compositions in Fuzzy Class Theory, *Fuzzy Sets and Systems,* 2009. [13]

The paper presents a uniform treatment of a large family of fuzzy relational and set-theoretic notions, based on their reduction to fuzzy relational sup-T and inf-R compositions, and a method for mass proofs of certain kinds of theorems on these notions. It formally elaborates the basic observation (already made by Bělohlávek in [16]) that the defining formulae (here, in FCT over MTL_Δ) of, e.g., the image or preimage of a fuzzy set under a fuzzy relations have similar forms to that of fuzzy relational (sup-T) composition [30]:

$$R^{\leftarrow}A = \{x \mid (\exists z)(Rxz \,\&\, Az)\}$$
$$S^{\rightarrow}A = \{y \mid (\exists z)(Az \,\&\, Szy)\}$$
$$R \circ S = \{xy \mid (\exists z)(Rxz \,\&\, Szy)\}$$

The only difference being the absence of a variable, the former definitions can be made instances of the latter by substitution of a dummy constant $\underline{0}$ for the missing variable. Formally, this corresponds to representing a fuzzy set A by a binary fuzzy relation $\boldsymbol{R}_A = A \times \{\underline{0}\}$; then $\boldsymbol{R}_{R^{\leftarrow}A} = R \circ \boldsymbol{R}_A$ or (identifying $\boldsymbol{R}_X$ with X), $R^{\leftarrow}A = R \circ A$. Similarly, $R^{\rightarrow}A = R^{\mathrm{T}} \circ A$, $A \times B = A \circ B^{\mathrm{T}}$, $\mathrm{Dom}\,R = R \circ \mathrm{V}$, and $\mathrm{Rng}\,R = R^{\mathrm{T}} \circ \mathrm{V}$. where $R^{\mathrm{T}} = \{xy \mid Ryx\}$ is the fuzzy relation converse to R.

Further notions are obtained by considering a relational representation of truth degrees. The paper shows that modulo certain metamathematical provisos, the semantic truth degrees can be internalized in FCT as elements of the crisp class $\mathrm{L} =_{\mathrm{df}} \mathrm{Ker}\,\mathrm{Pow}\{\underline{0}\}$, i.e., the kernel of the power class of a crisp singleton. Then the truth value of a formula φ is represented by the class $\{\underline{0} \mid \varphi\} \in \mathrm{L}$, and each $\alpha \in \mathrm{L}$ represents a truth value (e.g., of $\underline{0} \in \alpha$). Connectives and quantifiers are then represented by class operations on L (e.g., & by $\cap$, $\exists$ by $\bigcup$, etc.). Using the relational representation $\boldsymbol{R}_\alpha = \alpha \times \{\underline{0}\}$ for $\alpha \in \mathrm{L}$, further notions are expressible by means of $\circ$, incl. compatibility $A^{\mathrm{T}} \circ B$, conjunction $\alpha \circ \beta$, or height $A^{\mathrm{T}} \circ \mathrm{V}$.

A similar representation can be used for inf-R composition of fuzzy relations (also known as Bandler and Kohout's BK-product, [1]), $R \triangleleft S =_{\mathrm{df}} \{xy \mid (\forall z)(Rxz \to Szy)\}$. The family of notions expressible as inf-R composition then includes implication $\alpha \triangleleft \beta$, inclusion $A^{\mathrm{T}} \triangleleft B$, and several further notions known from the literature under varying names and often implicitly used in fuzzy applications (e.g., the BK-analog of preimage $R^{\hookleftarrow\!\!\triangleleft}A = R \triangleleft A$, i.e., the dual image $R^{\downarrow}A$ of [7], see p. 5).

One of the merits of the present approach is a systematization of the family of sup-T and inf-R representable notions, and introducing a uniform terminology. The identifications furthermore ensure automatic translation of certain kinds of graded theorems on fuzzy relational compositions (incl. $\subseteq$-monotony or $\bigcup$- and $\bigcap$-distribution) to all notions of the family. Moreover, a few simple identities of $\circ$ and $\triangleleft$, such as the associativity $R \circ (S \circ T) = (R \circ S) \circ T$, transposition $(R \circ S)^{\mathrm{T}} = S^{\mathrm{T}} \circ R^{\mathrm{T}}$, residuation $R \triangleleft (S \triangleleft T) = (R \circ S) \triangleleft T$, double transposition $R^{\mathrm{TT}} = R$, and a few more, provide a simple equational calculus for proving identities between expressions composed of the operations from the sup-T and inf-R families, thus yielding an infinite number of easy corollaries to the few properties of $\circ$ and $\triangleleft$. This nearly trivializes an important part of the theory of binary fuzzy relations in FCT. A few examples of the many corollaries listed in the paper follow:

$$
\begin{array}{rcl}
(R_1 \subseteq R_2) \rightarrow (R_1 {}^{\rightarrow} A \subseteq R_2 {}^{\rightarrow} A) & \text{by} & (R_1 \subseteq R_2) \rightarrow (R_1 \circ S \subseteq R_2 \circ S) \\
\bigcap_{R \in \mathcal{A}} (R {}^{\leftarrow\!\lhd} A) = \left(\bigcup_{R \in \mathcal{A}} R\right) {}^{\leftarrow\!\lhd} A & \text{''} & \bigcap_{R \in \mathcal{A}} (R \lhd S) = \left(\bigcup_{R \in \mathcal{A}} R\right) \lhd S \\
\bigwedge_{\alpha \in \mathcal{A}} (\alpha \rightarrow \beta) = \left(\bigvee_{\alpha \in \mathcal{A}} \alpha\right) \rightarrow \beta & \text{''} & \text{''} \\
R {}^{\rightarrow} \operatorname{Rng} S = \operatorname{Rng}(S \circ R) & \text{''} & R^{\mathrm{T}} \circ S^{\mathrm{T}} \circ \mathrm{V} = (S \circ R)^{\mathrm{T}} \circ \mathrm{V} \\
R {}^{\leftarrow\!\lhd} (S {}^{\leftarrow\!\lhd} A) = (R \circ S) {}^{\leftarrow\!\lhd} A & \text{''} & R \lhd (S \lhd A) = (R \circ S) \lhd A, \text{ etc.}
\end{array}
$$

L. Běhounek: Extensionality in graded properties of fuzzy relations, *Proceedings of IPMU,* 2006. [5]

In this conference paper, new definitions of graded properties of fuzzy relations are proposed that are relative to a given indistinguishability relation E:

$$
\begin{array}{rcl}
\operatorname{Refl}_E R & \equiv_{\mathrm{df}} & (\forall x x')(E x x' \rightarrow R x x') \\
\operatorname{Sym}_E R & \equiv_{\mathrm{df}} & (\forall x x' y y')(E x x' \mathbin{\&} E y y' \mathbin{\&} R x y \rightarrow R y' x') \\
\operatorname{Trans}_E R & \equiv_{\mathrm{df}} & (\forall x x' y y' z z')(E x x' \mathbin{\&} E y y' \mathbin{\&} E z z' \mathbin{\&} R x y \mathbin{\&} R y' z \rightarrow R x' z')
\end{array}
$$

and similarly for E-antisymmetry and E-functionality. The definitions are motivated by elimination of the undesirable crispness of identity hidden in multiple references to the same variable in the usual definitions of Refl, Sym, etc. It is shown in the paper that in the non-graded setting, the E-properties reduce to the usual (identity-based) properties under the presence of the (similarly motivated) well-known property of extensionality of R w.r.t. E,

$$\operatorname{Ext}_E R \equiv_{\mathrm{df}} (\forall x x' y y')(E x x' \mathbin{\&} E y y' \mathbin{\&} R x y \rightarrow R x' y'),$$

on condition that E is reflexive and symmetric, as FCT over MTL_Δ proves

$$\Delta \operatorname{Refl} E \mathbin{\&} \Delta \operatorname{Sym} E \mathbin{\&} \Delta \operatorname{Ext}_E R \rightarrow (\operatorname{Trans}_E R \leftrightarrow \operatorname{Trans} R)$$

(and similarly for reflexivity, symmetry, antisymmetry, and functionality). However, in the graded setting with the antecedents not required to be 1-true, this simple reduction no longer works, as in the case of transitivity we only get

$$\operatorname{Refl}^3 E \mathbin{\&} \operatorname{Sym}^3 E \mathbin{\&} \operatorname{Ext}_E^2 R \rightarrow (\operatorname{Trans}_E R \leftrightarrow \operatorname{Trans} R).$$

A counterexample shows that the double use of $\operatorname{Ext}_E R$ cannot be eliminated. Thus even though the property of extensionality has the same motivation as our E-properties, it ensures them straightforwardly only if non-graded properties of fuzzy relations are considered. The non-graded reducibility also explains why the E-properties have not been studied in the literature, except E-reflexivity used in Bodenhofer's similarity-based orderings [17], where it was however understood as the conjunction of reflexivity and extensionality due to the presence of transitivity, as

$$\Delta \operatorname{Refl} E \mathbin{\&} \Delta \operatorname{Sym} E \mathbin{\&} \Delta \operatorname{Trans} R \rightarrow (\Delta \operatorname{Refl}_E R \leftrightarrow \Delta \operatorname{Refl} R \mathbin{\&} \Delta \operatorname{Ext}_E R).$$

Further variants of indistinguishability-based properties are hinted at, e.g.,

$$\operatorname{Refl}_E^{\exists} R \equiv_{\mathrm{df}} (\forall x)(\exists x')(E x x' \mathbin{\&} R x x').$$

They can be justified game-theoretically by whether it is ‘us’ or ‘Nature’ who decides on the identity of indistinguishable elements.

L. Běhounek: Towards a formal theory of fuzzy Dedekind reals, *Proceedings of EUSFLAT,* 2005. [4]

In this conference paper, fuzzy real numbers are introduced in the framework of FCT as fuzzy Dedekind cuts over the crisp domain $(Q, \leq)$ of rational (or equally well, real) numbers. Fuzzy Dedekind cuts on $(Q, \leq)$ are defined as left-closed upper sets, i.e., sets $A \subseteq^{\Delta} Q$ such that

$$(\forall p, q \in Q)[(p \leq q) \to (p \in A \to q \in A)]$$
$$(\forall p \in Q)[(\forall q \in Q)(p > q \to q \in A) \to p \in A]$$

The class $\mathcal{R}$ of all such cuts lattice-completes Q w.r.t. ordering of cuts by $\supseteq$ (as FCT proves $\mathcal{A} \subseteq \mathcal{R} \to \bigcap \mathcal{A} \in \mathcal{R}$), and Q is embedded in $\mathcal{R}$ as the set of crisp cuts with the least elements $q \in Q$ (denoted $\overline{q} \in \mathcal{R}$). The provability of $q \in A \leftrightarrow A \supseteq \overline{q}$ confirms the motivation of cuts as distributions in Q of the 'fuzzy elements' A from $\mathcal{R}$.

Fuzzy intervals can then be introduced as fuzzy sets $[A, B]_* = A \cap_* B$, where A is an upper cut $\{q \mid A \supseteq \overline{q}\}$ and B is a lower cut $\{q \mid \overline{q} \supseteq B\}$. Their kernels $[A^{\leftarrow}, B^{\rightarrow}]$ where $A^{\leftarrow} = \inf\{\overline{q} \mid \Delta A q\}$ and $B^{\rightarrow} = \sup\{\overline{q} \mid \Delta A q\}$ are preserved by all arithmetical operations that extend the crisp ones. *Fuzzy numbers* $\widetilde{r}$ can be defined as (equivalence classes of) such fuzzy intervals that have $A^{\leftarrow} = B^{\rightarrow} = r$; this definition joins the competing traditional views of fuzzy numbers as uni-normal convex fuzzy sets and Dedekind cuts. Several observations on the properties and arithmetic of fuzzy Dedekind cuts and fuzzy intervals are made (a more comprehensive study of lattice completions by fuzzy Dedekind cuts and fuzzy MacNeille stable sets is given in [3], not included in the dissertation).

L. Běhounek: Fuzzification of Groenendijk–Stokhof propositional erotetic logic, *Logique et Analyse,* 2004. [2]

In the paper, a Groenendijk–Stokhof–style semantics for fuzzy yes–no questions is given in the framework of FCT. Classical Groenendijk and Stokhof's semantics [24], also known as the partition semantics of questions, identifies questions with partitions of a logical space (i.e., a set of possible worlds), where the blocks of the partition are the UCLA-propositions (i.e., subsets of the logical space) representing the direct answers to the question. Since the theory of fuzzy partitions in FCT was developed only later in [7], just yes–no questions (where the partition can be reduced to the subset representing the affirmative answer) were considered; while the classical Groenendijk–Stokhof logic of yes–no questions is trivial, for fuzzy question it is less so.

The prerequisite intensional semantics of propositional fuzzy logic was defined by interpreting atomic formulae p_i by class variables $\|p_i\| = A_i$ and complex formulae $\varphi(p_1, \ldots, p_n)$ by operations $\|\varphi(p_1, \ldots, p_n)\| = \mathrm{Op}_\varphi(\|p_1\|, \ldots, \|p_n\|)$. The soundness and completeness theorem ($L \vdash \varphi$ iff $\mathrm{FCT} \vdash W \subseteq \|\varphi\|$, for FCT over the logic L) and several observations on the semantics (e.g., the graded transitivity of the entailment defined as $\varphi \models \psi \equiv_{\mathrm{df}} W \cap \|\varphi\| \subseteq \|\psi\|$) were made in the paper.

Two interpretations of interrogative formulae $?\varphi$ were considered: (i) "What is the truth value of φ?", reflected in the following definition of answerhood

$$\psi \models_{\mathrm{t}} ?\varphi \equiv_{\mathrm{df}} (\forall w, w' \in W)[\Delta(w \in \|\psi\| \leftrightarrow w' \in \|\psi\|) \to \Delta(w \in \|\varphi\| \leftrightarrow w' \in \|\varphi\|)]$$

(or its fuzzified variants obtained by omitting the Δ's), and (ii) "Is it the case that φ?", which leads to the following definition of answerhood:

$$\varphi \models ?\psi \equiv_{\mathrm{df}} (\varphi \models \psi) \vee (\varphi \models \neg\psi).$$

Entailment between questions can in both cases be defined analogously to the classical theory, viz $?\varphi \models_{(\mathrm{t})} \psi \equiv_{\mathrm{df}} (\forall A)[(A \models ?\varphi) \to (A \models ?\psi)]$. Several observations (such as the graded transitivity of entailment between fuzzy questions) were made for both senses (i) and (ii), though it was argued in the paper that the latter is better suited to the treatment in FCT, as the former violates principles advocated in the manifesto [10].

L. Běhounek, T. Kroupa: (i) Topology in Fuzzy Class Theory: Basic notions, *Proceedings of IFSA,* 2007. (ii) Interior-based topology in Fuzzy Class Theory, *Proceedings of EUSFLAT,* 2007. [14, 15]

In these two conference papers, several graded notions of fuzzy topology were defined and their mutual relationships studied in the framework of FCT over MTL_Δ. Fuzzy topology based on open fuzzy sets was defined by the fuzzy predicate

$$\mathrm{OTop}^{e,v,i,u}_{m,n}(\tau) \equiv_{\mathrm{df}} (\emptyset \in \tau)^e \ \&\ (\mathrm{V} \in \tau)^v \ \&\ [(\forall A, B \in \tau)(A \cap B \in \tau)]^i \ \&\ [(\forall \nu \subseteq^m \tau)(\bigcup \bigcap_1^n \nu \in \tau)]^u$$

(the multiplicities of the conjuncts are a regular feature of graded definitions in FCT, see [12]; by convention they can be omitted if all equal to 1). Interiors and neighborhoods in open fuzzy topology can then be defined as follows:

$$\mathrm{Int}_\tau(A) =_{\mathrm{df}} \bigcup\{B \in \tau \mid B \subseteq A\}, \qquad \mathrm{Nb}_\tau(x, A) \equiv_{\mathrm{df}} (\exists B \in \tau)(x \in B \subseteq A).$$

Several observations on these notions are made in the paper, including the theorem

$$\mathrm{Crisp}(\mathcal{X}) \ \&\ (\forall \tau \in \mathcal{X})\left(\Delta\, \mathrm{OTop}(\tau)\right) \rightarrow \Delta\, \mathrm{OTop}\left(\bigcap \mathcal{X}\right)$$

which makes it possible to define an open fuzzy topology by specifying its subbase σ,

$$\tau_\sigma = \bigcap\{\tau' \mid \Delta(\mathrm{OTop}(\tau') \ \&\ \sigma \subseteq \tau')\};$$

the construction yields, e.g., an open fuzzy topology generated by the intervals of [4] (see p. 8).

In a similar manner, gradual notions of fuzzy topology represented as a system of neighborhoods or as a Kuratowski interior operator on fuzzy sets were defined as certain fuzzy predicates $\mathrm{NTop}^{i,j,k,l}(\mathrm{Nb})$ and $\mathrm{ITop}^{p,q,r,s,t}(\mathrm{Int})$. Open classes, neighborhoods, and interiors were then defined under each representation (e.g., $\tau_{\mathrm{Int}} =_{\mathrm{df}} \{A \mid A \subseteq \mathrm{Int}(A)\}$), and observations similar to those on open fuzzy topologies made. Particular examples include a neighborhood-based interval fuzzy topology (which differs from the open-based one), and the interior operators taking the kernel, plinth, or the relational opening studied in [7]. Mutual relationships between the variant gradual notions of fuzzy topology have been established; e.g., for OTop and ITop we obtain:

$$\mathrm{ITop}^{1,1,1,1,2}(\mathrm{Int}) \rightarrow \mathrm{OTop}_{2,1}(\tau_{\mathrm{Int}}) \ \&\ (\forall A)(\mathrm{Int}(A) \approxeq \mathrm{Int}_{\tau_{\mathrm{Int}}}(A))$$
$$\mathrm{OTop}^{0,1,1,1}(\tau) \rightarrow \mathrm{ITop}(\mathrm{Int}_\tau) \ \&\ (\forall A)(A \in \tau \leftrightarrow A \subseteq \mathrm{Int}_\tau(A)).$$

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