

LOGICAL FOUNDATIONS OF FUZZY MATHEMATICS

LOGICKÉ ZÁKLADY FUZZY MATEMATIKY

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Prohlašuji, že jsem disertační práci vypracoval samostatně s využitím uvedených pramenů a literatury.

I hereby declare that this dissertation is the result of my own work and that all sources have been duly acknowledged.

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Part I

Thesis description

Logical foundations of fuzzy mathematics

Preface

This is a commentary associated with the author's PhD thesis in logic at the Faculty of Arts, Charles University in Prague. The thesis is based on papers containing results of five years of the author's research in logic-based fuzzy mathematics. The papers have been published in peer-reviewed international journals [14, 30, 34, 26, 28, 41], proceedings of international conferences [33, 31, 32, 16, 17, 19, 22, 18, 37, 43, 42, 44, 29] and other volumes [15, 21, 25, 35]. By the time of the submission of this thesis and according to the author's knowledge, the papers have been cited 25 times in peer-reviewed international journals and 15 times in edited volumes and proceedings of international conferences, excluding auto-citations and citations by co-authors. Two co-authored conference papers won the Best Paper [32] and Distinguished Student Paper [42] awards (respectively at the 11th IFSA World Congress and the 5th Conference of EUSFLAT).

The work is part of a larger project in formal fuzzy mathematics, which is still in progress (cf. the end of Section 4.1 below): several important topics in formal fuzzy mathematics are being investigated by my colleagues and myself, with results not yet complete for publication. Therefore it seemed more appropriate to present the results of this research in the form of a commented collection of papers, rather than to compile a monographic text, as at the time of submission the topic was still under permanent construction and re-construction and not yet ripe for a book-style presentation.

Due to the brevity or purely expository nature of some of the conference papers and the overlap of their topics with full journal articles, only the six journal papers [34, 30, 28, 41, 26, 14] and four of the proceedings papers [16, 19, 43, 42] have actually been included in the thesis. The author's contribution to co-authored papers is indicated in Section 4.2.

The thesis is organized as follows: In the cover text (Part I), Section 1 provides a general introduction to the area of research. A broader context and the state of the art upon which the thesis is based is described in Section 2. The main features of the approach developed in the thesis and the significance of the topic are discussed in Section 3. The author's own contribution to the topic and the papers included in the thesis are then described in Section 4. The author versions of the published papers constitute the main body of the thesis (Part II). The thesis is concluded by mandatory annexes (Part III).

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(but not limited to) his advice, support, and friendship: by adopting me to his working group at the Institute of Computer Science, he has enabled me to participate in the scientific life of the relevant community; and nearly all of my knowledge of fuzzy logic has its ultimate roots in his lectures and tutoring.

I thank all my co-authors for fruitful co-operation. In alphabetical order they are Ulrich Bodenhofer, Petr Cintula, Martina Daňková, Rostislav Horčík, and Tomáš Kroupa. Without them, this thesis would either be much sparser, or would have to deal with different topics. I am grateful to people who supported my research by including me in their grant teams, supporting my fellowship applications, or providing funding for my participation at conferences: Petr Hájek, Jiří Šíma, Jeff Paris, Mirko Navara, Petr Jirků, Franco Montagna, and Andrzej Wiśniewski.

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I have also benefited from discussions with many colleagues and kind advice given to me by senior researchers in the field. Among those whose remarks helped me a lot are (in alphabetical order and besides those listed above) Christian Fermüller, David Makinson, Jeff Paris, and several anonymous referees. Thanks are due to Petr Cintula, Petr Hájek, Tomáš Kroupa, and Carles Noguera for comments on a draft version of the cover chapter. Many other people have helped me during the course of my PhD study with scientific and organizational matters; I appreciate all help I have been given.

1 Introduction

Fuzzy mathematics is the study of fuzzy structures, or structures that involve *fuzziness*—i.e., such mathematical structures that at some points replace the two classical truth values 0 and 1 with a larger structure of *degrees*. Often, the real unit interval $[0, 1]$ is employed as the system of degrees, but other options are common as well—a finite set, an arbitrary lattice, an algebra of some kind or other. The degrees are intended to provide more flexibility to a fuzzy mathematical structure than the two truth degrees provide to the corresponding classical (“crisp”) mathematical structure.

A simple example of a fuzzy mathematical structure is that of a *fuzzy set*. Instead of classical two-valued characteristic functions $\chi: X \rightarrow \{0, 1\}$, fuzzy sets employ real-valued membership functions $\mu: X \rightarrow [0, 1]$, where X is a fixed universe of discourse. While ordinary crisp sets clearly cut the elements of X between members and non-members, the richer system of degrees in fuzzy sets allows modeling *gradual change* between membership and non-membership.

Since the introduction of fuzzy sets by Zadeh [212] in 1965, a plethora of fuzzy mathematical structures have been proposed and investigated in the literature. The degrees that replace the classical truth values 0 and 1 (usually called *membership degrees*, as they serve

as values of membership functions) can appear at various places in such fuzzy structures. For instance in *fuzzy topology*, crisp families of open fuzzy sets (“fuzzy topologies”), fuzzy families of open crisp sets (“fuzzifying topologies”), and fuzzy families of open fuzzy sets (“bifuzzy topologies”) have all been investigated [55, 208, 130]. The membership degrees themselves can form various structures. For example, the original notion of $[0, 1]$ -valued fuzzy set was soon generalized to L -valued fuzzy sets for L an arbitrary lattice [95], and even more general (e.g., poset-valued) fuzzy sets and fuzzy structures are studied [184].

The large freedom in defining fuzzy mathematical notions correlates with freedom in interpreting the informal meaning of membership degrees. Depending on the intended interpretation, various structures of membership degrees and various definitions of fuzzy mathematical notions are appropriate. Vice versa, particular structures of membership degrees and particular definitions of fuzzy mathematical notions admit only some of all possible informal interpretations and applications of the fuzzified theory. Unfortunately, this fact is seldom reflected in the practice of the fuzzy community. The omission of such considerations can result in arbitrariness of definitions, inappropriateness of applications, and completely unclear methodology, for all of which fuzzy mathematics has often (and in many cases quite justly) been reproached and disrespected by the mainstream mathematical community.

The present work is not intended to contribute to the chaotic and methodologically confused development of the broad area of fuzzy mathematics. Instead, from the mixture of possible interpretations of membership degrees it selects one particular interpretation which has already been clarified enough to support a methodologically sound development: namely, the interpretation of membership degrees as degrees of *comparative truth*, which is studied by deductive fuzzy logic. Our approach to fuzzy mathematics can thus be characterized as *logic-based*.¹ More detailed methodological considerations justifying this approach have been presented in [26]; here we only stress the most important points.

Deductive (or *formal*, *symbolic*, *mathematical*) fuzzy logic follows the *modus operandi* of classical logic. Without necessarily claiming that the philosophical notion of truth as such is (or is not) many-valued, it employs semantical models that assign intermediary truth degrees to propositions. In deductive fuzzy logic, like in fuzzy mathematics in general, a richer structure of truth degrees enables to model gradual change between truth and falsity, which seems appropriate in many real-life situations. The interpretation of membership degrees in terms of truth, moreover, allows studying transmission of truth degrees in formalized arguments, in the same way as classical logic studies transmission of bivalent truth.

The study of transmission of partial truth (in the technical sense of “partial truth” as the graded quality preserved in sound arguments) is what in fact distinguishes *deductive* fuzzy logic from *traditional* fuzzy logic. Fuzzy logic in the traditional sense has emerged soon after the introduction of fuzzy sets [96], by generalizing the obvious correspondence between elementary set operations and logical connectives. If the transmission of partial truth is not taken into account, there are as many ways to define fuzzy logical connectives as there are possibilities for pointwise elementary fuzzy set operations. This makes traditional fuzzy logic subject to the same criticisms as fuzzy mathematics as a whole, especially for arbitrariness and unclear methodology. Moreover for most choices of logical

¹Other approaches to fuzzy mathematics exist, some of them quite far developed—for example category-theoretical (see [106]) or sheaf-theoretical (see [129]). As far as they can address the methodological issues hinted at above, they provide a legitimate grounding for those branches of fuzzy mathematics that are compatible with their methodological assumptions. The logic-based approach then complements rather than rivals such approaches.

connectives, the resulting logical systems have very poor logical properties, which leads to additional criticism from the point of view of formal logic.

It can nevertheless be shown [26, 22] that if certain basic principles governing the transmission of partial truth are observed, the resulting logical systems are well-behaved and well-motivated. These principles narrow down the choice of real-valued logical connectives to a class based on left-continuous t-norms and described by deductive systems of t-norm based fuzzy logics [110, 81]. These logics not only model the gradual change from truth to falsity like other kinds of fuzzy logic, but also have a “deductive face” and belong in a well-explored and well-behaved class of substructural logics [173, p. 208].

Since Hájek’s monograph [110], various propositional and first-order systems of deductive fuzzy logic have been defined and intensively studied. Nowadays the discipline is developed to the point that it is reasonable to construct and study axiomatic mathematical theories within the formal framework of deductive fuzzy logic. A systematic development of axiomatic fuzzy mathematics based on deductive fuzzy logic has been proposed as a research program in [34]; the present work can be viewed as a report on its implementation.

The strategy proposed in [34] was to utilize the similarities between deductive fuzzy logics and classical logic and employ the architecture that has proved useful in foundations of classical mathematics. The classical foundational approach consists in developing a sufficiently rich foundational theory that would harbor all (or almost all) other mathematical theories. In classical mathematics, the role of a foundational theory can be assumed, e.g., by some variant of set theory, type theory, or category theory. For the foundations of logic-based fuzzy mathematics, [34] proposes a fuzzy variant of Russell-style simple type theory that has been introduced in [30]. It can equivalently be characterized as *Henkin-style higher-order fuzzy logic* or a typed theory of cumulative fuzzy classes (i.e., Zadeh’s fuzzy sets of all finite orders). The apparatus of this foundational theory, also called Fuzzy Class Theory or FCT, is described in detail in [30, 31, 32, 35]; its methodological issues are further discussed in [26, 37]. The basics of the theory of fuzzy sets and relations, which are the prerequisites of all other branches of fuzzy mathematics, are developed in [30, 28, 41, 19]. Some more advanced topics of fuzzy mathematics have already been developed, too, including the (graded) theory of fuzzy lattices [17, 15], fuzzy intervals [16, 134], aggregation operators [64, 29], fuzzy filters [149], and fuzzy topology [43, 42, 44]. In [14, 38, 25], the apparatus is applied in metamathematics of fuzzified versions of other non-classical logics.

The results achieved so far have already demonstrated that this style of development of fuzzy mathematics is viable and can facilitate generalizing known theorems as well as discovering new results. Indeed, from the point of view of formal logic the methodology and foundational structure of the theory is quite standard and straightforward. On the other hand, from the point of view of traditional fuzzy mathematics the theory presents a radical shift of paradigm, embraced till now by very few authors (for notable exceptions see Section 2). The main reasons justifying the development of logic-based fuzzy mathematics are described in Section 3 below.

There are many open questions and areas for future research in formal fuzzy mathematics, as well as problems of philosophical and methodological nature. Even though these problems are still distant from the applied practice or topics of mainstream interest, their solution can give us better understanding of the phenomenon of gradedness and its role, as well as possible applications.

For the adequate perspective on the present work with respect to the whole of fuzzy mathematics, it is necessary to keep in mind the methodological restrictions of the logic-

based approach. The definite interpretation of membership degrees as degrees of truth transmitted under inference leads on the one hand to methodological clarity, but on the other hand it restricts meaningful definitions to those compatible with the deductive paradigm, and limits the scope of applicability of the results. For instance, as mentioned above and as shown in more detail in [26], the principles of deductive fuzzy logic restrict the choice of the conjunction connective on the interval $[0, 1]$ to left-continuous t-norms. Consequently, the operation of fuzzy set intersection can meaningfully be defined only by means of such conjunctions.² Other possible notions of intersection that may be meaningful in broader fuzzy mathematics, for instance those based on aggregation operators different from left-continuous t-norms, may well be definable in a sufficiently strong higher-order fuzzy logic (e.g., higher-order logic $L\Pi$), but are *ill-motivated* from the point of view of logic-based fuzzy mathematics. Thus, as argued in [26], even though the expressive power of higher-order fuzzy logic goes well beyond its intended scope, the strength of its apparatus is best manifested within the limits of its motivation. Logic-based fuzzy mathematics thus forms a specific, distinct part of fuzzy mathematics, which is based on the notion of deduction and which should not be confused with other areas of fuzzy mathematics that are based on different interpretations of membership degrees, such as degrees of uncertainty, belief, frequency, preference, etc. (cf., e.g., [90, 74, 73] for different interpretations of degrees and the concluding part of [26] for the need of their clear separation).

Another connection that should be clarified is that to the philosophy of vagueness. On the one hand, fuzzy logic is often claimed to be the logic of vague propositions, or the logic of vagueness. On the other hand, it is as often criticized by philosophers as a completely misled and inadequate theory of vagueness. Although this introduction is not a suitable place to discuss this issue in detail, it should be stressed that both claims are inaccurate and need certain qualifications. To be sure, fuzzy logic cannot claim to be *the* logic of vagueness, as vagueness is a phenomenon with many facets, most of which are not captured by deductive fuzzy logic (e.g., are not truth-functional). If anything, deductive fuzzy logic can claim to be *a* logic of a *certain kind* of vagueness, related to properties that can be understood as coming in degrees. Moreover, deductive fuzzy logic is only a *logic*, rather than a fully fledged *theory* of vagueness meeting all requirements of the philosophy of vagueness (including answers to questions not asked by logic, for instance about the objectivity of the truth degrees etc.). Still, it can be argued that deductive fuzzy logic *is* a good model of inference under (certain kinds of) vagueness and as such can serve as a *logical basis* for a (prospective) theory of vagueness, or at least can help shed light on some of its facets. The sweeping damnation of fuzzy logic by many philosophers of vagueness is therefore unjustified and is for the most part caused by the ignorance of recent advances in fuzzy logic.³

Finally, fuzzy mathematics is sometimes criticized by mainstream mathematicians as

²At least as long as we understand intersection as the operation expressing the fact that an element belongs to the first *and* the second fuzzy set, i.e., require that one can infer *both* $x \in A$ and $x \in B$ from $x \in A \cap B$ and vice versa.

³For instance, many criticisms are caused by an inappropriate use of weak conjunction instead of strong conjunction in Łukasiewicz fuzzy logic (which is by far the most popular fuzzy logic among philosophers of vagueness), cf., e.g., [205, §4] or [78, §3]. Bad logical properties of *some* systems of fuzzy logic which are defective from the deductive point of view (e.g., Zadeh's original system of connectives \min , \max , and $1-x$) induce many philosophers (e.g., [201]) to condemn fuzzy logic as a whole, without considering better options offered by present-day mathematical fuzzy logic. Further problems arise from misunderstanding the role of fuzzy logic and expecting it to be applicable to situations that are beyond its scope (e.g., related to probability, levels of belief, etc.).

giving nothing but cheap generalizations of classical results. One of the aims of the present work is to show that *indeed* large parts of fuzzy mathematics are trivial, and demonstrate their triviality by deriving them from easily provable metatheorems (e.g., [30, Th. 33–36] or theorems in [41]). This shows that unlike more traditional approaches, the deductive apparatus of higher-order fuzzy logic enables clearly to perceive the triviality of such results. At the same time it provides means for reaching less trivial theorems (cf., e.g., [28, §6–7]) and possibly for achieving higher levels of fuzzy mathematics. It can be hoped that this direction will eventually contribute to gaining a better reputation for fuzzy mathematics among mainstream mathematicians.

2 State of the art

The enterprise of logic-based fuzzy mathematics is not isolated from other areas of mathematics and logic. It is based on *formal fuzzy logic* and its metamathematics, and can be regarded as its higher-order extension. At the same time it can be regarded as a formalization, reconstruction, and further development of certain parts of *traditional fuzzy mathematics*. In a broader context it is part of *non-classical mathematics*, i.e., mathematics that uses a non-classical logic for reasoning. This section gives an overview of previous results upon which logic-based fuzzy mathematics in general and the author’s contribution in particular have built, as well as main results in related areas. However, due to the breadth of the field, this section cannot give an exhaustive survey or full historical account of all important works published in this area. Works which are most relevant to particular topics of this thesis are referred to in the articles it consists of; only a brief description of the state of the discipline at the time of the current project is given here, with a focus on works relevant for formal fuzzy mathematics.

2.1 Non-classical mathematics

Non-classical mathematics can be defined as the development of mathematical theories that employ some non-classical logic for informal reasoning or formal derivations. The area of non-classical mathematics comprises several independent branches, according to the kind of underlying logic used for mathematical reasoning. Each of these branches can further be divided into many particular theories over particular logics of the respective kind.

An example of non-classical mathematics is *paraconsistent mathematics* based on some variant of paraconsistent logic. The common feature of paraconsistent logics is that contradictions are in general not explosive (i.e., A and non- A do not in general entail an arbitrary B). This fact can be used, e.g., for the development of mathematical analysis based on the (contradictory) notion of infinitesimals [165]. Another application of paraconsistent mathematics is in naive set theory with full comprehension, where Russell’s paradox is not destructive thanks to paraconsistency (e.g., [47]).

Avoiding Russell’s paradox is one of the most important motivations for non-classical mathematics. Besides paraconsistency, there are several alternative ways in which Russell’s paradox can be eliminated by employing a non-classical logic. One option is based on the observation that the structural rule of contraction (see, e.g., [182, 175, 173, 174]) is essential for the derivation of contradiction from the definition of Russell’s set. It has indeed been proved that in various contraction-free substructural logics, set theory with the unrestricted axiom scheme of comprehension is consistent, and some of such theories

have indeed been developed—e.g., over variants of linear logic [195, 187].⁴ In fuzzy logics (which belong among contraction-free logics, cf. [173]), the consistency of unrestricted comprehension over Łukasiewicz logic was conjectured in 1957 by Skolem ([190], according to [204]). Skolem’s partial results [191] were later extended by Chang [53] and Fensstad [84], and the conjecture finally confirmed in 1979 by White [204]. The theory has recently been investigated by Hájek [115] and Yatabe [207]. It is still an open question whether the theory or some extension thereof is sufficiently strong to support non-trivial mathematics (as conjectured by Skolem): Hájek’s paper [115] contains some negative results on this question regarding arithmetic.⁵ Although certainly worth investigating, this style of fuzzy mathematics is very different from traditional fuzzy mathematics. Logic-based fuzzy mathematics presented in this thesis avoids Russell’s paradox by different means (namely in the style of type theory) and is only remotely related to fuzzy set theories with unrestricted comprehension.⁶

Another way to avoid Russell’s paradox by means of non-classical logic has been proposed by Krajíček in [146, 147], namely by adding (epistemically interpretable) modalities to the language of set theory. The resulting theory is an example of *modal mathematics*, which in general can employ various kind of modalities serving various purposes. Modal mathematics related to the phenomenon of vagueness (and thus remotely to fuzzy logic) is proposed in [136].

Probably the most influential of all branches of non-classical mathematics is *intuitionistic mathematics*; related to the latter is *constructive mathematics* which usually uses some variant of intuitionistic reasoning, plus or minus some principles considered (non)constructive. The informal development of intuitionistic mathematics by Brouwer and his followers (cf. its formalization [143]) and constructive mathematics by constructivists can be considered the first non-classical mathematics ever developed. The later development of formal theories over intuitionistic logic (e.g., [77, 183]) is of special importance for logic-based fuzzy mathematics, since deductive fuzzy logics can be characterized as prelinear contraction-free intuitionistic logics;⁷ informally speaking, fuzzy logics show in general intuitionistic features (especially in the behavior of quantifiers and negation).

The most important parts of intuitionistic mathematics for the foundations of fuzzy mathematics are set theories over intuitionistic logic. Since they are directly connected with the development of set theories over fuzzy logic, they will be described together with the latter in Section 2.4.

Especially strong links exist between mathematics over intuitionistic logic and that over Gödel fuzzy logic (for which see [76, 135, 110, 4]), as Gödel logic extends intuitionistic

⁴Related theories with unrestricted comprehension were originally studied by Grishin over the logic known as (classical) logic without contraction, Grishin’s logic, or Ono’s CFL_{ew} (classical full Lambek calculus with exchange and weakening, see [173]). A description and further elaboration of Grishin’s work [108] can be found, e.g., in [51].

⁵Further negative results [181, 121] regard the related question (which classically is a variation of Russell’s paradox or the Liar) whether a truth predicate can be added to arithmetic over Łukasiewicz logic.

⁶It should be noted that for the consistency of unrestricted comprehension, a necessary condition on the underlying logic is that no bivalent connective be definable. Consequently, fuzzy set theories with full comprehension cannot be based on Gödel or product logics (as they have bivalent negation), nor any fuzzy logic with the Baaz Δ connective. Many important fuzzy logics are therefore excluded from this style of fuzzy mathematics. Fuzzy set theory with unrestricted comprehension is thus a very specific theory rather than a universal formalization of traditional fuzzy mathematics.

⁷More precisely, the weakest deductive fuzzy logic MTL, which is arguably [26] the weakest fuzzy logic suitable for formal fuzzy mathematics, arises by adding the axiom of prelinearity to the intuitionistic calculus LJ without the rule of contraction.

logic just by Dummett’s axiom of prelinearity $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$ and the first-order axiom of constant domains $(\forall x)(\varphi \vee \psi(x)) \rightarrow (\varphi \vee (\forall x)\psi(x))$. Fuzzy mathematics over Gödel logic is thus stronger (i.e., closer to classical mathematics) than intuitionistic mathematics and most results in intuitionistic mathematics are readily transferrable to Gödel fuzzy mathematics. Since the connectives of Gödel logic are available in all extensions of the fuzzy logic MTL_Δ , the results in Gödel fuzzy mathematics have also some relevance in general fuzzy mathematics.

Kripke semantics for predicate Gödel logics, characterized in [10] as countable linear Kripke frames for intuitionistic logic with constant domains, with a possible modal interpretation of epistemic states of the idealized mathematician (or Brouwer’s creating subject, cf. [199]) provides an additional link between fuzzy, modal, and intuitionistic mathematics. The Kripke semantics can be extended to non-contractive first-order fuzzy logics [163, 164] along the lines of [174] (i.e., equipping the Kripke frame with a monoidal operation, or a ternary accessibility relation). Kripke semantics can provide another possible link, beside that based on the algebraic semantics of (linear) residuated lattices, of logic-based fuzzy mathematics to other substructural (e.g., relevant [159, 88]) mathematical theories; this option has not yet been investigated, though. Since furthermore intuitionistic logic is the inner logic of topoi (see, e.g., [98]), intuitionistic mathematics may also provide a link between the logic-based and category-theoretic or sheaf-theoretic approaches to fuzzy mathematics [106, 129]. This link, however, has not yet been investigated, either.

2.2 Formal fuzzy logic

Logic-based fuzzy mathematics could not be developed without previous sufficient advancement of formal fuzzy logic. The requisite advances in formal fuzzy logic were achieved only in the past decade,⁸ even though there were some (rather isolated) predecessors to this development.

Logics now regarded as belonging to the family of fuzzy logics were defined and studied from about 1920 on by several logicians, including Łukasiewicz [154], Wajsberg [200],⁹ Gödel [94], Dummett [76], Hay [125], Belluce and Chang [45], Horn [135], and others. Fuzzy logic related to Zadeh’s idea of a fuzzy set first occurred in Goguen’s 1969 paper [96], motivated by the obvious correspondence between elementary fuzzy set operations and logical operations on truth degrees. In subsequent years, however, the term “fuzzy logic” was used either in a very broad sense (cf. the distinction between fuzzy logic in broad and narrow sense made by Zadeh in [214]), or only in reference to the semantical truth tables defining some (often rather arbitrarily chosen) operations on truth degrees.

The first formal calculus specifically devised for fuzzy logic, later proved to be equivalent to Łukasiewicz fuzzy logic with real truth constants [110, 122], was given in 1979 by Pavelka [177]. This line of research, further pursued and extended to first-order logic by Novák [167, 166], studies the so-called *fuzzy logic with evaluated syntax*—a specific kind of labeled-deduction calculus for fuzzy logic that enjoys the so-called Pavelka-style completeness (i.e., the correspondence between syntactic provability degrees of formulae and their semantic truth degrees). The logical foundations of fuzzy mathematics presented here are, however, based on systems of fuzzy logic with traditional logical syntax rather

⁸This explains why the logic-based approach to fuzzy mathematics started to be systematically investigated only a few years ago and almost forty years after fuzzy mathematics itself.

⁹The historical papers by Łukasiewicz and Wajsberg are cited according to [110] and [172], respectively.

than evaluated syntax, and utilize its similarity to classical Boolean logic and classical foundations of mathematics.¹⁰

The best known fuzzy logics with traditional syntax are those that use left-continuous t-norms as the truth functions of conjunction and their residua as the truth functions of implication (see, e.g., [144] for the theory of t-norms). Members of this family of *t-norm fuzzy logics* have systematically been studied since Hájek’s milestone 1998 monograph [110], in which the ‘basic’ fuzzy logic BL of all continuous t-norms and its most important extensions, both propositional and first-order, were described in detail. Since then, a plenitude of formal systems of t-norm logics have been defined and their basic metamathematical properties (incl. general and standard completeness of their axiomatic systems, arithmetical or computational complexity, functional representation, etc.) investigated.

Of these systems, the most important for our present investigation are the logic MTL (or its variant MTL_{Δ}) of all left-continuous t-norms [81] and the logic LII (or its variant $LII_{\frac{1}{2}}$) joining the three basic t-norms [83, 59]. As argued in [26], MTL is the weakest fuzzy logic suitable for the deductive style of fuzzy mathematics, thus providing the largest generality within certain reasonable constraints. The logic $LII_{\frac{1}{2}}$, on the other hand, is the most expressive system among best-known fuzzy logics that still possesses very good metamathematical properties: a deduction theorem [59], introduction and elimination of Skolem functions [60, 30], etc. Besides other features, its standard semantics contains all basic arithmetical operations on truth degrees; thus it is a good approximation of the needs of traditional fuzzy logic. Moreover, a broad class of propositional t-norm logics is interpretable in LII; thus it can serve as a common framework for integration of fuzzy mathematics over more specialized fuzzy logics.¹¹ Nevertheless, various modifications of these logics can be useful for more specific purposes within the project (for example, the involutiveness of negation was needed in [43]; therefore, $IMTL_{\Delta}$ was employed as the ground logic). Higher-order logic and formal fuzzy mathematics can be based on any t-norm fuzzy logic, and all of them may be useful for this purpose in specific situations.

General algebraic semantics of well-behaved propositional t-norm logics consists of suitable quasivarieties of residuated lattices (possibly enriched with additional operators). Consequently, t-norm fuzzy logics belong to the family of *substructural logics*, as the latter can be identified with logics of (classes of) residuated lattices [173]. Both the theory of residuated lattices [139, 89] and substructural logics [175, 182] thus provide a broader background for the more specific study of t-norm fuzzy logics. In particular, t-norm logics fall within *contraction-free* substructural logics [174], since their local¹² consequence relation in general fails to satisfy the structural law of contraction (or the idempotence of

¹⁰One of the reasons for the choice of traditional rather than evaluated syntax is the necessary condition for the Pavelka-style completeness that implication be continuous, which limits fuzzy logic with evaluated syntax to variants of Łukasiewicz logic.

¹¹These were the reasons why LII was chosen as the ground logic of the foundational Fuzzy Class Theory in the original paper [30], while later most of the more particular disciplines of logic-based fuzzy mathematics have for the sake of generality been developed in Fuzzy Class Theory over the logic MTL_{Δ} (as combinations of connectives pertaining to different t-norms turned out to be used only rarely).

¹²Like in modal logics or other logics with partially ordered truth values, local and global consequence can be distinguished in t-norm fuzzy logics [117, 26]. Even though the global consequence relation (which transmits the full truth of fuzzy propositions) is more commonly studied in formal fuzzy logic, it is the *local* consequence relation between partially true premises and a partially true conclusion which is more important for formal fuzzy mathematics, as it allows deriving graded results with imperfectly true premises. In the practice of formal fuzzy mathematics, we derive theorems of the form $\varphi_1 \& \dots \& \varphi_n \rightarrow \psi$ by the rules of global consequence, which is axiomatized by the usual systems of fuzzy logic; the latter form internalizes precisely the local consequence between the premises $\varphi_1, \dots, \varphi_n$ and the conclusion ψ .

conjunction), while the laws of exchange and weakening do hold in t-norm fuzzy logics. Related systems that lack some of the latter structural laws, e.g., Metcalfe’s uninorm logic UL which drops weakening or logics with non-commutative conjunction like pseudo-BL or the flea logic, are studied as well [157, 150, 71, 114]. With appropriate changes, higher-order logic and formal fuzzy mathematics can be developed over these related systems, too.

As contraction-free substructural logics with exchange and weakening, t-norm fuzzy logics extend Ono’s logic FL_{ew} (full Lambek calculus with exchange and weakening, see, e.g., [173]), also known as affine multiplicative additive intuitionistic linear logic [157], Höhle’s monoidal logic [127] or intuitionistic logic without contraction [1], i.e., the logic of commutative bounded integral residuated lattices. The distinctive feature of t-norm fuzzy logics among contraction-free logics is the validity of the axiom of prelinearity $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$. In [36] we argued that prelinearity (in a more general form) can be regarded as a characteristic feature of the class of *fuzzy logics* (i.e., not just t-norm based) among Cintula’s weakly implicative logics [62]. A general theory of weakly implicative fuzzy logics as the logics of classes of linearly ordered logical matrices is also described in [62].

The conditions of weakly implicative fuzzy logics, however, only ensure suitable properties of implication as the principal connective in formulae true to degree 1. For the deductive style of fuzzy mathematics aimed at graded theorems that transmit partial truth, further conditions are needed that ensure that implication and conjunction respectively internalize the local consequence relation and cumulation of premises. The resulting class of *deductive fuzzy logics* [26] can be characterized as the intersection of the classes of Cintula’s weakly implicative fuzzy logics and Ono’s substructural logics (optionally with exchange and weakening, which will be assumed further on), or as the class of fragments or expansions of MTL (or Metcalfe’s [156] uninorm logic UL when working without weakening) with all connectives congruent w.r.t. bi-implication. Deductive fuzzy logics (which include all common t-norm logics) are the intended background logics for logic-based fuzzy mathematics as studied in the present project.

Propositional fuzzy logic is of course insufficient for the development of formal fuzzy mathematics, which besides fuzzy logical connectives also needs some means for (preferably fuzzy) quantification over its individuals. Some first-order systems of particular fuzzy logics were developed already during the pre-fuzzy and early fuzzy era [125, 45, 194]. A systematic treatment of first-order variants of t-norm based fuzzy logics has started with Hájek’s book [110]. Those first-order fuzzy logics that are most important for logic-based fuzzy mathematics are described in [110, 59, 81]; a comprehensive survey of first-order t-norm fuzzy logics is [118]. The basics of model theory for t-norm fuzzy logics have been developed in [119] and [118, §6]. Metamathematical properties of first-order fuzzy logics relevant to the important question of their completeness w.r.t. the standard real-valued semantics can be found in [161, 113, 116, 163].¹³ Initial steps toward a general theory of first-order weakly implicative fuzzy logics are given in [60], which largely conforms with Rasiowa’s general approach to first-order implicative logics [180]. Higher-order systems of fuzzy logics and axiomatic theories over first-order fuzzy logics will be mentioned in Section 2.4.

The quantifiers in all of the first-order systems studied in the papers mentioned above are the *lattice* quantifiers, corresponding to lattice conjunction and disjunction.¹⁴ This

¹³Of the most important first-order fuzzy logics, the standard completeness holds only for MTL and Gödel logic. The standard incompleteness of other logics is usually proved by showing that the arithmetical complexity of the set of their standard real-valued tautologies is larger than Σ_1 .

¹⁴Recall that in contraction-free substructural logics there are two different meaningful conjunctions and

is fully sufficient for the needs of formal fuzzy mathematics, since the first-order systems with just the lattice quantifiers turn out to be strong enough to enable the construction of Henkin-style higher-order logic, in which various classes of strong (multiplicative) quantifiers become definable [39, 65, 64].

First-order fuzzy logics containing a strong quantifier as a logical symbol have been studied by Montagna in [160]; cf. also another strong quantifier introduced for the real-valued semantics by Thiele [196, 197] (as described in [160]). However, these quantifiers do not coincide with the *weakest* quantifier Π that allows the inference $(\Pi x)\varphi(x) \rightarrow \&\mathcal{X}_{t \in M} \varphi(t)$ for any finite multiset M of terms.¹⁵ According to our knowledge, the latter ‘inferentially optimal’ multiplicative quantifier has so far only been sketched in our abstract [65].

Besides the special case of multiplicative quantifiers, also a general notion of quantifier is of importance for formal fuzzy mathematics. Generalized quantifiers formally studied in a logic-based setting by Novák [168, 171] and Holčapek [132] are motivated mainly by modeling natural language (cf. generalized quantifiers in classical logic [203, 178] and linguistically-motivated fuzzy quantifiers in traditional fuzzy mathematics [215, 93]) or applications in fuzzy control. From the point of view of formal fuzzy mathematics, crisp generalized quantifiers such as *for infinitely many x* , fuzzy counting quantifiers like *for many x* , or quantifiers relativized to a fuzzy mathematical condition like *for all large numbers x* are of the greatest interest; however, these have apparently not yet been systematically studied in the framework of formal fuzzy logic, although some of the linguistic-oriented approaches mentioned above are undoubtedly applicable in this area as well. Such quantifiers are nevertheless implicit in many constructions of formal fuzzy mathematics: for example, Bandler and Kohout’s “local properties” of fuzzy relations [8, 9] are instances of fuzzily relativized quantifiers. The initial studies on fuzzy quantifiers in formal fuzzy logic mentioned above mostly employ higher-order systems (Novák’s fuzzy type theory [169] or our higher-order fuzzy logic [30]), since fuzzy quantifiers can be regarded as fuzzy sets of fuzzy sets. Apart from strong quantifiers mentioned above, the possibility of having generalized fuzzy quantifiers as primitives in the logical language has probably not been considered in formal fuzzy logic yet; for the development of formal fuzzy mathematics they are not indispensable, as they can be introduced internally in higher-order systems that are based on lattice quantifiers.

Perhaps even more important for formal fuzzy mathematics than various kinds of strong and generalized quantifiers is the related notion of *exponentials*. Exponentials (in our sense of [65]) can be seen as propositional counterparts of truth-functional strong quantifiers.¹⁶ They are motivated by similar considerations as Girard’s exponentials for linear logic [91], which are used generally in substructural logics (see, e.g., [175]). Exponentials studied in fuzzy logic so far include Montagna’s storage operator of [160], which corresponds to his strong quantifier mentioned above, and Baaz’s operator Δ , introduced

disjunctions. One of the pair of connectives is in the literature variably called *weak*, *lattice*, *comparative*, *extensional*, or *additive* and the other one *strong*, *group*, *parallel*, *intensional* or *multiplicative*. For the difference between them see, e.g., [175]. The distinction can be extended to quantifiers, see [176].

¹⁵A multiset, since the strong conjunction $\&$ is not contractive (idempotent); therefore multiple occurrences of the same term have to be taken into account. Thiele’s quantifier only ensures the property for a *set* of terms, while Montagna’s quantifier additionally ensures the idempotence of $(\Pi x)\varphi(x)$ w.r.t. conjunction; the latter quantifier therefore coincides with the optimal one in extensions of BL, but not generally in extensions of MTL, as was already proved in [160], even though the optimal quantifier was only implicit there.

¹⁶They can be defined from the latter by dummy quantification, i.e., $\varphi^* \equiv_{\text{df}} (\Pi x)\varphi$ if x is not free in φ . As propositional modifiers they are special *hedges* in the terminology of [151] (followed in [112]), or *modalities* in that of [58].

in [2] for Gödel logic, transferred to the most important fuzzy logics in [110, 81], and generalized for all weakly implicative fuzzy logics in [62]. A proof-theoretical investigation of logics expanded with certain classes of exponentials is given in [58].

The importance of exponentials for formal fuzzy mathematics stems from their role as lower estimates for the truth values of propositions $\varphi, \varphi^2, \varphi^3, \dots$ (where φ^n is the n -tuple strong conjunction of φ). Formulae of the form φ^n occur frequently in formal fuzzy mathematics due to the general non-idempotence of strong conjunction; exponentials $*$ such that $\varphi^* \rightarrow \varphi^n$ for any n then provide a common bound on the strength of φ^n for all n . Although Baaz Δ can always be taken for such an estimate, in many cases it is too strong (e.g., if there is an idempotent w.r.t. $\&$ below the truth value of φ). Montagna’s storage operator provides a better alternative, but is still unnecessarily strong in some cases (cf. footnote 15). The inferentially optimal exponential φ^ω , related to the inferentially optimal multiplicative quantifier mentioned above, has by now only been sketched in [65].¹⁷

The general state of the art of formal fuzzy logic can be characterized as follows: Chagrov [52] distinguishes three stages in the development of a new area of non-classical logic. In the first stage, the concepts and logics of the area emerge without a clear methodology or well-developed metamathematics. In the second stage, when the methodology and metamathematics has become available, the area is systematically explored: often, many new logics are defined and their properties studied by advanced techniques. The third stage then offers a synthesizing view on the area, when common properties of whole classes of logics are obtained by generalized methods, and unifying insights are achieved by mature understanding of the area. The three phases need not be sharply separated and may chronologically overlap. As observed by Chagrov, this account, though abstracted from the particular development of modal logic, can be applied to the history of most disciplines of non-classical logic.

In formal fuzzy logic, these three stages can be found as well. The first phase commenced with the early study of Łukasiewicz and Gödel–Dummett logics in the 1920–60’s, and continued by the informal development and applications of fuzzy logic since the 1970’s. The second phase was announced by the first works on formal fuzzy logic since the late 1970’s, especially those by Pavelka [177], Novák [166], and Gottwald [102]. The heyday of the second stage came after Hájek’s 1998 monograph [110], when an explosion of new systems of fuzzy logic and their systematic metamathematical study has begun. Now we find ourselves in the maturity of the second stage and the beginning of the third, as the exploration of the fuzzy-logical landscape is far advanced (though new logics still do emerge—recently, e.g., uninorm [156, 157] and weakly cancellative [162] logics) and the properties of known fuzzy logics have already been deeply investigated (including their arithmetical [113, 116] and computational [3, 123, 124] complexity, expansions by various kinds of connectives [82, 160, 80, 58], standard completeness theorems [138, 79, 133], proof theory [57, 158], etc.). One of the first works that clearly belongs to the third stage is Cintula’s [62], in which a unified metamathematical treatment of all weakly implicative fuzzy logics is given. This framework was further generalized to weakly implicational (fuzzy) logics in [66]; a narrower class of (Δ -)core fuzzy logics [119] was further studied

¹⁷As a primitive symbol of propositional logic, the exponential ω is axiomatizable by a straightforward infinitary rule. It can moreover be approximated by a finitary axiomatization such that the finitarily axiomatized exponential coincides with the optimal one if the latter does exist on the algebra of truth values (which in general need not be the case). The exponential ω is of course definable in higher-order fuzzy logic, though only with the qualification that the Henkin-style axiomatization of higher-order fuzzy logic admits its non-intended models (it is nevertheless the optimal *internal* exponential in the theory).

in [63]. The second stage, however, cannot be considered completed, as is apparent for instance from the as yet insufficient investigation of exponentials in fuzzy logic.

An indispensable precondition for the development of logic-based fuzzy mathematics was to achieve at least an advanced phase of Chagrov's second stage in formal fuzzy logic. In particular, it was the extensive exploration of the logical landscape in the fuzzy area that enabled finding the most suitable systems of fuzzy logic that could support formal fuzzy mathematics (esp. the logics MTL_{Δ} , $L\Pi$, and t-norm fuzzy logics in general), allowed applying their basic metamathematical properties in the development of the formalism and helped clarify the area of their applicability. The emergence of logic-based fuzzy mathematics indeed coincides with this stage of development of formal fuzzy logic. The above considerations can partly explain why it had not appeared earlier during the four decades of the existence of traditional fuzzy mathematics.

2.3 Traditional fuzzy mathematics

As fuzzy mathematics has been developed by many researchers for more than forty years, it is impossible to present an overview of all its developments in this brief survey. Therefore we shall only deal with those areas of fuzzy mathematics to which the papers included in this thesis are related, namely the theory of fuzzy sets and fuzzy relations, fuzzy topology, and fuzzy numbers. The developments in other areas of traditional fuzzy mathematics are described, e.g., in the surveys [73, 142]. A compendium of application-oriented traditional fuzzy mathematics is, e.g., [145]. For each of the relevant disciplines of fuzzy mathematics, only the works that initiated the research and recent representative books or surveys of the area will be mentioned here. Approaches that are close to logic-based fuzzy mathematics, where they exist, will also be noticed. Further details can be found in the introductions and references to the papers included in this thesis, and in the literature cited in the surveys.

The theory of *fuzzy sets* (and fuzzy mathematics as the whole) is usually considered to have started with Zadeh's 1965 paper [212], which introduced the concept of fuzzy set (and coined the term *fuzzy*), identifying fuzzy sets with membership functions from a crisp ground set to $[0, 1]$. There have, nevertheless, been several predecessors who proposed similar or identical concepts, most notably Max Black [48], Abraham Kaplan and Hermann Schott (see [73, §1.2.4]), Karl Menger (ibid.), and Dieter Klaua (see [105]). In 1967, the notion of fuzzy set was generalized to lattice-valued membership functions by Goguen [95]; since then, various structures of membership degrees have been considered.

Graded properties of fuzzy sets have been considered mainly in the setting related to or based on formal fuzzy logic (esp. by Bandler and Kohout [7] and Gottwald [99]). For axiomatic theories of fuzzy sets based on formal systems of fuzzy logic, in which graded properties of fuzzy sets appear quite naturally, see Section 2.4.

Important monographs with chapters on fuzzy sets include [166, 145, 104]. An overview of basic notions in the theory of fuzzy sets is given, e.g., in [73]. Besides the direct representation by means of membership functions, various alternative foundations for the notion of fuzzy set have been considered in the literature: category-theoretical approaches to fuzzy sets are surveyed in [131, 106], and categories of fuzzy sets are treated in detail in [206] and [172, Ch. 7]. A sheaf-theoretic foundation of fuzzy sets is described in [129]. Axiomatic theories of fuzzy sets based on formal fuzzy logic are described in more detail in Section 2.4 below.

The notion of *fuzzy relation* was defined already in Zadeh's first paper on fuzzy sets [212]. It was generalized to lattice-valued relations in Goguen's 1967 paper [95],

in which several important fuzzy-relational concepts (including, e.g., sup-product compositions) were studied. Many important concepts, incl. fuzzy similarity and fuzzy ordering, were introduced in Zadeh's 1973 paper [213]. Important contributions to the theory of fuzzy relations were made by Bandler and Kohout, esp. regarding generalized relational products [6]. Further references to the vast literature on fuzzy relations can be found in the papers on fuzzy relations included in this thesis [28, 41].

Graded properties of fuzzy relations, which are fundamental in logic-based theory of fuzzy relations (cf. [28]), were first proposed by Gottwald in [101]. They are systematically studied in Gottwald's monographs [102] and [104, §18.6] and Bělohlávek's book [46]. The graded approach has been applied by Gottwald to the solvability of fuzzy relational equations in [103]. Several graded notions of fuzzy function have been studied by Demirci [70].

The discipline of *fuzzy topology* was established in the 1960's and 1970's in papers by Chang [55], Goguen [97], Lowen [152], and others.¹⁸ It has been given a considerable attention throughout the history of fuzzy mathematics and elaborated by a number of researchers. Several approaches to fuzzy topology have been developed: besides those based on membership functions and fuzzy sets, the most prominent are the (point-free) lattice-theoretical and categorial treatments. Various definitions of fuzzy topology were surveyed by Höhle and Šostak in [130]. Many results are also surveyed in the (more recent, but somewhat self-promoting) historical overview [142, §6]. A detailed exposition based on the categorial viewpoint is given in Höhle's monograph [128].

An early example of logic-based fuzzy topology is Ying's investigation of fuzzifying and bifuzzy topologies in the early 1990's [208, 209, 210]. His definitions and proofs were based on the semantics of Łukasiewicz predicate logic (or complete residuated lattices later in [211]), which naturally led him to graded fuzzy topological notions and theorems. Graded topological notions (of compactness and connectedness) had even earlier been studied by Šostak (see the references in [70], esp. to [192]).

Two main competing approaches to *fuzzy numbers* have originally been proposed: one of them treats fuzzy numbers as fuzzy intervals (Mizumoto and Tanaka 1979, see [142, §11]), while the other regards them as (certain equivalence classes of) distribution functions (Rodabaugh 1982, see [142, §11]).¹⁹ In the interval approach, Dubois and Prade (1980, see [142, §11]) have added further conditions of monotony and continuity. The distribution-based approach has been extensively studied in relation to the construction of fuzzy real numbers and the topology of the fuzzy real line, by Lowen, Höhle and others [153, 126].

The interval-based approach was recently criticized by Dubois and Prade [75] as representing the fuzzified notion of interval rather than number. Their proposal to define a fuzzy number as a gradual element, i.e., a function from truth values to the domain of discourse rather than vice versa, is discussed from the point of view of formal fuzzy logic in [26, §2] included in this thesis.

2.4 Formal fuzzy set theories

In this section we shall describe the state of the art in formal theories of fuzzy sets. We shall leave aside set theories with the unrestricted comprehension scheme, mentioned in Section 2.1, as these are very specific theories, unrelated to logic-based mathematics as

¹⁸These papers are cited according to [130].

¹⁹It can be observed that the approach of [16] (included in this thesis) in fact combines both approaches, since it treats fuzzy numbers as intervals between two distribution functions (or fuzzy Dedekind cuts).

presented here and constrained to a very narrow class of fuzzy logics (cf. footnote 6 on page 9 above). Instead, we shall focus on theories which are closest to our approach to fuzzy set theory, namely such theories over fuzzy logics whose axioms ensure that basic set-theoretical constructions (such as forming unions, intersections, singletons, or power sets) can be carried out. As this is exactly the motivation of the axioms of classical Zermelo–Fraenkel set theory, we shall call such theories *ZF-style fuzzy set theories*. Akin to ZF-style fuzzy set theories are *fuzzy type theories* (including Fuzzy Class Theory upon which our logic-based fuzzy mathematics is founded), since their hierarchy of types and the comprehension axioms (or the mechanism of λ -abstraction in Church-style type theories) are aimed at ensuring the availability of basic set-theoretic constructions, too (and vice versa, the set-theoretical constructions, esp. those of power set and union, guaranteed by the axioms of ZF-style set theories usually impose a cumulative structure on the universe of sets analogous to the hierarchy of types in type theories).

At least two strands can be recognized in the history of ZF-style theories of fuzzy sets.²⁰ One of them attempted at formal treatment of fuzzy (or many-valued) sets from the outset, while the other originated in ZF-style set theories over intuitionistic logic, whose methods were subsequently transferred to fuzzy logics (with systems close to Gödel logic as an intermediary step).

Early works in the former strand are due to Dieter Klaua (in 1965–1973, see Gottwald’s survey [105]), who defined (variants of) a cumulative hierarchy of fuzzy sets using definitions based on Łukasiewicz logic. This approach was followed and further modified by Siegfried Gottwald [99, 100] who derived many results on fuzzy sets in this framework.

While Klaua’s and Gottwald’s fuzzy set theories were essentially based on (Łukasiewicz) fuzzy logic, other early axiomatizations of fuzzy sets were based on membership functions and classical logic. Chapin’s axiomatic fuzzy set theory [56] considered a ternary membership predicate, with the third argument representing the degree of membership. An important feature of Chapin’s theory was a homogeneity of its objects (which is desirable in foundational theories—cf. classical set theory, where all objects are sets), as the membership degrees were not external objects different from fuzzy sets: rather, the role of membership degrees was played by some of the fuzzy sets themselves (hence the papers’ title ‘Set-valued set theory’). Basic parts of the formal theory of so defined fuzzy sets were derived from the proposed axioms in the two parts of the paper (the announced third part was never published). A similar setting was presented by Weidner [202], whose system (called Zadeh–Brown set theory ZB) aimed at emending some features of Chapin’s axioms; to this effect, the ordering relation between the degrees was taken as an additional primitive notion besides the ternary membership predicate. Consistency of ZB was shown by constructing a Boolean-valued model in ZF.

The construction of formal ZF-models valued in an appropriate structure of degrees was also a main motive in a series of papers, by several different authors, that originated in set theory over intuitionistic logic and subsequently shifted towards formal theories of fuzzy sets. In his 1975 paper [179], Powell constructs a syntactic interpretation (called the *inner model*) of classical ZF in a certain reformulation of ZF over intuitionistic logic (Int). To this end, he first needs to introduce and investigate various set-theoretical notions (e.g., ordinal numbers) and prove several results (e.g., the transfinite recursion theorem) within the formal intuitionistic set theory.²¹ Grayson’s 1979 paper [107] studies in detail various properties of ordinal numbers in a similar reformulation of ZF over Int, and shows

²⁰A much more refined classification of formal theories of fuzzy sets can be found in Gottwald’s recent survey [105], which also includes approaches distant from ours.

²¹A similar Heyting-valued model was much later studied by Shimoda [186].

counterexamples to some classically valid theorems on ordinals in a Heyting-valued sheaf model of the theory (which is a generalization of Boolean-valued models of Scott and Solovay). Even though the main results of those papers are metamathematical (viz, mutual interpretability of the theory and classical ZF), the theory itself is proposed as a prospective axiomatic setting for intuitionistic set theory in the sense of non-classical mathematics (see Section 2.1 above), and the theorems derived within the theory are regarded as results on intuitionistic (Heyting-valued) sets. This attitude differs from that of an earlier book [85] by Fitting, who only employs a reformulation of ZF over Int for metamathematical purposes, namely as a means for classical independence proofs by forcing. Driven mainly by this motivation, Fitting’s axioms arise from the classical axioms of ZF by replacing all occurrences of \forall by $\neg\exists\neg$, which yields a rather unintuitive axiomatic system from the point of view of non-classical set theory over Int.

Powell’s results and methods were in 1984 adapted by Takeuti and Titani [193] for a ZF-style set theory over a variant of Gödel logic (with a rule ensuring density of truth values). Besides the main result on mutual interpretability with classical ZF by means of inner models, they developed some parts of the formal theory, incl. the properties of real numbers.²² In 1992 the same authors [194] presented a ZF-style set theory over a richer logic which contained further connectives besides those of Gödel logic, representing the basic arithmetical operations (except division) in the standard $[0, 1]$ -interpretation of the logic. Again they constructed a cumulative $[0, 1]$ -valued model of their theory and proved mutual interpretability with classical ZFC. The paper also contains the construction of internal truth values (adapted in [41] for FCT over MTL) and various definitions and results within the theory.

Takeuti and Titani’s definitions mostly employ Gödel connectives as primary ones, and make use of the arithmetical operations only where necessary for their metamathematical purposes (mainly in the construction of internal truth values); the theory thus retains the structure of intuitionistic and Gödel set theories of the previously mentioned papers. Titani’s 1999 lattice-valued set theory of [198] is also largely based on lattice connectives in the underlying logic (although the implication connective is generalized so that it also admits a quantum-logic interpretation). The results and methods of Gödel set theory are, however, hardly transferable to other fuzzy logics, as they depend heavily on the idempotence (i.e., contractivity) of the minimum conjunction. The step to non-contractive fuzzy logics was undertaken by Hájek and Haniková in their 2003 paper [120], in which they adapt the previous methods (using also ideas from Shirahata’s work on set theory over linear logic [188]) for a set theory over the logic BL_{Δ} .²³

A (Church-style) *fuzzy type theory* FTT over the logic $IMTL_{\Delta}$ has been introduced in 2004 by V. Novák [169]. Although it has been mainly used as a formal background for linguistic modeling [171, 170], some parts of fuzzy mathematics have necessarily been developed in its framework, too (e.g., the theory of feasible natural numbers, [170, §3.5.3]).

Fuzzy Class Theory FCT of [30], which is the foundational theory of logic-based fuzzy mathematics as studied in this thesis, can be regarded as a (Henkin-style) simple type theory (of Russell’s type), too.²⁴ The developments of the theory by the present author

²²Basic notions of set theory over Gödel logic, motivated by both intuitionistic and fuzzy considerations, have also been developed in the present author’s master thesis [12] (in Czech; a short English summary can be found in [13]).

²³The Δ connective is used for limiting the size of powersets, which otherwise would be inconsistently large, and ensuring full existence of postulated sets, which is justifiable by the Skolem function equivalents of the axioms.

²⁴FTT and FCT (in logics over which FTT has been defined) seem to be mutually faithfully interpretable.

and his co-authors (U. Bodenhofer, P. Cintula, M. Daňková, R. Horčík, S. Saminger-Platz, and T. Kroupa) are described in detail in other parts of this thesis. Further works that contributed to the study of FCT and formal fuzzy mathematics developed within its framework are [64, 149, 148, 134] by P. Cintula, R. Horčík, and T. Kroupa.

3 Significance of the area of research

Logic-based fuzzy mathematics is a minor, rather than mainstream, current in fuzzy mathematics. This fact may raise questions about the meaningfulness of the enterprise and the significance of this area of research. In this section we shall summarize some reasons why the study of logic-based fuzzy mathematics is worthwhile.

There are general arguments in favor of the importance of any kind of non-classical mathematics (cf. also Section 2.1). Changing the logical principles that underlie mathematical reasoning may reflect some *external motivation* under which these principles are no longer valid—compare, e.g., the rejection of certain laws of classical reasoning by intuitionistic mathematics; this is also the case in fuzzy mathematics, as for instance the law of excluded middle is in general implausible for graded propositions. Non-classical mathematics can, however, also be justified independently of such ‘applied’ motivations, and be studied for the intrinsic reason of developing an *alternative view on classical mathematics*, as removing some assumptions of classical logic may reveal various kinds of dependencies between classical notions and present classical mathematical structures as special (or degenerate) cases of more general non-classical structures. This enables us, for instance, to compare the robustness of various mathematical definitions and theorems with respect to the changed logical assumptions.²⁵ The splitting of classically equivalent notions in weaker logics (in which their equivalence may no longer be provable), can shed light on classically indistinguishable aspects of the notions and provide a better understanding of the interdependencies between such aspects. (For an illustration, compare the various notions of finiteness in intuitionistic mathematics, cf. [87, §IV.6] and [77]—or, for that matter, in classical mathematics without the axiom of choice, see, e.g., [137, §4.6].) In the particular case of logic-based fuzzy mathematics, the change of the underlying logic yields linear-valued (and often continuous-valued) mathematical structures as semantical models, variants of which have been studied—mainly for such intrinsic reasons rather than for the sake of applications—since the 1960’s [54, 55, 152].

Besides being a sub-area of non-classical mathematics, logic-based fuzzy mathematics is also a specific sub-area of the theory of fuzzy sets. The importance of fuzzy sets for certain kinds of engineering applications is beyond doubt. In such applications, the richer system of membership degrees allows modeling the gradual change of a property, using it as a feedback measure for fine-tuning the value of the property by approximation steps—which would not be enabled by a crisp jump from 0 to 1 without intermediate values. Giving a formal foundation to various engineering fuzzy methods was one of the original motivations for the development of formal fuzzy logic, e.g., in [110, p. 2]. Although logic-based fuzzy mathematics does not directly address all methods of engineering-applicable fuzzy mathematics (cf. [26]), it provides a unifying framework for at least some of its parts [34, 26]. A consistent application of the logic-based approach moreover yields certain

²⁵For example, it is known [107] that the axiom of choice entails bivalence already in very weak set theories over intuitionistic logic, while Zorn’s lemma does not do so even in rather strong intuitionistic set theories. This shows, not only that the classical theorem on their equivalence uses the law of double negation in an essential way, but also that Zorn’s lemma is a more robust variant of the axiom of choice with respect to the behavior of negation.

favorable features of the resulting theory that provide further reasons for developing fuzzy mathematics in this specific manner; we shall list some of them in the next paragraphs.

One of the most important characteristics of the logic-based approach to fuzzy mathematics is the universal gradedness of defined notions. Traditional fuzzy mathematics employs classical logic for mathematical reasoning, therefore its defined concepts are by default crisp; gradedness has to be intentionally introduced in each definition. In logic-based fuzzy mathematics, on the contrary, defined notions are introduced by formulae of many-valued logic, and therefore are by default many-valued. This applies not only to fuzzy structures themselves (where adding gradedness is usual even in traditional fuzzy mathematics), but also to their *properties*, which in logic-based fuzzy mathematics are naturally fuzzy as well. Such graded properties of fuzzy structures have occasionally been studied in traditional fuzzy mathematics, too, often in partially logic-based setting. In fuzzy relations they have been first studied by Gottwald [101, 102, 104] and later by Bělohlávek [46]. Graded properties of fuzzy structures are also met in fuzzifying topology [208, 209, 211, 185]. Measures of defects of a broad range of mathematical properties, though motivated by other than logic-based considerations, were studied by Ban and Gal in [5]. A few properties like fuzzy set inclusion are commonly introduced as graded even in mainstream fuzzy mathematics [6, 7]. The idea of gradedness is also very strong in Pavelka-style fuzzy logic with evaluated syntax [177, 167, 172], which fuzzifies even the concept of provability. Full gradedness is a general feature of fuzzy mathematics based on formal fuzzy logic, be it a higher-order logic like FCT, a fuzzy type theory [169], or a formal fuzzy set theory [194, 120]. There are many reasons why graded properties of fuzzy structures are important; some of them are given in [19, §1], [35, §2.1], and [28, §1]. A reason which has not been stressed in these papers is that graded properties, like all fuzzy sets, enable one to *optimize* the property that is only imperfectly satisfied, where the degrees give a feedback for the optimization that cannot be provided by a crisp jump from 0 to 1. (Generalized fuzzy quantifiers would provide more kinds of logically meaningful measures of graded properties, thus enabling more kinds of optimization besides that on the infimum; however, a logic-based theory of generalized quantifiers is only in its beginnings, see Section 2.2.)

A related feature of logic-based fuzzy mathematics is a smooth accommodation of fuzzy sets of fuzzy sets. This is desirable in many branches of fuzzy mathematics: prototypically in fuzzy topology, as topological structures are usually formed of sets of sets (namely, sets of open sets, systems of neighborhoods, etc.), but also in other areas (consider, e.g., fuzzy sets of fuzzy numbers, of fuzzy points, of fuzzy events, etc.). Since formal fuzzy set theories axiomatize fuzzy sets of all kinds, their theorems apply as well to fuzzy sets of complex structures as to simple fuzzy sets of atomic urelements. Thus even though their semantical models are as complex as required, the syntactic logic-based apparatus that describes them is much simpler than their direct semantical description that is usual in traditional fuzzy mathematics. This demonstrates an advantage of the strict separation of syntax from semantics in the approach based on formal logic.

A related advantage of logic-based fuzzy mathematics ensues from its radical axiomatic approach, which contributes to its methodological clarity. An axiomatic approach has proved beneficial in countless fields of mathematics; it has occasionally been employed in traditional fuzzy mathematics, too (cf., for instance, de Luca and Termini's [67] axioms for fuzzy entropy or various axioms for aggregation operators—see, e.g., [145, Ch. 3]). However, grounding the axiomatic method on formal fuzzy logic offers an additional advantage for fuzzy mathematics, as the assumptions on the structure of truth degrees are then isolated and encapsulated in the logic itself, rather than re-introduced at each

definition. Fuzziness is thus only introduced into the theory by the set of rules that can generally be employed for sound reasoning about fuzzy propositions.²⁶ By this treatment, the general properties of fuzziness are removed from a particular theory to the level of logic; the theory itself can then deal with its specific notions only, and need not be concerned at the same time with the properties of truth degrees.

By hiding truth degrees in the semantics of fuzzy logic, logic-based fuzzy mathematics also alleviates one of the criticisms of traditional fuzzy mathematics, namely the artificial over-precision of fuzzy sets: an argument often raised against fuzzy set theory points out that instead of giving less information about the membership in a vague concept, fuzzy sets provide more (indeed too much) information by specifying its value to a precise real number (or an element of another lattice of truth degrees). However, by screening off direct references to truth degrees, logic-based fuzzy mathematics avoids (in a principled way) computing with particular truth degrees: not only are such calculations absent from formulae of the theory, but the theory in fact *abstracts* from them, in consequence of the definition of validity in formal logic by generalization over all models.

Another appealing consequence of hiding fuzziness into the rules of logic is the resulting similarity of formulae of fuzzy mathematics to those of classical mathematics. Since deductive fuzzy logics are not too different from classical logic, many concepts of classical mathematics can be naturally transferred to fuzzy mathematics simply by reinterpreting the logical connectives that appear in their formal definitions (cf. [126, §5]).²⁷ Quite often, classical definitions reinterpreted in fuzzy logic yield useful and interesting notions of fuzzy mathematics. The meaning of fuzzy concepts obtained in this way can be clarified by taking the meaning of fuzzy connectives and quantifiers into account. For instance, fuzzy inclusion $A \subseteq B \equiv_{\text{df}} (\forall x)(Ax \rightarrow Bx)$, defined by the same formula as in classical mathematics (since Ax is just an abbreviation for $x \in A$), is not just *some* measure of inclusion of fuzzy sets, as it is understood in traditional fuzzy mathematics, but is *the* strongest measure which allows for any x to infer²⁸ Bx from Ax & $(A \subseteq B)$, which is a transparent generalization of the same idea underlying the classical notion of inclusion. The parallel with classical logic and the meaning of fuzzy connectives thus provides additional motivation and guidance in defining concepts of fuzzy mathematics, besides the criteria of traditional fuzzy mathematics (which in practice often fail to prevent ad hoc definitions).

A further consequence of the closeness between classical and fuzzy logic is the fact that the three-layer architecture of classical mathematics (with the layers of logic, foundations, and particular theories) can be paralleled in fuzzy mathematics. (This was the leading idea of the position paper [34], included in this thesis.) The layer of foundations, provided by a sufficiently general formal theory over fuzzy logic, establishes a common language and a unifying framework for different disciplines of fuzzy mathematics. The foundational theory thus facilitates the exchange of concepts and results across the subfields of fuzzy mathematics.

As stressed above (p. 19), logic-based fuzzy mathematics directly formalizes only a limited part of traditional fuzzy mathematics. Nevertheless, its clearly isolated pre-

²⁶Particular sets of inference rules—i.e., particular fuzzy logics—then reflect special assumptions on the structure of degrees.

²⁷Though of course not too mechanically, as there are usually more options for finding a fuzzy counterpart to a crisp notion (e.g., if classically equivalent definitions are no longer equivalent in fuzzy logic). Some selection is needed, based on pragmatic criteria; often it leads to splitting classical notions, cf. [37, §4].

²⁸I.e., to ensure that the truth degree of the consequent is at least as large as that of the antecedent. This kind of inference in deductive fuzzy logic is based on the local consequence relation, cf. [26].

theoretical assumptions, captured in the form of the axioms of the background fuzzy logic, enable it to find applications even beyond the traditional realm of fuzzy logic, namely in the areas where these extracted assumptions are applicable. An example of such application is the interpretation of deductive fuzzy logics in terms of resources or costs [11], similar to the resource-based interpretation of linear logic (cf. [92]). Under this interpretation of deductive fuzzy logic, the semantic values of formulae represent (prelinear) resources or costs rather than degrees of truth.²⁹ Deductions in formal fuzzy logic then preserve costs rather than partial truth, and particular fuzzy logics correspond to different ways how the costs can be summed by conjunction. The resource-based applications of deductive fuzzy logic (esp. in epistemic, deontic, and dynamic logics) are yet to be elaborated: currently they are just sketched in the present author’s conference abstracts [20, 23, 25].

The latter connection between fuzzy logic and linear logic is just an instance of similar connections between fuzzy logic and other substructural logics. These links follow from the fact that deductive fuzzy logics are specific substructural logics (namely those with the law of prelinearity), and so usual motivations for having dropped structural rules apply to them as well. Logic-based fuzzy mathematics can therefore model specific (namely, prelinear) situations modeled by substructural logics. Even though substructural logics are mostly studied in their propositional forms (because of the problems with strong quantifiers, see Section 2.2), it is clear that more complex situations modeled by substructural logics would require first- or higher-order language. This motivates the need for substructural mathematics (cf. Section 2.1), of which logic-based fuzzy mathematics is a specific and important part. Possible generalizations of the methods of logic-based fuzzy mathematics to broader classes of higher-order substructural logics with a wider area of applications thus give another reason for the development of fuzzy mathematics in the logic-based setting.

The above paragraphs summarized *motivations*, i.e., “ex ante” reasons for developing logic-based fuzzy mathematics. However, there is also an “ex post” reason, namely the results already achieved in the framework of Fuzzy Class Theory. As witnessed by the papers included in this thesis (esp. [30, 41]), the logic-based approach is capable of trivializing certain parts of traditional fuzzy mathematics. This demonstrates that logic-based fuzzy mathematics is capable of providing powerful tools for traditional fuzzy set theory (which in turn is directly applicable in engineering practice).

4 Description of the author’s contribution

This section provides a commentary on the papers comprising this thesis, with a special focus on several points. First, the relation of each paper to the topic of the thesis and to other papers connected with the project is explained. Second, some of the older papers are commented from the point of view of the later development of the theory. Finally, the author’s contribution to the joint papers included in this thesis is indicated. (The co-authors have read the descriptions of author contribution and explicitly confirmed their accuracy by email.)

In order also to clarify the author’s contribution to the project of logic-based fuzzy mathematics itself, a short history of the development of Fuzzy Class Theory is given first. Though unavoidably subjective, it tries to describe the emergence of ideas related to the project in as accurate way as possible. For the account of predecessor ideas and results upon which the project has been built see Section 2.

²⁹Parts of this idea arose in discussions with Petr Cintula.

4.1 A short history of Fuzzy Class Theory

The author's master thesis [12] dealt with axiomatic set theory over Gödel logic.³⁰ The thesis was written shortly after the period (ca. in 2000–2001) when a small semi-regular seminar was organized in Prague by Petr Hájek, which was devoted to developing formal set theories over t-norm fuzzy logics and in which the present author actively participated.³¹ The attempt at investigation of some basic disciplines of fuzzy mathematics (with fuzzified set theory and arithmetic as first choices) seemed to be a natural next step after first-order fuzzy logic and its metamathematics had advanced enough [110] to provide a meaningful machinery for such theories. The study of set theory based on Gödel logic was a reasonable choice as the latter logic extends intuitionistic logic in which successful variants of set theory had been built [179, 107, 85], and is closely related to the (slightly stronger) logic in which Takeuti and Titani's fuzzy set theory [193] had been constructed; also Takeuti and Titani's subsequent variant of fuzzy set theory [194], even though defined over a much stronger logic similar to LII, employs mainly Gödel operations in definitions, and therefore most of its constructions can be modified for set theory over Gödel logic, too. The seminar was, nevertheless, partly devoted to set theories and arithmetics over other fuzzy logics (esp. Łukasiewicz and BL), which later resulted in Hájek's study of Cantor–Łukasiewicz set theory with full comprehension over Łukasiewicz logic [115] and the construction of Hájek and Haniková's ZF-style set theory over BL [120]. The seminar laid stress on actual developing mathematics formally within the theories (in the spirit of Klaua's, Chapin's [56], and Gottwald's [99, 100] papers), and not just on the metamathematical study of their properties. Although the seminar stopped meeting in 2001, several participants continued investigating formal fuzzy set theory individually (including the present author, whose master thesis on the topic was defended in 2002). An attempt by Cintula, Hájek, and the present author to revive the seminar in 2003 led instead to the employment of the present author at the Institute of Computer Science (where the former two were working) and a close collaboration by the three on the topic, and eventually to the development of Fuzzy Class Theory and the current research project.

Fuzzy Class Theory was conceived in discussions between Petr Cintula and the present author during their research stay in Barcelona (at IIIA CSIC, Bellaterra) in October 2003. At that stage, only the first-order classes over the logic LII were considered, and the aim was to construct a common framework for the study of elementary operations and relations on fuzzy sets and fuzzy relations (such as various kinds of intersection, union, inclusion, etc.) over first-order fuzzy logic. The authors' motivations for this study, however, slightly differed from each other. P. Cintula had shortly before (in 2002) solved a problem on fuzzy orderings, presented to him by U. Bodenhofer, by means of first-order fuzzy logic (so in fact by using first-order classes) and wanted to continue the study of fuzzy orderings to see how far could the theory be developed with the limited means of elementary theory of fuzzy classes. The present author, on the other hand, had the experience from his work on Gödel set theory that a very large number of concepts of applied fuzzy mathematics can be defined and investigated just by means of first-order fuzzy classes (i.e., without considering membership of fuzzy sets in fuzzy sets). Even though elementary class theory consists for the most part just in translating the first-order predicate calculus into the set-

³⁰The thesis was written in Czech; an English overview of its topic and methodological principles can be found in [13].

³¹Establishing the new focused seminar followed a series of talks on the same topic at the Seminar in Applied Mathematical Logic (an activity of the Czech Society for Cybernetics and Informatics) held at the Institute of Computer Science of the Academy of Sciences of the Czech Republic. Regular attendants were P. Cintula and Z. Haniková (then students of Petr Hájek), K. Bendová (later the supervisor of the author's master thesis related to the topic of the seminar), A. Sochor, K. Trlifajová, and several others.

theoretical language, it is actually just a theory of fuzzy *classes* which is mostly used in applied fuzzy set theory, rather than a fully fledged fuzzy *set* theory.³² In particular, such concepts as the empty and universal class; the relations of inclusion, equality, disjointness, and compatibility; the properties of fuzziness and crispness, normality, and height; the unary class operations of complement, kernel, and support; the binary class operations of intersection, union, and difference; the properties of reflexivity, symmetry, transitivity, antisymmetry, and functionality of fuzzy relations; the operations of composition and inversion of fuzzy relations; and many other important relations and operations on fuzzy sets and fuzzy relations can all be expressed and investigated in the theory of first-order classes, not needing a theory with fuzzy sets of higher ranks (or orders).

The aim therefore was to have an axiomatic framework for the study of such concepts, with the possibility of quantification over classes (rather than just over atomic elements as in first-order fuzzy logic) and with the apparatus for handling tuples in order to internalize fuzzy relations (besides fuzzy classes). Since the tuples were intended just to represent multiple arguments of predicates with arities larger than 1, there was no need to fuzzify tuples, and the classical axioms for crisp tuples (regarded as crisp logical functions in the sense of [111]) could be adopted.³³ As fuzzy classes were to be treated in the same manner as crisp classes in classical second-order logic, axioms analogous to those of classical second-order logic could be adopted to describe them: the axiom scheme of comprehension, ensuring that each fuzzy property expressible in a fixed formal language defines a fuzzy class; and the axiom of extensionality, ensuring that a fuzzy set is uniquely determined by its members (i.e., by the truth values of membership of x in A for all x —in other words, by its membership function). The fuzzy logic LII was chosen for the background logic in order to have full arithmetic power over the system of truth degrees³⁴ in a system that would still enjoy good metamathematical properties. The logic was also suitable as a unified framework for the investigation of many different fuzzy set operations, by virtue of the representability of a large class of truth functions in the standard LII-algebra.

Initially, the theory was expected to provide little more than a convenient framework for easy proofs of schematic theorems on several kinds of intersection, union, inclusion, etc. However, the full potential of the theory was realized soon (before the end of 2003). The present author observed that the fragment of class theory reducible to propositional logic [30, Th. 33–36] is so large that it covers most interesting elementary theorems of traditional fuzzy set theory. Jointly we observed that by iterating the machinery for classes of higher orders, the expressive power of the resulting simple fuzzy type theory (FCT) is sufficient for a large part of traditional fuzzy mathematics, as classical higher-order theories are interpretable in FCT [30, L. 41] (so we can assume any crisp structure on the universe of discourse), and moreover such concepts of fuzzy set theory as Zadeh’s extension principle become definable objects of FCT [30, Def. 39]. In this form, the theory

³²By a class theory we mean the study of classes that contain atomic individuals from some fixed domain, but the membership of classes in classes is not considered. Set theory proper, on the other hand, is the study of sets as objects that contain other sets or objects and are themselves members of other sets.

³³Subsumption of sorts of variables had to be introduced for convenient handling of tuples; this was done by Petr Cintula when our discussions convinced us that other possibilities would probably not be more easily implementable. Even though sorted first-order languages had been used before [110, 46], subsumption of one sort by another had not yet been considered in formal fuzzy logic.

³⁴Insufficient expressive power could lead to the undefinability of various notions of traditional fuzzy mathematics. For instance in set theory over Gödel logic without Δ , even such basic concepts as the normality and the crisp kernel of a fuzzy set are undefinable [12].

was presented at the FSTA conference in January 2004, where also the first version of the paper [30] was finished.

The expressive power of the theory suggested the possibility of its foundational role for fuzzy mathematics, analogous to that of Russell's simple type theory for classical mathematics. Furthermore, the fact that the underlying logic of the theory was fuzzy offered a consistent methodology of fuzzification of classical notions by a (controlled) reinterpretation of classical defining formulae in fuzzy logic (cf. [126, §5]), building upon the corresponding roles of classical and fuzzy logical symbols (cf. p. 21 above). The idea of a foundational research program based on this methodology and implemented by means of FCT emerged in a discussion between P. Cintula and the present author at an institutional workshop in March 2004. The foundational program was then described in the manifesto [34] (presented at *The Challenge of Semantics* in Vienna, July 2004) and the research program was elaborated into a grant proposal in April 2004. The grant was awarded for 2005–2007 and the grant team included, besides P. Cintula (the principal investigator) and the present author, also T. Kroupa (as the principal co-investigator) and R. Horčík. The latter two focused on the development of particular disciplines of fuzzy mathematics in FCT: R. Horčík on fuzzy intervals [134] and fuzzy quantifiers [65, 64] and T. Kroupa on fuzzy filters [149] and fuzzy topology [43, 42, 44]. The description [32] of the foundational program won the Best Paper Award at the 11th IFSA World Congress in Beijing, July 2005.

The next task after the development of the basic apparatus of FCT was to advance a formal theory of fuzzy relations within its framework, as fuzzy relations are indispensable in all disciplines of fuzzy mathematics. Following P. Cintula's previous contacts in this area, in November 2004 we started a cooperation with Ulrich Bodenhofer, focusing on basic properties of fuzzy preorders and similarities. The first joint results [33, 49] were presented at the Linz Seminar in February 2005, and the cooperation eventually led to the comprehensive paper [28], finished in 2007.

Since 2005, the investigation of particular disciplines of fuzzy mathematics has begun and the project participants turned their interests to various directions; only a sketchy description of these activities can be given here.

A different approach to basic properties of fuzzy relations, making them relative to a fuzzy relation representing indistinguishability of elements, was proposed by the present author at IPMU 2006 [19]. In 2005, the present author started working with M. Daňková on properties of fuzzy relational operations that had not been covered by his joint paper with Cintula and Bodenhofer. It was soon realized that many relational operations had a form similar to either Zadeh's [213] sup-T relational composition or Bandler and Kohout's [6] BK-product (i.e., inf-R composition) of fuzzy relations. The informal correspondence was made precise by means of internalized truth values (cf. [194]) and formal interpretations [21] by the present author, and systematically explored in a joint paper with M. Daňková [41]. The method described in the paper provides a reduction to a simpler calculus for fuzzy relational operations, in a similar manner as the metatheorems of [30] do for class operations.

The internalization of truth values described in [41] initiated later (in 2007) an investigation of graded properties of truth-value operators (e.g., t-norms, copulas, etc.) under a Czech–Austrian project on aggregation operators. The first results (by U. Bodenhofer, P. Cintula, S. Saminger-Platz, and the present author) were presented at the Linz Seminar 2008 [29]; a full paper is in preparation. In 2004–5, the first steps were also done in the logic-based theory of measures on clans of fuzzy set by T. Kroupa [148] and fuzzy Dedekind–MacNeille lattice completion and fuzzy Dedekind reals by the present au-

thor [15, 16]. An application of the formalism to the fuzzified logic of questions, sketched by the present author at the VlaPoLo workshop in Zielona Góra as early as in November 2003, was turned into a full paper [14] in 2004. Several further areas are currently under investigation; for an overview of the work in progress and future plans see the end of this section.

During the work on formal fuzzy mathematics, several peculiar features of axiomatic theories over fuzzy logic have been noticed which are not met in classical nor mainstream fuzzy mathematics. These features, due mainly to the non-idempotence of strong conjunction and thus common to mathematical theories in all contraction-free substructural logics, have been summarized in [37]. The different style of fuzzy mathematics ensuing from these peculiarities has been gradually introduced in papers since 2006, cf. [19, 28, 29, 43, 42]. This also emphasized the need of exponentials and generalized fuzzy quantifiers for fully fledged formal fuzzy mathematics (to be worked out yet, with initial results in [65, 64]) and directions for further elaboration of the basic apparatus of FCT.

The experience with fuzzy mathematics also helped to analyze fundamental differences between the fundamental assumptions of mainstream fuzzy mathematics and logic-based fuzzy mathematics. As argued by the present author in [26], logic-based fuzzy mathematics directly addresses only a very specific portion of traditional fuzzy mathematics, and even though its apparatus is powerful enough to encompass a much larger area of traditional fuzzy mathematics, the advantages of the logic-based approach are manifested best in problems close to its own principles and motivation (i.e., logical inference preserving the degrees). The scope of the logic-based approach thus should be specified more narrowly than originally in the manifesto [34]. Nevertheless, its applicability is still broad enough to make it a significant part of mainstream fuzzy mathematics, with clear methodology and interpretation.

At present, the project of logic-based foundations of fuzzy mathematics is by no means finished and continues to be under permanent progress. Among the proximate future tasks is the elaboration of the theory of fuzzy quantifiers and their application in all disciplines of logic-based fuzzy mathematics (which would include revisiting areas that have already been developed, and a thorough study of new notions defined by means of such quantifiers). Another important topic is the notion of fuzzy function, which has not yet been sufficiently investigated in FCT, either. The notion can then be employed for defining in FCT the concepts of fuzzy cardinality (based on fuzzily bijective fuzzy functions) and fuzzy morphism of fuzzy structures. Various properties of fuzzy orderings have not yet been systematically studied, for instance linearity, directedness, or well-foundedness. Fuzzy topology, fuzzy aggregation operators, and fuzzy interval arithmetic are currently under study; fuzzy lattices, measures, and metric spaces are possible candidates for forthcoming topics of research in FCT.

4.2 The papers comprising the thesis

This section comments on the papers comprising the thesis. The papers are grouped and ordered by topic rather than chronologically, in order to give an exposition of the theory proceeding in a logical way from the methodological assumptions and the basic apparatus of FCT to more advanced disciplines of fuzzy mathematics. The texts of the papers were recompiled for inclusion in the thesis, and may therefore differ from the published versions in such details as formatting, numbering of footnotes or references, etc.³⁵ Several typos that occurred in the published papers have also been fixed in the present version.

³⁵The applicable copyright transfer agreements allow including the papers in a thesis.

L. Běhounek: *On the difference between traditional and deductive fuzzy logic* [26]. The paper analyzes methodological principles of logic-based fuzzy mathematics and demonstrates them to be fundamentally different from those of traditional fuzzy mathematics. The paper shows that even though most concepts of traditional fuzzy mathematics can be modeled in higher-order fuzzy logic (as its expressive power includes classical mathematics), the logic-based rendering of notions that are based on principles alien to deductive fuzzy logic is rather artificial and gives little advantage over studying such notions by traditional methods. Therefore, the logic-based approach is best suited to a specific area of fuzzy mathematics consonant with its methodological assumptions (namely those related to the deductive treatment of partial truth), and its foundational significance is smaller in other areas of fuzzy mathematics.

The paper was based on several years of experience with developing logic-based fuzzy mathematics; therefore it could make distinctions that had not been recognized in the Manifesto [34] written at the beginning of the research program. Even though the more precise delimitation of the scope of the foundational program could be seen as a retreat from the too optimistic tone of the Manifesto (which purported to give foundations to all fuzzy mathematics), it can on the other hand be interpreted as a clarification of the fact that traditional fuzzy mathematics actually deals with several phenomena that are too different from each other, and therefore it in fact comprises several different fields of research. The field in which the logic-based approach is most fruitful is marked by a clear interpretation of membership degrees as degrees of truth (preserved under inference), while other areas of fuzzy mathematics work with a mixture of several different conceptions of membership degree (cf. [74]), often not clarified enough. Naturally, logic-based methods apply in a less straightforward manner to such fields. The paper thus presents a more precise delimitation of the area of research, rather than a retreat from the foundational program.

Although the paper uses the term *partial truth* frequently, it was not meant to engage in the philosophical dispute on the nature of truth and its (un?)necessary bivalence:³⁶ the term should be understood in the technical sense of “the (gradual) quality of propositions that is preserved under the deductions in fuzzy logic”. The gradual quality is in the paper called “partial truth” in analogy with the (bivalent) quality transmitted in deductions of classical logic, which is usually called—and understood as—*truth*. Whether we call the gradual quality “partial truth” or another name has no effect on the observations made in the article: the only important thesis is that, similarly as classical logic operates *salva veritatis*, deductive fuzzy logics infer their conclusions *salvo gradu*—i.e., preserving the grades assigned to propositions,³⁷ no matter whether the grades are interpreted as degrees of truth, a measure of the underlying attributes [140], utility values [90], costs [11, 25], or grades of any other kind.

The term *deductive fuzzy logic* is in the paper used for logic-based fuzzy mathematics in general (i.e., not only for formal fuzzy logic in the strict sense), since the intended audience usually employs the term *fuzzy logic* (both in Zadeh’s [214] broad and narrow sense) in the broader sense of *fuzzy mathematics*. The term is in the paper additionally given a concrete mathematical meaning of the logics of linear residuated lattices, which delimits the class of logics upon which logic-based fuzzy mathematics in our sense can be built.

³⁶This was not stressed in the paper, as the intended audience were researchers in traditional fuzzy logic rather than philosophers.

³⁷*Preserving* should here be understood in the sense of the local consequence in substructural logics (cf. footnote 12 on page 11 above and see [26] for details), not in the sense of [50, 86].

In the paper, the structural rule of exchange (i.e., commutativity of conjunction) is assumed for deductive fuzzy logics. This assumption was based on the idea that while the absence of the rules of contraction and weakening can be motivated by considerations about truth degrees,³⁸ non-commutativity of conjunction is motivated by (e.g., temporal) considerations that are not related to degrees of truth. However, since fuzzy logics can [11, 20, 23, 25] also be motivated as logics of resource-aware reasoning (or logics of costs), and the rule of exchange can fail for resources (i.e., fusion of resources need not be commutative), it is more reasonable to discard in general the assumption of commutativity, too.

In [86], Josep Maria Font proposed to call the intervals $\{\beta \in L \mid \beta \geq \alpha\}$ for each $\alpha \in L$ truth *degrees*, as opposed to the truth *values* $\alpha \in L$. Then one can say that it is a truth degree what is preserved by fully true implication in deductive fuzzy logics, rather than a truth value. Font’s distinction is consonant with the considerations presented in the discussed paper and provides a better formulation of what in the discussed paper is described as “guaranteed degrees of truth”, “guaranteed truth thresholds”, etc.

L. Běhounek, P. Cintula: *From fuzzy logic to fuzzy mathematics: A methodological manifesto* [34]. The paper was written in June 2004 and presented at the workshop *The Challenge of Semantics* in Vienna in July 2004. The main motivation for writing the paper was to have a concise description of the methodology of logic-based fuzzy mathematics (called *Hájek’s program* in the Manifesto) that could be referred to in subsequent papers. The contents of the paper arose from extensive discussions between both authors and is their joint work. The structure and actual wording of the paper was drafted by the present author and finalized by both.

At the moment of writing it was assumed that deductive fuzzy logics could provide foundations for the whole of traditional fuzzy mathematics. While this is true to some extent, the best-suited area of applicability of the approach was later clarified in [26]; see the previous paragraph on [26] for details.

A skeptical attitude towards the methodology described in the Manifesto (and towards non-classical many-valued mathematics in general) was expressed by D. Dubois in [72, p. 195–6]:

Although some may be tempted to found new mathematics on many-valued logics [34], this grand purpose still looks out of reach if not delusive. It sounds like a paradox of its own since we use classical mathematics to formally model many-valued logic notions. What could be named “many-valued mathematics” essentially looks like an elegant way of expressing properties of many-valued extensions of Boolean concepts in a Boolean-like syntax. For instance, the transitivity property of similarity relations is valid in Łukasiewicz logic, and, at the syntactic level, exactly looks like the transitivity of equivalence relations, but should be interpreted as the triangular inequality of distances measures.

To answer the criticism, the following clarification should be given first. Formal fuzzy mathematics based on the methodology of [34] can essentially be understood in any of the following two ways:

³⁸Namely, by observing that combining imperfect truths combines their imperfection, which justifies the general non-idempotence of conjunction; and that there can be degrees of full truth (e.g., in such predicates as *acute angle*)—i.e., that the residuated lattice of truth degrees need not be integral.

- In a “*traditionalist view*”, logic-based fuzzy mathematics is just a methodologically advantageous treatment of traditional fuzzy mathematics, where in exchange for voluntary restrictions on the language and methods we obtain a formalism that enables us to derive theorems of certain forms more easily (cf. [30, §3.4] or [41]). Under this view, the formulae of higher-order fuzzy logic indeed describe the behavior of membership functions valued in the real unit interval or more generally in an appropriate semilinear residuated lattice.

One of the benefits of this approach indeed comes from the fact that many notions of traditional fuzzy mathematics turn out to be expressed by formulae of exactly the same form as analogous notions of classical mathematics—e.g., T-transitivity by a formula expressing classical transitivity, only reinterpreted in many-valued logic. This enables to treat fuzzy notions in a similar way as classical notions: e.g., the proofs of theorems often just copy classical proofs. Moreover, it allows us to extrapolate this observed correspondence and find new important notions of fuzzy mathematics by reinterpreting classical definitions in fuzzy logic. Furthermore, when employing many-valued logic, all defined notions become naturally graded, which radically facilitates the study of graded properties (in the sense of [101]) of fuzzy notions.

- Alternatively, in a “*foundationalist view*”, logic-based fuzzy mathematics presents a fundamental treatment of fuzzy mathematics (indeed a “new mathematics”, as called by Dubois in the cited passage of [72]), based on non-classical logics. This interpretation understands fuzzy sets as a *primitive notion*, axiomatized (or governed) by the axioms and rules of the non-classical logic, in a similar manner as crisp sets are governed (and can be axiomatized) by the axioms and rules of classical logic.

Under this approach, fuzzy sets are *not* represented or modeled by their membership functions, but are primitive objects *sui generis*. Pre-theoretical considerations (cf. [22, 117]) about (certain kinds of) vague propositions suggest that they can be assumed to be governed by the laws of the fuzzy logic MTL or some of its variations. Importantly, the justification of the logical laws governing vague propositions is pre-theoretical and independent of any model of fuzzy sets in classical mathematics. Based on the axioms and rules of fuzzy logic, a formal theory of fuzzy sets can be developed, with the intended informal semantics of *actual* fuzzy sets, i.e., unsharply delimited collections of objects—similarly as the intended informal semantics of classical sets is that of sharply delimited collections of objects. The *formal* semantics of fuzzy logic is then formed by fuzzy sets described by (a fragment of) the very same theory itself—similarly as the semantics of classical logic is formed by sets described by (a fragment of) classical set theory (i.e., the same form of ‘circularity’ is encountered as in the foundations of classical mathematics).

It turns out that, *incidentally*, the theory of fuzzy sets can be *formally interpreted* in classical mathematics: this formal interpretation is what more usually is called “the many-valued semantics” of fuzzy set theory, in which fuzzy sets become interpreted by “membership functions”. Although classical mathematics is thus, by means of the formal interpretation, capable of faithful modeling fuzzy mathematics, it does not establish its priority over fuzzy mathematics, as both theories can be founded independently of each other and are faithfully interpretable in each other.³⁹

³⁹A formal interpretation of classical mathematics in fuzzy mathematics can be done by means of the propositional connective Δ —which is no wonder as the connective is intended to represent crisp propositions among fuzzy ones and is axiomatized by the laws valid for crisp sets.

Both classical and fuzzy mathematics are therefore of equal standing as foundational theories, and precedence can be given to one of them only on the basis of some pragmatic criteria. Classical mathematics may be preferred because of our long experience with it or because of its simplicity (as it only considers crisp sets). Fuzzy mathematics, on the other hand, can be preferred in vague contexts because it renders vaguely delimited sets more directly, and because of the advantages of its apparatus in proving theorems on fuzzy sets as described under the traditionalistic view above.

It can be seen that the three points of the above criticism of many-valued mathematics from [72, pp. 195–196], namely that

1. “we use classical mathematics to formally model many-valued logic notions”,
2. “what could be named ‘many-valued mathematics’ essentially looks like an elegant way of expressing properties of many-valued extensions of Boolean concepts in a Boolean-like syntax”, and that
3. “the transitivity property of similarity relations [...] should be interpreted as the triangular inequality of distances measures”,

only apply to the traditionalistic view of the non-classical theory. The second statement is explicitly admitted in the Manifesto [34, p. 643]:

Admittedly, a formal theory over fuzzy logic is just a notational abbreviation of classical reasoning about the class of all models of the theory.

Still, the advantages of the logic-based approach fully justify the development of logic-based fuzzy mathematics even under the traditionalistic interpretation. The possibility of the foundationalist attitude, however, shows that the non-classical theory need not be regarded just as formally modeling many-valued notions while still using classical mathematics. Rather, the non-classical notions can be regarded as primitive and independent of classical mathematics: since the theory is *syntactical*, it does not need to presuppose that classical mathematics has been developed first. And under the foundationalist approach, the transitivity of similarity relations is *not* interpreted as the triangular inequality of distance measures, but indeed as *transitivity* of unsharply delimited relations (regarded as *primitive* entities). The application of the name “transitivity” to fuzzy relations is then justified by the fact that $\text{Trans } R \equiv_{\text{df}} (\forall xyz)(Rxy \ \& \ Ryz \rightarrow Rxz)$ is the necessary and sufficient graded condition ensuring that Rxz can for any instances of x, y, z be inferred⁴⁰ from Rxy and Ryz (which is exactly the property we usually call “transitivity”). Only accidentally the property coincides, when fully true, with the notion of T-transitivity that is known from traditional fuzzy mathematics and that expresses the triangle inequality of distance measures.

In sum, Dubois’ criticism of [72] only applies to the traditionalist understanding of logic-based fuzzy mathematics, and not to the foundationalist one. But even under the traditionalistic view, logic-based fuzzy mathematics has undisputable advantages described above, which fully justify its development.

⁴⁰In the graded way, i.e., preserving the truth degrees in the sense of the local consequence of deductive fuzzy logics, see footnote 12 on page 11 or [26].

L. Běhouněk, P. Cintula: *Fuzzy class theory* [30]. This chronologically first paper on Fuzzy Class Theory introduced its apparatus, demonstrated its expressive power (by interpreting notions of fuzzy and classical mathematics) and hinted at benefits of its formal methods (by reducing a large part of graded elementary theory of fuzzy classes to propositional calculations).⁴¹

In the paper, the logic $\mathbb{L}\Pi$ was used as the background logic of the theory, because of its expressive power. The aim of the paper was to construct a unified framework for most of fuzzy mathematics, which required having a large class of t-norm based propositional connectives interpretable in the underlying logic. The logic $\mathbb{L}\Pi$ which interprets all finite ordinal sums of the three basic continuous t-norms (\mathbb{L} , \mathbb{G} , and \mathbb{I}) and many left-continuous t-norms (e.g., $\mathbb{N}\mathbb{M}$) as well as their residua while still retaining good metamathematical properties provided a suitable compromise between the expressive power and simplicity of the logic. For the sake of generality, all notions were in the paper defined relative to an arbitrary $\mathbb{L}\Pi$ -representable t-norm (intended to interpret the connectives in the defining formula), and theorems and proofs were formulated schematically, with connectives indexed by $\mathbb{L}\Pi$ -representable t-norms. Later the practice showed that connectives pertaining to different t-norms are seldom mixed in particular disciplines of logic-based fuzzy mathematics, and that it is therefore more convenient to work in a fragment of $\mathbb{L}\Pi$ containing just the connectives needed for the particular purpose. The schematic formulae indexed by an $\mathbb{L}\Pi$ -representable t-norm can then be replaced by formulae of the logic MTL_Δ , which is sound for any left-continuous t-norm. If needed, the logic MTL_Δ can be strengthened by additional assumptions (e.g., to IMTL_Δ if the involutiveness of the residual negation is required) or expanded by additional connectives (e.g., to MTL_\sim if an independent involutive negation is needed). Currently, therefore, logic-based fuzzy mathematics is mostly done in FCT over MTL_Δ or a similarly weak logic rather than over $\mathbb{L}\Pi$.⁴² It is obvious that the apparatus of FCT can straightforwardly be transferred to any deductive fuzzy logic that extends MTL_Δ . Fuzzy Class Theory over $\mathbb{L}\Pi$, nevertheless, remains being the common framework for the study of notions pertaining to different $\mathbb{L}\Pi^{\frac{1}{2}}$ -representable t-norms in one theory, and thus (a candidate for) the common foundational theory for logic-based fuzzy mathematics.

The paper, though necessarily technical, was also aimed at the audience not specialized in formal fuzzy logic; therefore some technical details were only sketched (e.g., the apparatus of tuples) or not discussed at all (for instance, that the comprehension schema should be extended to formulae of the enriched language if new symbols are added to the language, as in Section 6 of the paper). As it was sufficient for the basic development of logic-based fuzzy mathematics, only the axioms of extensionality and comprehension (and, optionally, fuzziness) were considered in the paper, although it was already clear that advanced topics in logic-based fuzzy mathematics will sooner or later require some forms of the axiom of choice or similar principles. Since only the basics of fuzzy mathematics have been investigated by now, such a need has not arisen yet. The expected complexity of the relationships between possible variants of choice principles over fuzzy logic makes them another topic for future investigation.

⁴¹Theorems on fuzzy sets that are typically found on the first several dozens of pages in standard textbooks in fuzzy set theory (e.g., [166, 145]) are corollaries of the metatheorems [30, Th. 33–36] and simple theorems of propositional fuzzy logic. Since usual propositional deductive fuzzy logics are decidable, the metatheorems show that basic properties of fuzzy sets could easily be generated by a computer program. A similar comment applies to the theorems on fuzzy relations from the paper [41] described below.

⁴²Working in MTL_Δ is slightly more general than the schematic work in $\mathbb{L}\Pi$, since it admits interpreting the connectives by all left-continuous t-norms rather than only those representable in $\mathbb{L}\Pi$. Notice, however, that *propositional* MTL is complete w.r.t. all left-continuous t-norms representable in $\mathbb{L}\Pi^{\frac{1}{2}}$ [155].

The paper was written in January 2004; the authors' motivations for the study of the theory are described in Section 4.1. In the actual preparation of the paper, all parts were extensively discussed by both authors, and most sections are their joint work. Of particular developments that are due mainly to one of the authors, the apparatus of subsumption of sorts in first-order fuzzy logic was prepared by Petr Cintula, while the metatheorems of Section 3 were observed by the present author.

L. Běhounek, U. Bodenhofer, P. Cintula: *Relations in Fuzzy Class Theory: Initial steps* [28]. The paper treats basic properties of fuzzy relations in the graded setting of FCT. Since relations occur in all parts of mathematics, the investigation of basic properties of fuzzy relations in FCT is an indispensable prerequisite for all disciplines of logic-based fuzzy mathematics. A parallel aim of the paper was to present Fuzzy Class Theory to researchers in traditional fuzzy mathematics and introduce to them the fully graded approach in fuzzy mathematics. To this end we wanted to recast in FCT known theorems on graded properties of fuzzy relations (esp. those from Gottwald's monograph [104, Ch. 18]), and to give graded generalizations of some representative non-graded results of traditional fuzzy mathematics; a few previously unknown concepts and results were discovered along the way, too. As it was impossible to cover the whole area of fuzzy relations, the paper focused mainly on fuzzy preorders and similarities; but even with this reduction of scope, the paper could only treat a selection of their most basic properties. Further classes of fuzzy relations (e.g., fuzzy orderings or functions) still wait for a thorough investigation.

Several parts of the paper have preliminary versions in conference proceedings [33, 49, 61, 16, 27]. In order to keep the introduction to the paper short, a primer in Fuzzy Class Theory [35] was written and made freely available online as a research report.

The paper was written over the period of more than three years (2004–7), mostly during several research stays of the Czech co-authors at Johannes Kepler University in Linz. All parts of the paper were edited, discussed, and checked for correctness by all of the co-authors. Particular contributions of the co-authors (so far as they can be determined) were as follows: Ulrich Bodenhofer provided the examples and links to known results of traditional fuzzy mathematics (cf. [49]), wrote most of the Introduction, edited many passages in other sections, and collaborated on several parts of the paper (esp. in Sections 4, 6, and 7). Most of the introductory Section 2 and the Appendices were written by Petr Cintula and the present author (cf. [37, 35]); the latter is also responsible for Section 5 (on bounds, cf. [16]) and smaller parts of other sections. Section 6 (on Valverde representation, cf. [27]) is a joint work of all three co-authors. Section 7 (on partitions, cf. [61]) is mostly due to Petr Cintula, who also produced most proofs in Sections 3 and 4 (all of these proofs were presented in the paper in order to keep the exposition self-contained, even though some of the properties follow independently from more general theorems of [41]).

The clumsy proof of Corollary 4.11(I52) in the published version of the paper resulted from a trivial mistake discovered only when the final version was already submitted. The statement has in fact a trivial proof that uses just the monotony of the opening and closure operators and of min-intersection and max-union with respect to inclusion.

L. Běhounek, M. Daňková: *Relational compositions in Fuzzy Class Theory* [41]. The paper, written in 2006–2007, was originally intended to deal with properties of relational notions not covered by [28] (then under preparation) such as Cartesian products or preimages. However, it was soon realized by the present author that most of

such notions are just instances of (sup-T or inf-R) relational compositions for arguments with lesser arities, and that this relationship, which had only informally been sketched in Bělohávek’s monograph [46, Rem. 6.16], can be made precise in the formal framework of FCT by means of syntactic interpretations (cf. [21]). The systematic exploitation of the correspondence (including classes of arity 0 that represent truth values, cf. [18, §2]) resulted in a systematic and uniform description of more than 30 relational notions, with many properties translated automatically from a few basic properties of relational compositions. A large class of properties of these notions furthermore turns out to be derivable from a few identities in a simple equational calculus for fuzzy relations. The method thus provides a reduction of a large fragment of the elementary theory of fuzzy relations to a much simpler calculus, comparable to the reduction of a fragment of the theory of fuzzy classes to fuzzy propositional calculus by the metatheorems of [30, §3]. The fact that the fuzziness of the relations under consideration does not play a significant role in the application of the equational calculus to the relational notions further supports the thesis that with a suitable apparatus (here, of deductive fuzzy logic), the generalization of some parts of classical mathematics to fuzzy sets is rather straightforward (cf. the end of Section 1).

Even though a larger part of the paper is due to the present author, Martina Daňková had an indispensable role in the exhaustive derivation of relational properties in the equational calculus and providing links to the applied practice (esp. Examples 5.12–13). She also prepared and presented the preliminary conference version [40] of the paper and made a search for relevant literature. All parts of the paper were discussed and checked for correctness by both co-authors.

L. Běhouněk: *Extensionality in graded properties of fuzzy relations* [19].

The conference paper, presented at IPMU 2006, offers new definitions of basic graded properties of fuzzy relations, relative to a fuzzy indistinguishability relation between the objects of discourse. The approach is part of the effort to avoid hidden crispness in definitions, suggested already in the original FCT paper [30, §7]: the proposed definitions eliminate the implicit crisp identity of traditional graded definitions that is hidden in using multiple references to the same variable, and replace it by an explicit fuzzy indistinguishability relation E ; the traditional definitions are then the special cases for E equal to the crisp identity relation Id . The paper gives arguments supporting the need for such definitions, answers the counter-argument referring to an infinite regress, and shows that the traditional property of extensionality of a fuzzy relation w.r.t. an indistinguishability, which in the non-graded setting has the same motivation as the new definitions, cannot substitute the new definitions in the graded setting (although it can do so in the non-graded setting). The paper furthermore offers an explanation why only some of the indistinguishability-based properties have previously been defined in the non-graded setting.

It was not mentioned in the paper, though it should have been, that also E -functionality had been defined in the traditional non-graded setting (alongside several variant definitions of a fuzzy function) by Demirci [68, 69].

The conference paper only gave results relevant to its main theses, rather than a comprehensive list of properties of indistinguishability-based properties; these will be given in a full paper, which is currently under construction.⁴³ The full paper will also

⁴³Incidentally, all results included in the conference paper were first-order; therefore just first-order logic (MTL_Δ) could be employed. The higher-order setting is, nevertheless, needed for the study of

extend the game-theoretically motivated generalization sketched in Section 4 of the paper and a systematization of the properties in terms of sup-T and inf-R compositions.

L. Běhounek: *Towards a formal theory of fuzzy Dedekind reals* [16]. The conference paper, presented at EUSFLAT 2005, presents a sketch of a theory of fuzzy real numbers and fuzzy intervals based on the Dedekind completion of an underlying structure of crisp numbers. Besides purely theoretical motivations, one of the aims of the investigation was to model the traditional notion of fuzzy number (cf. Section 2.3 above, p. 16) in the logic-based framework. The (fuzzified) lattice completeness of the resulting fuzzy real numbers, construed as fuzzy Dedekind cuts, is proved, and the transition from Dedekind cuts to fuzzy intervals representing traditional fuzzy numbers is sketched.

Only fuzzy reals satisfying the defining conditions to degree 1 were considered in the paper, partly for simplicity and partly in order to adhere to the motivation that Dedekind cuts express the distribution of the fuzzy real number (which would be violated by any misbehavior of its distribution function, and therefore the condition should be satisfied to the full degree). The definitions are thus regarded as the *axioms* for fuzzy Dedekind cuts, rather than graded conditions. Results on the graded notion of fuzzy real would for a large part be obtainable by replacing Δ 's by suitable exponents, but this generalization might not be very interesting for mainstream fuzzy mathematics, as traditional fuzzy reals form a crisp class, too.

Even though only crisp rationals were considered for the underlying structure in this paper (as they are sufficient for generating a structure of fuzzy reals), the results obviously hold for each dense linearly ordered field of numbers (e.g., crisp real numbers, which are more often used for a construction of fuzzy numbers in the mainstream fuzzy mathematics). Fuzzy lattice completions of crisp dense linear orders are studied in more detail in the author's workshop paper [15], where two methods of obtaining a fuzzy lattice from such crisp orderings (viz, by Dedekind cuts and MacNeille stable sets, which differ in the fuzzy setting) are described.

The fuzzy lattice completion employed in the paper differs from fuzzy lattice completions described earlier in the literature [126, 46]: while [126] and [46] study the *minimal* fuzzy lattice completions of *fuzzy* orderings (achieved by MacNeille stable fuzzy sets), the present paper is concerned with a fuzzy completion of a *crisp* ordering, which need not be minimal, but should contain all fuzzy Dedekind cuts. Despite different settings and definitions in [126, 46], some results are nevertheless similar (for more details see [15]).

A similar approach to fuzzy numbers (or intervals) has later been taken by Horčík in his paper [134] on fuzzy interval analysis, where analogous results on representation and arithmetic of fuzzy intervals have been derived.

The results of [28, §5] on suprema were originally derived by the present author for the purposes of the discussed conference paper [16]. Therefore most proofs omitted from [16] can be found in [28, §5]. A full paper on this topic is still in progress; the main obstacle to finishing it is an as yet unclarified suitable definition of multiplication of fuzzy Dedekind cuts. (Observe that Horčík [134] also defines just multiplication by a scalar, i.e., a crisp number, which is unproblematic.) The aim is to extend the operations from the underlying crisp numbers to fuzzy cuts A in such a way that Aq can be interpreted as the truth value of $A \leq q$ (or a measure of the distribution of the fuzzy real A in $(-\infty, q]$), with the ordering preserved by the extended operations (cf. [16, §4]). Zadeh's extension

preservation of the indistinguishability-based properties w.r.t. class unions, intersections, etc., which will be given in the full paper.

principle works to this effect only if the original operation on crisp numbers is monotone; a suitable definition of multiplication of fuzzy cuts therefore has to separate positive numbers from negative ones (cf. the definition of multiplication for crisp Dedekind cuts), but the details of the construction that would capture the informal motivation correctly are not yet clear enough.⁴⁴ The ‘game-theoretical’ considerations on the interpretation of the truth values of the operands hinted at in Horčík’s paper [134] (similar to those sketched in [19, §4]) should also be taken into account. At present, a sound treatment of the extended operations on fuzzy intervals remains a subject for future work.

L. Běhounek: *Fuzzification of Groenendijk–Stokhof propositional erotetic logic* [14]. This early (and in many respects premature) paper is included in the dissertation in order to demonstrate a possible application of logic-based fuzzy mathematics as a formal semantics for fuzzified non-classical logics. By defining a fuzzified consequence relation of a non-classical logic in Fuzzy Class Theory, the fuzzified non-classical logic gets formally interpreted in FCT (i.e., in higher-order fuzzy logic). The apparatus of FCT then provides a well-defined framework for introducing semantical notions of the non-classical logic and deriving its metamathematical properties.

The paper avoids the problem of quantification over a fuzzy domain W by requiring the crispness [14, §6] or full contractivity [14, §7] of W ; an adequate account for arbitrary fuzzy logical spaces would need a better understanding of quantification over a fuzzy domain (cf. footnote 45 and comments on [43, 42] below). The paper only deals with yes–no questions, since yes–no partitions of a logical space are explicitly definable by means of negation; a logic-based theory of fuzzy partitions, needed for choice questions and first-order fuzzy erotetic logic, had not yet been developed in the time of writing the paper. A possible extension to fuzzy choice questions or to first-order fuzzy erotetic logic, generalizing the framework of [109], could use the results of [28, §7] on graded T-partitions: by [28, §7], T-partitions correspond to fuzzy equivalences (even in the graded manner); graded T-partitions thus provide a well-motivated basis for a partition semantics of fuzzy questions. This approach would enable to fuzzify the notion of question itself, by considering the fuzzy notion of T-partition. (In the discussed paper [14], the concept of fuzzy yes–no question itself is crisp, although it admits fuzzy answerhood.)

Further applications of the apparatus of FCT in the semantics of non-classical logic are sketched by the present author in the workshop paper [25] on fuzzified propositional dynamic logic (employed for modeling costs of program runs) and the Czech conference papers [23, 24] on fuzzified epistemic logic (employed for modeling feasible and vague knowledge). Full journal papers based on these conference papers are being prepared.

L. Běhounek, T. Kroupa: *Topology in Fuzzy Class Theory: Basic notions* [43]; *Interior-based topology in Fuzzy Class Theory* [42]. The conference papers [43] and [42] were presented, respectively, at the IFSA World Congress 2007 and the Conference of EUSFLAT 2007 (where the latter paper won the Distinguished Student Paper Award). The papers present the first treatment of fuzzy topology in the framework of FCT: they investigate the mutual relationships between alternative graded definitions of a fuzzy topology, namely by open or closed fuzzy sets [43, §3], fuzzy neighborhoods [43, §4], and fuzzy interior operators [42].

⁴⁴The definition of multiplication for fuzzy cuts over the discrete domain of integers, extending a cardinality-based multiplication of natural numbers, could help to clarify the matter; however, the theory of fuzzy functions and cardinalities has to be developed first.

Only fuzzy topologies over crisp universes have been considered in [43, 42]. This restriction has rather been chosen for methodological than technical difficulties. Even though it would be technically quite straightforward to generalize the notions for fuzzy universes (in exchange for several more exponents in definitions), the meaningfulness of such definitions would need a much more thorough discussion. For example, fuzzy domains are not preserved under the usual (sup-T) composition \circ of fuzzy mappings, which makes many straightforward constructions over fuzzy topological spaces ill-defined (as the mapping $F \circ G$ is defined on *another* fuzzy space, different from the domain of F) or ill-motivated. The motivational discussion needed to avoid ad hoc solutions exceeds the scope of fuzzy topology, as the questions encountered here are particular instances of more general problems of quantification over a fuzzy domain, which have not yet been satisfactorily addressed by deductive fuzzy logic.⁴⁵ The discussion of topologies over a fuzzy universe in the fully graded setting of FCT thus remains a task for future work, which can only be successfully solved after the general questions of quantification over fuzzy domains are addressed.

Although the underlying logic employed in [43] was IMTL, most results hold generally in MTL: the involutiveness of negation was only used to obtain the duality between open and closed fuzzy sets (which, moreover, seems to be inessential for fuzzy topology: cf. the successful development of constructive topology [183] where the duality fails as well).

The general approach and the definitions of basic concepts arose from joint discussions of both authors. Actual derivations of most particular properties listed in the papers have been done by T. Kroupa (some of them followed easily from his results in [149] on fuzzy filters), while the present author is responsible for most of the examples. Both authors participated in writing the papers and checking the results for correctness. P. Cintula gave us a hint on the importance of the inner exponent in the definition of U_{2c} [43, Def. 4.3]. The papers have been followed by an abstract [44] on the notions of continuity in the present setting; a full paper summarizing these results is under construction.

The present author's current view (which may not be shared by his co-author) on how logic-based fuzzy topology should further be developed differs somewhat from that presented in the above papers and is based on a more radical reading of [37, §7] on deprecating fixed preconditions in definitions. Obviously, the notions of fuzzy topology as presented in [43, 42] have to be parameterized by several indices that determine the multiplicities of conjuncts in the compound notion. The list of such indices, which is already too long, can further grow if more special properties of fuzzy topologies (like stratification [128] or separation axioms) are considered. Even the defining conditions proposed for open fuzzy topology (OTop) in our papers [43, 42] are themselves disputable as they are not independent (since $\emptyset \in \tau$ is implied by the union-closedness of τ); yet it would not be reasonable to omit the condition $\emptyset \in \tau$ in favor of the union-closedness, as the latter is much stronger and many properties only need the former. It is not at all clear which are the 'right' counterparts in fuzzy topology of the classical conditions that the empty set and the whole space are open. This suggests that the notion of fuzzy topological space is even less rigid than in classical mathematics or in traditional fuzzy mathematics (cf. the plenitude of variant kinds of topological spaces defined in both), and that there is no predetermined set of properties which together would form a well-motivated and sufficiently stable notion of fuzzy topology. Rather, there is a vague

⁴⁵Observe that the apparatus of deductive fuzzy logic itself only considers crisp domains of discourse, and thus is best suited to modeling fuzzy structures over crisp universes. Possibly, a proper use of strong quantifiers (see Section 2.2) might provide an adequate treatment of quantification over fuzzy domains: a detailed investigation in this direction has yet to be done.

informal set of independent ‘topologically flavored’ graded properties of *arbitrary* fuzzy systems of fuzzy subsets (or of fuzzy neighborhoods), and various combinations of such properties should be studied without restricting our attention in advance to a fixed set of conditions. Under this approach, fuzzy topology would not ask the properties of a pre-defined notion of fuzzy topology, but rather proceed in a reverse manner, by deriving preconditions ensuring such ‘topologically flavored’ properties (cf. the research program of reverse mathematics, e.g., in [189, Ch. 1]). This reverse style of logic-based fuzzy topology may well be the right manner of developing logic-based fuzzy mathematics in general, as the problem of too many indices and the absence of a fixed set of defining conditions are not specific for fuzzy topology (being due just to the non-contractivity of conjunction).

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Part II

Papers comprising the thesis

Note: Papers omitted in this copy

Part II (pp. 53–250), consisting of the published papers comprising the thesis, is omitted in this copy. See the full version of the thesis or refer to the original articles (preprints are freely available online).

Part III

Mandatory annexes

Abstracts

English abstract

The dissertation consists of the author's published papers on logic-based fuzzy mathematics. It is accompanied with a cover study (Part I of the thesis), which introduces the area of logic-based fuzzy mathematics, argues for the significance of the area of research, presents the state of the art, indicates the author's contribution to the field, and comments on the papers comprising the thesis.

Fuzzy mathematics can be characterized as the study of *fuzzy structures*, i.e., mathematical structures in which the two values 0, 1 are at some points replaced by a richer system of degrees. Under the *logic-based* approach, fuzzy structures are formalized by means of *axiomatic theories* over suitable systems of *fuzzy logic*, whose rules replace the rules of classical logic in formal derivation of theorems. The main advantages of the logic-based approach are the general gradedness of defined notions, methodological clarity provided by the axiomatic method, and the applicability of a foundational architecture mimicking that of classical mathematics. Logic-based fuzzy mathematics is part of a broader area of non-classical mathematics (i.e., mathematical disciplines axiomatizable in non-classical logics), as well as a specific subfield of general fuzzy methods. Following earlier isolated developments in logic-based fuzzy set theory and arithmetic, a systematic logic-based study of fuzzy mathematics was made possible by recent advances of first-order fuzzy logic that opened the way for Henkin-style higher-order fuzzy logic (or simple fuzzy type theory), which is capable of serving as a foundational theory for logic-based fuzzy mathematics. The author's contribution to the development of logic-based fuzzy mathematics has been presented in the published papers that comprise the main body of the thesis.

The paper *On the difference between traditional and deductive fuzzy logic* clarifies methodological assumptions of formal fuzzy logic, contrasts them to those of traditional fuzzy mathematics, and indicates necessary conditions on systems of fuzzy logic suitable for logic-based fuzzy mathematics as developed in this thesis. The paper *From fuzzy logic to fuzzy mathematics: a methodological manifesto* (co-authored by P. Cintula) formulates methodological guidelines for logic-based fuzzy mathematics and proposes a foundational architecture analogous to that of classical mathematics, with three layers formed by first-order fuzzy logic, a foundational theory axiomatized in fuzzy logic, and particular mathematical disciplines developed within the foundational theory.

The paper *Fuzzy class theory* (co-authored by P. Cintula) introduces Henkin-style higher-order fuzzy logic (also called Fuzzy Class Theory or FCT) as an axiomatic approximation of Zadeh's notion of fuzzy set, and proposes it as a foundational theory for logic-based fuzzy mathematics. Metatheorems are proved for FCT that reduce a large part of elementary fuzzy set theory to propositional fuzzy logic, and the interpretability

of classical higher-order theories in FCT (by which classical mathematical structures are available within the theory) is shown in the paper.

The paper *Relations in Fuzzy Class Theory: initial steps* (co-authored by U. Bodenhofer and P. Cintula) develops the basic theory of fuzzy relations in FCT, which is a prerequisite for all other parts of formal fuzzy mathematics. The topics studied include basic graded properties of fuzzy relations, relational images and bounds, Valverde characterization theorems, and fuzzy partitions. The paper *Relational compositions in Fuzzy Class Theory* (co-authored by M. Daňková) reduces a large family of fuzzy relational and set-theoretical notions to fuzzy relational compositions, and presents methods for mass proofs of theorems on these notions. The paper *Extensionality in graded properties of fuzzy relations* introduces indistinguishability-relative graded properties of fuzzy relations and studies their relationship to the property of extensionality, to which they reduce in traditional fuzzy mathematics, but not in the logic-based setting.

The paper *Towards a formal theory of fuzzy Dedekind reals* constructs fuzzy real numbers as the lattice completion of the classical real line by fuzzy Dedekind cuts and gives some hints for logic-based fuzzy interval arithmetics. The paper *Fuzzification of Groenendijk–Stokhof propositional erotetic logic* employs FCT as the formal semantics for a logic of fuzzy questions. Finally, the papers *Topology in Fuzzy Class Theory: basic notions* and *Interior-based topology in Fuzzy Class Theory* (both co-authored by T. Kroupa) introduce logic-based notions of fuzzy topology defined respectively by open or closed sets, neighborhoods, and interior operators, and study their mutual relationships.

Český abstrakt (Czech abstract)

Předložená disertační práce sestává z autorových publikovaných článků o logických základech fuzzy matematiky, doplněných shrnující studií (tvořící úvodní část disertace), ve které je představen na formálnělogický přístup k fuzzy matematice. Dále je v ní dokládána důležitost výzkumu v tomto oboru a charakterizován jeho současný stav, popsán autorův příspěvek k oboru a podány komentáře k jednotlivým článkům, z nichž se disertace skládá.

Fuzzy matematiku lze vymezit jako studium *fuzzy struktur*, tj. takových matematických struktur, v nichž je dvojice hodnot 0, 1 na některých místech nahrazena bohatším systémem stupňů. V přístupu založeném na formální logice jsou fuzzy struktury zachyceny prostřednictvím *axiomatických teorií* ve vhodných systémech *fuzzy logiky*, jejichž pravidla jsou použita pro formální odvozování teorémů namísto pravidel klasické logiky. Hlavními výhodami logického přístupu k fuzzy matematice jsou všeobecná gradualita definovaných pojmů, metodologická čistota daná aplikací axiomatické metody a použitelnost podobné základové architektury jako v klasické matematice. Na logice založená fuzzy matematika je součástí neklasické matematiky (tj. rodiny matematických teorií axiomatizovatelných v neklasických logikách), a zároveň tvoří specifickou část širšího oboru fuzzy metod. Systematické zkoumání fuzzy matematiky v přístupu založeném na logice, navazující na předchozí ojedinělé výzkumy podobného přístupu k teorii fuzzy množin a aritmetice, bylo umožněno nedávným pokrokem v oblasti prvořádkové fuzzy logiky. Díky němu bylo možno vyvinout henkinovskou fuzzy logiku vyššího řádu (čili jednoduchou fuzzy teorií typů), jež může sloužit jako základová teorie pro formální fuzzy matematiku. Autorovy příspěvky k výzkumu logických základů fuzzy matematiky byly publikovány v člancích, které tvoří hlavní část disertace.

Článek *On the difference between traditional and deductive fuzzy logic* (K rozdílu mezi tradiční a deduktivní fuzzy logikou) vyjasňuje metodologické předpoklady formální fuzzy logiky ve srovnání s předpoklady tradiční fuzzy matematiky a stanovuje požadavky na systémy fuzzy logiky vyhovující takovému přístupu k fuzzy matematice, jaký je rozvíjen v této disertaci. V článku *From fuzzy logic to fuzzy mathematics: a methodological manifesto* (Od fuzzy logiky k fuzzy matematice – metodologický manifest, spoluautor P. Cintula) jsou formulovány metodologické zásady na logice založeného přístupu k fuzzy matematice a je navržena její základová architektura způsobem analogickým k základům klasické matematiky, se třemi vrstvami tvořenými prvořádovou fuzzy logikou, v ní axiomatizovanou základovou teorií a jednotlivými matematickými disciplínami vyvíjenými v rámci této základové teorie.

V článku *Fuzzy class theory* (Teorie fuzzy tříd, spoluautor P. Cintula) je zavedena henkinovská fuzzy logika vyššího řádu (zvaná též teorie fuzzy tříd, zkr. FCT z angl. Fuzzy Class Theory), jakožto axiomatická aproximace Zadehova pojmu fuzzy množiny. Tato teorie je zde navržena za základovou teorii pro formální fuzzy matematiku. V článku jsou dokázány metavěty FCT, které redukuje značnou část elementární teorie fuzzy množin na výrokovou fuzzy logiku, a je ukázána interpretovatelnost klasických teorií vyššího řádu v FCT (díky níž jsou v FCT k dispozici klasické matematické struktury).

V článku *Relations in Fuzzy Class Theory: initial steps* (Relace v teorii fuzzy tříd – počáteční kroky, spoluautoři U. Bodenhofer a P. Cintula) jsou v rámci FCT vybudovány základy teorie fuzzy relací, jež tvoří nezbytný předpoklad zkoumání ostatních partií fuzzy matematiky. V článku se zkoumají zejména základní graduální vlastnosti fuzzy relací, obrazy, závory, valverdovské charakterizace a fuzzy rozklady. V článku *Relational compositions in Fuzzy Class Theory* (Skládání relací v teorii fuzzy tříd, spoluautorka M. Daňková) popisuje redukcí rozsáhlé rodiny pojmů teorie fuzzy relací a fuzzy množin na pojem skládání fuzzy relací a ukazuje metodu hromadných důkazů vět o těchto pojmech. Článek *Extensionality in graded properties of fuzzy relations* (Extenzionalita u graduálních vlastností fuzzy relací) zavádí graduální vlastnosti fuzzy relací definované relativně vůči dané relaci nerozlišitelnosti a studuje jejich vztah k vlastnosti extenzionality, s níž v tradiční fuzzy matematice splývají, v přístupu založeném na logice se však od ní liší.

Článek *Towards a formal theory of fuzzy Dedekind reals* (Předběžné poznámky k formální teorii dedekindovských fuzzy reálných čísel) podává konstrukci fuzzy reálných čísel pomocí svazového zúplnění klasické reálné číselné osy fuzzy dedekindovskými řezy a uvádí některé výsledky potřebné k vybudování fuzzy intervalové aritmetiky. V článku *Fuzzification of Groenendijk–Stokhof propositional erotetic logic* (Fuzzifikace výrokové Groenendijkovy–Stokhofovy erotetické logiky) je aparát FCT použit jako formální sémantika pro logiku fuzzy otázek. V závěrečných článcích *Topology in Fuzzy Class Theory: basic notions* (Topologie v teorii fuzzy tříd – základní pojmy) a *Interior-based topology in Fuzzy Class Theory* (Topologie definovaná pomocí operátoru vnitřku v teorii fuzzy tříd, spoluautor obou článků T. Kroupa) jsou v rámci přístupu založeném na logice zavedeny pojmy fuzzy topologie definované pomocí otevřených či uzavřených množin, okolí bodů a operátoru vnitřku a prozkoumány jejich vzájemné vztahy.